Is the category of finite sup semilattices compact $*$-autonomous?

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Abstract

The category of finite distributive sup semi-lattices (distributive lattices and sup-preserving morphisms) is $*$-autonomous but not compact. We also show that the category lacks equalizers.

The category $\mathcal{S}up$

The category $\mathcal{S}up$ of complete sup semi-lattices and sup preserving morphisms is well known to be $*$-autonomous. In the talk, I will be concentrating on the finite sup semi-lattics. A finite (or complete) sup semi-lattice is obviously a lattice and so it makes sense to ask that it be distributive. We will denote by $\mathcal{FDS}up$ the category of finite distributive semi-lattices. We will show that it is $*$-autonomous, but not compact.

Primer on $*$-autonomous categories

Recall that a $*$-autonomous is a category that is closed monoidal with $\multimap$ as internal hom, $\otimes$ as tensor, $\top$ as tensor unit, and a "dualizing object" $\perp$ satisfying the usual conditions, which include that $(A \otimes B) \multimap C \cong A \multimap (B \multimap C)$ and, in particular, with A^* defined as $A \multimap \perp$, the canonical map $A \longrightarrow A^{**}$ is an isomorphism. From these isomorphisms several others follow.

In the case of $\mathcal{Sup}$, the dual $A^* = A^{\mathrm{op}}$. If $f : A \longrightarrow B$ is a morphism, it is cocontinuous between cocomplete categories. Since they are posets, the solution set condition is automatic and hence f has a right adjoint $g : B \longrightarrow A$. Then $f^* = g^{\mathrm{op}} : B^{\mathrm{op}} \longrightarrow A^{\mathrm{op}}$. Note that although g is continuous from $B \longrightarrow A$, $f^* = g^{\mathrm{op}} : B^* \longrightarrow A^*$ is cocontinuous.

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Compact $*$-autonomous categories

A $*$-autonomous category is said to be *compact* if for all objects A and B we have $A \multimap B \cong A^* \otimes B$. Not very many such categories are known, the prime example being finite dimensional vector spaces over a field. Finite abelian groups would be another example, except that neither the tensor unit nor the dualizing object are finite.
An interesting example of a $*$-autonomous is the category of complete sup semi-lattices. Both $\top$ and $\perp$ are the 2 element chain. At some point as I was preparing my monograph on $*$-autonomous categories, someone (Max Kelly?) asked whether it was compact.

Interlude on $*$-autonomous categories

1.

$$(A \otimes B)^* = (A \otimes B) \multimap \perp \cong A \multimap (B \multimap \perp) = A \multimap B^*$$

whence $A \otimes B \cong A \multimap B^*$ and that can be taken as a definition if $\multimap$ and * are already given.

2.

$$A \multimap B \cong A \multimap B^{**} \cong A \multimap (B^* \multimap \perp)$$

$$\cong (A \otimes B^*) \multimap \perp = (A \otimes B^*)^*$$

which could be taken as a definition if $\otimes$ and * are already given.

Interlude on compact ∗-autonomous categories

1. $$(A \otimes B)^* \cong A \multimap B^* \cong A^* \otimes B^*$$

2. $$(A \multimap B)^* \cong (A \otimes B)^* \cong A^* \otimes B \cong A^* \multimap B^*$$

The counter-example for finite sup semi-lattices

Compactness would imply that

$$(A \multimap B)^* \cong (A^* \otimes B) \multimap \bot \cong A^* \multimap (B \multimap \bot) = A^* \multimap B^*$$

Now let

$A = A^* =$ $\quad B =$

Thus would imply that $A \multimap B \cong A \multimap B^*$. But it is possible to directly calculate that there are exactly 88 sup-preserving maps $A \longrightarrow B$, while there are 94 such maps $A \longrightarrow B^*$.

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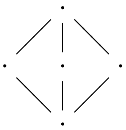
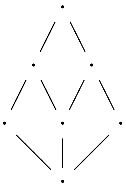
$A = A^* =$  $B =$ 

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Distributive lattices

Thinking about this recently, it occured to me that A and B are not distributive. So what about the case of finite distributive lattices? I leave the general complete case untouched. The methods I will describe later clearly do not work outside of the finite case, even leaving aside the question of what kind of distributivity.

There are two separate quesions to be raised here.

1. Is $\mathcal{FDSup}$ $*$-autonomous?
2. Assuming it is, is it compact?

It turns out that the answer to the first question is yes and to the second is no. The rest of this lecture will describe how to show these.

Bases

Let A be a finite sup semi-lattice. By an **order basis** (or simply **basis**) for A, we mean a subset $S \subseteq A$ such that

1. Every element of A is a sup of elements of S
2. If B is another finite sup semi-lattice, then any order preserving map $S \longrightarrow B$ extends to a unique sup-morphism $A \longrightarrow B$

Likely the first condition follows from the uniqueness in the second.
Theorem. *Every finite distributive sup semi-lattice has an order basis.*

Proof of theorem

If $A_0 \subseteq A$ is a (full) sup subsemi-lattice and $a \in A$, let $A_0[a]$ denote the sup subsemi-lattice generated by A_0 and a. Obviously, it consists of all elements $a_0 \vee a$, with $a_0 \in A_0$. Note that this is not necessarily a sublattice since it might not have the same infs as A, but by hypothesis it has the same sups.

The theorem obviously follows from:

Lemma. *Suppose $A_0 \subseteq A$ is a proper ideal that has a basis. Then there is some element $a \in A$ such that $A[a]$ is an ideal and has a basis.*

Note that an ideal is a sub semi-lattice, although a proper ideal cannot be a sublattice.

Proof of lemma

Since A is finite and A_0 is an ideal, there is at least one element $a \in A$ all of whose predecessors lie in A_0. We claim that $A_0[a]$ is an ideal. In fact, if $a' \leq a_0 \vee a$, with $a_0 \in A_0$, then $a' = a' \wedge (a_0 \vee a) = (a' \wedge a_0) \vee (a' \wedge a)$. Now $a' \wedge a_0 \in A_0$ since A_0 is an ideal. As for $(a' \wedge a)$ it is either a or it is in A_0 by minimality of a. In either case, $a' \in A[a]$.
Now let S_0 be a basis of A_0. Then I claim that $S = S_0 \cup \{a\}$ is a basis for $A_0[a]$. That it sup-generates $A_0[a]$ is obvious.

Proof of lemma, continued

So suppose that $f : S \longrightarrow B$ is an order preserving map. Let $f_0 : A_0 \longrightarrow B$ be the unique extension $f|S_0$ and set $b = f(a)$. In order to extend f to $A_0[a]$ we have to let $f(a_0 \vee a) = f_0(a_0) \vee b$. We must show that the extended f is well-defined. We claim that if $a_0 \in A_0$ is an element such that $a_0 \leq a$, then $f_0(a_0) \leq b$. For we can write $a_0 = x_1 \vee \cdots \vee x_k$, $x_i \in S_0$. But then all these $x_i \leq a_0 \leq a$ so that $f(x_i) \leq b$ which implies that $f_0(a_0) = f(x_1) \vee \cdots \vee f(x_n) \leq b$. Now if $a_0 \vee a = a_0' \vee a$, we have

$$a_0 = a_0 \wedge (a_0 \vee a) = a_0 \wedge (a_0' \vee a) = (a_0 \wedge a_0') \vee (a_0 \wedge a)$$

and both terms belong to A_0. It follows that $f_0(a_0) = f_0(a_0 \wedge a_0') \vee f_0(a_0 \wedge a)$. But then

Proof of lemma, completed

$$f_0(a_0) \vee b = f_0(a_0 \wedge a_0') \vee f_0(a_0 \wedge a) \vee b = f_0(a_0 \wedge a_0') \vee b$$

since $f_0(a_0 \wedge a) \leq b$. Similarly $f_0(a_0') \vee b = f_0(a_0 \wedge a_0') \vee b$ and so the extension of f is well defined.
Finally, we must show that f preserves sup. We have

$$f((a_0 \vee a) \vee (a_0' \vee a)) = f(a_0 \vee a_0' \vee a)$$

$$= f_0(a_0 \vee a_0') \vee b = f_0(a_0) \vee f_0(a_0') \vee b$$

$$= (f_0(a_0) \vee b) \vee (f_0(a_0') \vee b) = f(a_0 \vee a) \vee f(a_0' \vee a)$$

and a similar computation if only one of the two involves a.

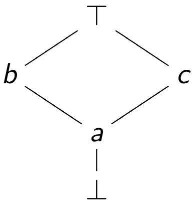
$\mathcal{FDSup}$ is closed

We must show that when A and B are distributive, so is $A \multimap B$. But given a basis $S \subseteq A$ a map $f : A \longrightarrow B$ is completely determined by its restriction $f|S$. Moreover, $f \leq g$ iff $f|S \leq g|S$. It follows immediately that $(f \vee g)|S = f|S \vee g|S$ and $(f \wedge g)|S = f|S \wedge g|S$. But it is important to note that while the pointwise inf of f and g is not the inf in $A \multimap B$ since that will not usually preserve sup, the pointwise inf of $f|S$ and $g|S$ is still order preserving and thus must be $(f \wedge g)|S$. This is the significance of having a basis.

Since the inf and sup of $f|S$ and $g|S$ are calculated pointwise, the distributivity $A \multimap B$ follows from that of B.

$\mathcal{FDS}up$ is not compact

We begin with the distributive lattice A



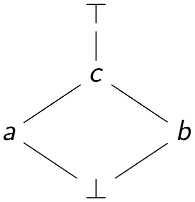
Let B be an arbitrary lattice. A sup morphism $f : A \longrightarrow B$ must satisfy $f(\bot) = \bot$, $f(\top) = f(b) \vee f(c)$ and $f(a) \leq f(b) \wedge f(c)$. Thus if we let $f(b) = x$, $f(c) = y$, and $f(a) = z$, we see that

$$A \multimap B = \{(x, y, z) \in B^3 \mid z \leq x \wedge y\}$$

.

$\mathcal{FDS}up$ is not compact, continued

Now let us do the same for A^*:



If B is an arbitrary lattice a sup morphism $f : A^* \longrightarrow B$ must satisfy $f(\bot) = \bot$, $f(c) = f(a) \vee f(b)$ and $f(\top) \geq f(a) \vee f(b)$. Thus if we let $f(a) = x$, $f(b) = y$, and $f(\top) = z$. We have that

$$A^* \multimap B = \{(x, y, z \in B^3) \mid z \geq x \vee y\}$$

$\mathcal{FDSup}$ is not compact, continued

To repeat:

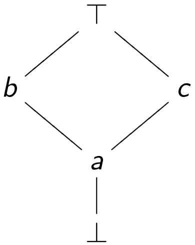
$$A^* \multimap B = \{(x, y, z \in B^3) \mid z \geq x \vee y\}$$

But this implies that

$$A^* \multimap B^* = \{(x, y, z) \in B^3 \mid z \leq x \wedge y\}$$

In other words $A^* \multimap B^* \cong A \multimap B$. But in order that $\mathcal{Sup}$ be compact it is necessary that $A^* \multimap B^* \cong (A \multimap B)^*$. Thus if we can find a distributive lattice B for which $A \multimap B$ is not isomorphic to its dual, it follows that $\mathcal{FDSup}$ is not compact.

$\mathcal{FDS}up$ is not compact, concluded

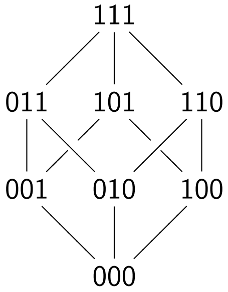


We can, in fact, take $B = A$. The elements of $A \multimap A$ consists of all (x, y, z) such that $z \leq x \wedge y$, with the product order. It follows that the atoms of $A \multimap A$ are $(a, \bot, \bot)$ and $(\bot, a, \bot)$. But $A \multimap A$ has four coatoms: $(b, \top, b)$, $(\top, b, b)$, $(c, \top, c)$, and $(\top, c, c)$. Thus $(A \multimap A)^* \ncong (A^* \multimap A^*)$ and we conclude:

Theorem. *$\mathcal{FDS}up$ is not compact $*$-autonomous.*

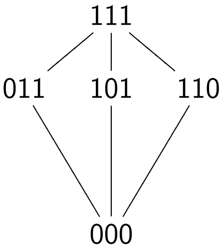
Is $\mathcal{FDSup}$ closed under finite limits?

It is not hard to show that if it were, it would have to be equational which would mean there was an equation involving only sup and $\perp$ that forced distributivity. This seems unlikely. I will start with a counter example and then show how I found it.
We begin with 2^3 which can be described as the following lattice:



Finite limits?

The set of atoms is obviously a basis for 2^3. We let $f : 2^3 \longrightarrow 2^3$ be the map for which $f(001) = 110$, $f(010) = 101$, and $f(100) = 110$. It follows that $f(000) = 000$ and $f(110) = f(101) = f(011) = f(111) = 111$. Let $g : 2^3 \longrightarrow 2^3$ be determined by $g(001) = g(010) = g(100) = 111$, from which it follows that $g(000) = 000$ and $g(110) = g(101) = g(011) = g(111) = 111$. It is obvious that the equalizer of f and g is the lattice



which is not distributive.

Finite limits?

I did not pull these maps out of thin air. As I mentioned above, if $\mathcal{FDSup}$ had been closed under finite limits, it would be equational over finite sets. The underlying functor has an adjoint $X \mapsto 2^X$ and then every object would be a quotient of a power of 2. In particular, we have a surjection $2^3 \longrightarrow A$, that takes the three atoms to the three middle atoms of A. The kernel pair is also a quotient of a power of 2, so we have a coequalizer $2^X \rightrightarrows 2^3 \longrightarrow A$, which dualizes to an equalizer $A \longrightarrow 2^3 \rightrightarrows 2^X$ since all three are self-dual. I then constructed a simple example using $X = 3$ and dualized it to give the present one.

What about $A \multimap B$ when A is infinite?

I don't know the answer but if time permits I will show an attempt that failed. First, the above argument works just as well when B is infinite, so the only case that matters is when A is infinite. A good categorist tries to write $A = \operatorname{colim} A_i$ over all finite sublattices. The arrows in this diagram are lattice inclusions and $A \multimap B = \lim(A_i \multimap B)$. It looks like the RHS is a diagram of distributive lattices and it is. The problem is that for this argument to work, we must have the transition maps be lattice homomorphisms. So we must show that if $A_1 \longrightarrow A_2$ is a lattice homomorphism of finite distributive lattices, then $A_2 \multimap B \longrightarrow A_1 \multimap B$ is a lattice homomorphism. Is it?

Map induced by a lattice homomorphism

No, it is not true even when $B = 2$. We give an example to show $A_2^* \longrightarrow A_1^*$ is not a lattice homomorphism while A_1 is a sublattice of A_2.

$$A_1 = \begin{tikzcd}[row sep=large] \top \arrow[d, dash] \\ a \arrow[d, dash] \\ \bot \end{tikzcd} \qquad A_2 = \begin{tikzcd} & \top \arrow[dl, dash] \arrow[dr, dash] & \\ b \arrow[dr, dash] & & c \arrow[dl, dash] \\ & \bot & \end{tikzcd}$$

Let $f : A_1 \longrightarrow A_2$ be the inclusion such that $f(a) = b$. Then f has a right adjoint g for which $g(\top)$ and $g(\bot) = \bot$, $g(b) = \bigvee\{x \mid f(x) \leq b\} = a$, while $g(c) = \bigvee\{x \mid f(x) \leq c\} = \bot$. This is not a lattice homomorphism and hence neither is its opposite map which is f^*.