Completions of subcategories of domains

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http://www.math.mcgill.ca/barr/papers

Abstract

We have been studying the limit completion, in the category of commutative rings, of various subcategories of integral domains. Since any limit of domains is a semiprime ring (only nilpotent is 0), we will concentrate on the limit closure in that subcategory. This will complement the talk Bob gave two weeks ago

Some subcategories of domains

- $\mathcal{A}_{\mathrm{dom}}$, the category of domains;
- $\mathcal{A}_{\mathrm{fld}}$, the category of fields;
- $\mathcal{A}_{\mathrm{pfld}}$, the category of perfect fields;
- $\mathcal{A}_{\mathrm{ic}}$, the category of integrally closed domains;
- $\mathcal{A}_{\mathrm{bez}}$, the category of Bézout domains;
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- $\mathcal{A}_{\mathrm{icp}}$, the category of perfect integrally closed domains;
- $\mathcal{A}_{\mathrm{per}}$, the category of perfect domains;
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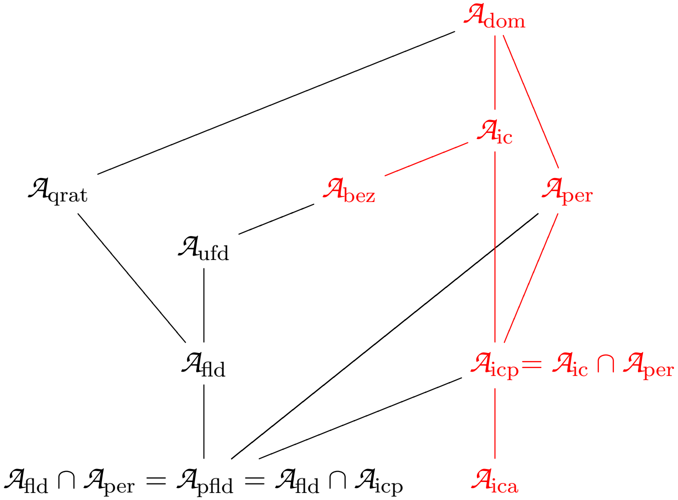
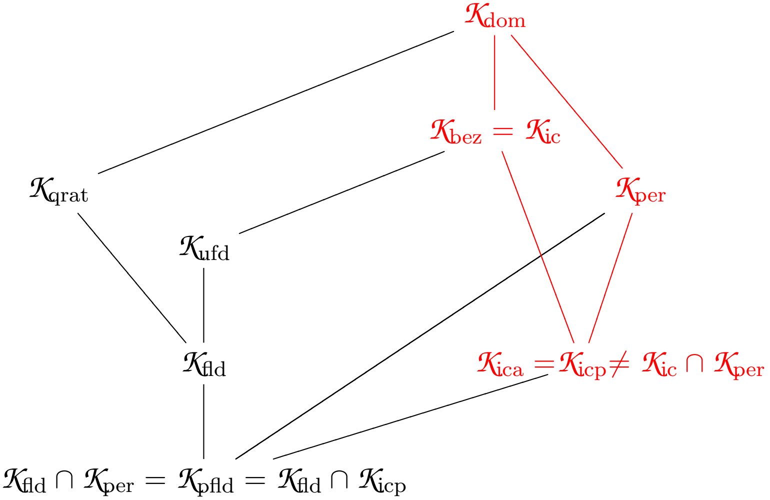
Relations among the subcategories



Relations among their limit closures.



Basic assumptions

- $\mathscr{A}$ is a category of domains (such as one of the above).
- $\mathscr{K}$ is the limit closure of $\mathscr{A}$ in commutative rings.
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Construction of K and G

$K : \mathcal{SPR} \longrightarrow \mathcal{K}$ is the adjoint to the inclusion of $\mathcal{K}$ into the category of semiprime rings, easily shown to exist.

G is more interesting. Let $\mathcal{B} \subseteq \mathcal{K}$ consist of all domains in $\mathcal{K}$. In most cases it is larger than $\mathcal{A}$.

Example: Define D as the pullback $\mathbf{Z}[x] \times_{\mathbf{Z}_2[x]} \mathbf{Z}_2[x^2]$. Then $D \in \mathcal{B}_{\mathrm{ic}}$ but is not integrally closed since $x \notin D$ satisfies the integral equation $t^2 - x^2$ with coefficients in D.

For a domain D we let $G(D)$ denote the intersection of all objects of $\mathcal{B}$ that contain D. There is at least one since there is a field in $\mathcal{A}$ that contains D.

- Suppose $D \subseteq F \in \mathcal{A}$ with F a field. Then $G(D)$ is the intersection of all $\mathcal{B}$-subobjects of F that contain D.

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Some properties of G and K

- $G(D)$ is a subring of the perfect closure of the field of fractions of D.
- The inner adjunction $R \longrightarrow K(R)$ is an injection.
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2. $P \subseteq D$, *there is a map* $G(D) \longrightarrow G(D/P)$.
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2. $P \subseteq D$, *there is a map* $G(D) \longrightarrow G(D/P)$.
3. *The map* $\mathrm{Spec}(G(D)) \longrightarrow \mathrm{Spec}(D)$ *is surjective.*
4. G *is a functor on domains.*
5. $K(D)$ *is a domain.*
6. $\ker(K(R) \longrightarrow K(R/P))$ *is prime.*
7. $\mathrm{Spec}(K(R)) \longrightarrow \mathrm{Spec}(R)$ *is an order isomorphism.*
8. $R \longrightarrow K(R)$ *is essential.*
9. $R \subseteq S \subseteq K(R)$ *implies* $K(S) = K(R)$.
10. $R \subseteq S \subseteq K(R)$ *implies* $R \hookrightarrow S$ *is epic.*
11. $R \hookrightarrow K(R)$ *is integral.*
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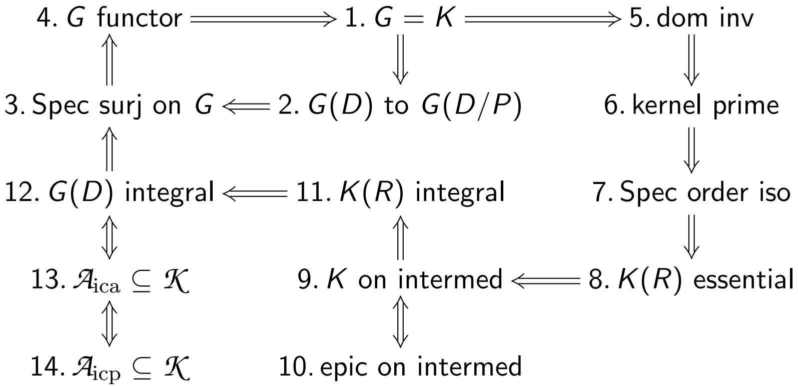
Diagram of logical inferences



Sample results

- A semiprime ring satisfies the (2,3)-condition if whenever $r^3 = s^2$, there is a t (provably unique) such that $t^2 = r$ and $t^3 = s$. To prove uniqueness, compute $(t - u)^3$.
- It is interesting, although not important, to note that the (2,3)-condition is equivalent to the (k,n)-condition whenever $k > 1$ and $n > 1$ are relatively prime integers.
- Every integrally closed domain D satisfies that condition. The element $t = s/r$ of the field of fractions solves it and is integral over D.
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Sample results, continued

- A semiprime ring satisfies the DL-condition if whenever $r^3 = s^2$ and r is a square mod every prime ideal, then there is a t (provably unique) such that $t^2 = r$ and $t^3 = s$.
- Using the compactness of Spec in the domain topology, you can prove that the condition of being a square mod every prime is equivalent to the existence of a set $\{t_1, \ldots, t_n\}$ such that $(r - t_1^2) \cdots (r - t_n^2) = 0$.
- Every domain trivially satisfies the DL-condition.
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These conditions are essentially algebraic

- Aside from the operations defining commutative rings, we let α be the unary partial operation whose domain consists of $\{r \mid r^2 = 0\}$, subject to the equations $\alpha(r) = r$ and $\alpha(r) = 0$. The algebras for this theory is just the semiprime rings.
- Add a binary operator β whose domain is $\{(r, s) \mid r^3 = s^2\}$ and subject to the equations $\beta(r, s)^2 = r$ and $\beta(r, s)^3 = s$. The algebras for this theory are the (2,3)-closed rings.

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These are essentially algebraic, cont'd

- Aside from the operations and the partial operation defining the semiprime rings, we add, for each $n > 0$, a partial $(n+2)$-ary operation β_n whose domain is

$$\{(r, s, t_1, \dots, t_n) \mid r^3 = s^2 \text{ and } (r - t_1^2) \cdots (r - t_n^2) = 0\}$$

  subject to the equations that, for $t = \beta_n(r, s, t_1, \dots, t_n)$, then $t^2 = r$ and $t^3 = s$.
- In all cases the values of the partial operations are unique, subject to the equations, and therefore the subcategory of models is full in the category of commutative rings.

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