

# BECK MODULES FOR MONOIDS

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## 1. Introduction

The notion of Beck module was introduced in [Beck, 1967]. If  $X$  is an object of the category  $\mathcal{X}$ , Beck defined an  $X$ -module to be an abelian group object in the slice category  $\mathcal{X}/X$ . He then showed what this amounted to in several categories:

1. When  $\mathcal{X}$  is the category of associative algebras with unit, a Beck module over a ring  $X$  is a 2-sided  $X$ -module.
2. When  $\mathcal{X}$  is the category of groups, a Beck module over a group  $X$  is a right  $X$ -module.
3. When  $\mathcal{X}$  is the category of Lie algebras, a Beck module over a Lie algebra  $X$  is just an  $X$ -module.

Since the abelian group objects form a category, it is also of note that in these three cases the category of Beck modules is exactly the category of modules in the usual sense.

Recently, a student from Paris named Léonard Guetta asked me if I knew what a Beck module for a monoid is. This note answers that question. It is not what you would usually think of as module for the monoid.

## 2. The objects

As is well-known, if a category  $\mathcal{X}$  has finite limits, an abelian group object  $Y \in \mathcal{X}$  is given by an addition  $+$  :  $Y \times Y \rightarrow Y$ , a negation  $-$  :  $Y \rightarrow Y$  and a zero map  $0$  :  $1 \rightarrow X$ , subject to the identities that characterize an abelian group. In the case of a slice category  $\mathcal{X}/X$ , the addition is a map  $Y \times_X Y \rightarrow Y$  over  $X$ , the negation must commute with  $p$  and the zero map  $0 : X \rightarrow Y$  has to split  $p$ .

If  $X$  is a monoid, let  $p : Y \rightarrow X$  be a monoid homomorphism. An abelian group structure on  $Y$  requires maps  $+$  :  $Y \times_X Y \rightarrow Y$ ,  $-$  :  $Y \rightarrow Y$  and  $0 : X \rightarrow Y$ , all over  $X$  and satisfying the abelian group axioms. If  $M_x = p^{-1}(x)$ , then  $Y \times_X Y$  is just the disjoint union of the  $M_x$  and the abelian group structure is on the fibres. Since the minus map also preserves the fibres, it is clear that each fibre is an abelian group so that an  $X$ -module is an  $X$ -indexed family of abelian groups.

In the following, we will let  $x, x', x''$  denote elements of  $X$ ,  $m, m_1, m_2$  denote elements of  $M_x$ ,  $m', m'_1, m'_2$  denote elements of  $M_{x'}$  and  $m'' \in M_{x''}$ . For reasons that will become clear, we introduce the notation  $(x, m)$  for an element of  $M_x$ . Then  $Y = \{(x, m) \mid x \in X, m \in M_x\}$ . Since  $p$  is a monoid homomorphism, the product  $(x, m)(x', m')$  must have first coordinate  $xx'$  and we will explore what its second coordinate is. Since the 0 map, which takes  $x$  to  $(x, 0)$  is also a monoid homomorphism, we must have  $(x, 0)(x', 0) = (xx', 0)$ .

### 2.1. LEMMA.

$$(x, m_1)(x', m'_1) + (x, m_2)(x', m'_2) = (x, m_1 + m_2)(x', m'_1 + m'_2)$$

PROOF. Since  $+$  is a homomorphism of monoids, the diagram

$$\begin{array}{ccc} (Y \times_X Y) \times (Y \times_X Y) & \xrightarrow{+\times+} & Y \times Y \\ \downarrow & & \downarrow \\ Y \times_X Y & \xrightarrow{+} & Y \end{array}$$

in which the vertical maps are the monoid multiplication, must commute. The monoid structure in  $Y \times_X Y$  is coordinate-wise. That is,  $((x, m_1), (x, m_2))((x', m'_1), (x', m'_2)) = ((x, m_1)(x', m'_1), (x, m_2)(x', m'_2))$ . If we follow the element  $((x, m_1), (x, m_2), (x', m'_1), (x', m'_2))$  around the two paths, we get the desired equation.

For any  $x, x' \in X, m \in M_x$ , the element  $(x, m)(x', 0) \in M_{xx'}$  and we denote it  $(xx', mx')$ . Similarly, for  $m' \in M_{x'}$ , we write,  $(x, 0)(x', m') = (xx', xm')$ .

### 2.2. THEOREM.

- (1)  $(m_1 + m_2)x' = m_1x' + m_2x'$ ;
- (2)  $x(m'_1 + m'_2) = xm'_1 + xm'_2$ ;
- (3)  $(xx')m'' = x(x'm'')$ ;
- (4)  $(mx')x'' = m(x'x'')$ ;
- (5)  $(xm')x'' = x(m'x'')$ ;
- (6)  $(x, m)(x', m') = (xx', mx' + xm')$

PROOF.

- (1) We use 2.1

$$\begin{aligned} (x, (m_1 + m_2)x') &= (x, m_1 + m_2)(x', 0) = ((x, m_1) + (x, m_2))((x', 0) + (x', 0)) \\ &= (x, m_1)(x', 0) + (x, m_2)(x', 0) = (x, m_1x') + (x, m_2x') = (x, m_1x' + m_2x') \end{aligned}$$

(2) Dual.

(3) We use associativity in  $Y$ :

$$\begin{aligned} (xx'x'', (xx')m'') &= (xx', 0)(x'', m'') = (x, 0)((x', 0)(x'', m'')) \\ &= (x, 0)(x'x'', x'm'') = (xx'x'', x(x'm'')) \end{aligned}$$

(4) Dual

(5) Again from associativity in  $Y$ :

$$\begin{aligned} (xx'x'', (xm')m'') &= (xx', xm')(x'', 0) = ((x, 0)(x', m'))(x'', 0) \\ &= (x, 0)((x', m')(x'', 0)) = (x, 0)(x'x'', m'x'') \\ &= (xx'x'', x(m'x'')) \end{aligned}$$

(6) We use 2.1

$$\begin{aligned} (x, m)(x', m') &= ((x, m) + (x, 0))((x', 0) + (x', m')) \\ &= (x, m)(x', 0) + (x, 0)(x', m') = (xx', mx') + (xx', xm') \\ &= (xx', mx' + xm') \end{aligned} \quad \blacksquare$$

To show that these properties characterize Beck modules for a monoid, let  $X$  be a monoid and suppose we are given, for each  $x \in X$  an abelian group  $M_x$  and for each  $x$  a function  $M_x \rightarrow M_{xx'}$ , denoted  $m' \mapsto xm'$ , and for each  $x' \in X$  a similar function  $M_x \rightarrow M_{xx'}$ . Let  $Y = \sum M_x$  (disjoint union) and denote an element of  $M_x$  by  $(x, m)$ . Define a product on  $Y$  by  $(x, m)(x', m') = (xx', mx' + m'x)$ .

**2.3. THEOREM.** *Suppose  $Y$  is as above and assume that the right and left multiplication functions satisfy the conclusions of Theorem 2.2. Then the first coordinate function  $p : Y \rightarrow X$  is an abelian group object over  $X$ .*

**PROOF.** The first two conditions give additivity. We note that any function between groups that preserves multiplication also preserves the identity (the only idempotent) and then inverse. The next three conditions readily imply associativity of the multiplication. So each fibre is an abelian group. The only thing left is to show that the abelian group structure is a monoid homomorphism. This requires showing the formula 2.1(6):

$$((x, m_1) + (x, m_2))((x', m'_1) + (x', m'_2)) = (x, m_1)(x', m'_1) + (x, m_2)(x', m'_2)$$

The left hand side of that equation is

$$(x, m_1 + m_2)(x', m'_1 + m'_2) = (xx', m_1x' + m_2x' + xm'_1 + xm'_2)$$

while the right hand side is

$$(xx', m_1x' + xm'_1) + (xx', m_2x' + xm'_2) = (xx', m_1x' + xm'_1 + m_2x' + xm'_2) \quad \blacksquare$$

### 3. The morphisms

Let  $Z = \bigcup N_x$  be another Beck module over  $X$ . Then an element of  $Z$  is a pair  $(x, n)$ ,  $n \in N_x$  and  $(x, n)(x', n') = (xx', nx' + xn')$ . A map  $f : Y \rightarrow Z$  over  $X$  must have the form  $f(x, m) = (x, f_x m)$  in order to be over  $X$ . In order to be a map of abelian group objects, it must be an additive map  $M_x \rightarrow M_{x'}$ . But it must also be a monoid homomorphism. Thus we must have that

$$f((x, m)(x', m')) = f(xx', mx' + xm') = (xx', f_{xx'}(mx' + xm')) = (xx', f_{xx'}(mx') + f_{xx'}(xm'))$$

is the same as

$$f(x, m)f(x'm') = (x, f_x m)(x', f_{x'} m') = (xx', (f_x m)x' + x(f_{x'} m'))$$

If we take the case  $m = 0$ , this implies that  $f_{xx'}(xm') = x(f_{x'} m')$  and similarly  $f_{xx'}(mx') = (f_x m)x'$ .

Conversely, the same computations show that an  $X$ -indexed family of homomorphisms satisfying the last two equations will give us a morphism of additive group objects. Thus we conclude

**3.1. THEOREM.** *Let  $M = \{M_x\}$  and  $N = \{N_x\}$  be  $X$ -modules and  $Y \rightarrow X$ ,  $Z \rightarrow X$  be the corresponding abelian group objects over  $X$ . An  $X$ -indexed family of group homomorphisms  $\{f_x : M_x \rightarrow N_x\}$  determines a morphism of abelian group objects over  $X$  if and only if  $f_{xx'}(xm') = x(f_{x'} m')$  and  $f_{xx'}(mx') = (f_x m)x'$ . ■*

**3.2. GUETTA'S DESCRIPTION OF THE CATEGORY.** Léonard Guetta, looking at the description of Beck modules above, came up with a different way of looking at them. Suppose  $x, x', t, t' \in X$  are elements of  $X$  such that  $x' = t'xt$ . Then there is a map  $M_x \rightarrow M_{x'}$  given by  $m \mapsto t'mt$ . What Guetta noticed is that  $(t, t')$  is a map in the **twisted arrow category** of  $X$ . Like the arrow category, its objects are the elements of  $X$ , that is, the arrows of the single object category  $X$ , but whose morphisms are the “twisted arrows”, commutative diagrams

$$\begin{array}{ccc} \bullet & \xrightarrow{x} & \bullet \\ \uparrow t & & \downarrow t' \\ \bullet & \xrightarrow{x'} & \bullet \end{array}$$

If we denote this category by  $\text{TwAr}(X)$  then an  $X$ -module is simply a functor  $\text{TwAr}(X) \rightarrow \mathcal{A}\mathcal{b}$ , the category of abelian groups. The morphisms described are simply the natural transformations between such functors.

After the above was written, Guetta discovered the paper [Frankland, 2010], see Example 5.5 (in which what he called the factorization category is exactly the same as the twisted arrow category).

## 4. Derivations

Another question that Beck answered with his concept of modules was the meaning Der. If  $X$  is a ring and  $M$  an  $X$ -module, a map  $d : X \rightarrow M$  is called a **derivation** if it is additive and  $d(xx') = (dx)x' + x(dx')$ . If  $X$  is a group and  $M$  a right  $X$ -module, then a derivation is a function  $d : X \rightarrow M$  such that  $d(xx') = (dx)x' + dx'$ . In that case it is also called a **crossed homomorphism**. Note that if the right action is also trivial, it is simply a homomorphism to the abelian group  $X$ . Beck showed that the group  $\text{Der}(X, M)$  is canonically isomorphic to the abelian group  $\text{Hom}_X(X, M)$  in both cases (also for the Lie algebra case which has its own definition of derivation). It is just as easy to show that if  $Z \rightarrow X$  is any object of  $\mathcal{X}/X$  and  $M$  is made into a  $Z$ -module via  $Z \rightarrow X$ , then  $\text{Der}(Z, M) \cong \text{Hom}_X(Z, M)$ . These two questions, what is a module and what is a derivation were the two questions that he had to answer to carry out his program of defining the cotriple cohomology. If  $\mathbf{G} = (G, \epsilon, \delta)$  is a cotriple on  $\mathcal{X}$ , the cohomology of  $X$  with coefficients in  $M$  is the cohomology of the cochain complex associated to the cosimplicial set

$$\text{Der}(GX, M) \rightrightarrows \text{Der}(G^2X, M) \rightrightarrows \text{Der}(G^3X, M) \rightrightarrows \dots$$

in which the cofaces  $\partial_n^i : \text{Der}(G^n, M) \rightarrow \text{Der}(G^{n+1}, M)$  are induced by the maps  $G^i \epsilon G^{n-i} : G^{n+1} \rightarrow G^n$ .

## 5. Examples

The first example is that of a monoid  $X$  and an abelian group  $M$  equipped with an  $X$ -action in the usual way, that is an  $X$ -module. Then we let  $M - x = M$  independent of  $x \in X$ . We can identify  $Y = X \times M$  and let  $\lambda - x', x : M_x \rightarrow M_{x'x}$  be simply act as left multiplication by  $x$  and similarly for  $\rho_{x', x} : M_x \rightarrow M_{xx'}$ . Then  $(x, m)(x', m') = (xx', mx' + xm')$ .

The second example requires a monoid  $X$  in which there are no invertible elements except the identity. That being the case, let  $M$  be an arbitrary abelian group and let  $M_1 = M$  and  $M_x = 0$  for all  $x \neq 1$ . Then for  $x \neq 1$ , both left and right multiplication by  $x$  is the zero map.

## 6. Groups

It is known that if  $X$  is a group, a Beck module for  $X$  is just a right module in the usual sense. But it is interesting to see what the development above means in this case. First we note that if  $X$  is a group and  $Y \rightarrow X$  is an abelian group object over  $X$ , then  $Y$  is also a group since  $(x, m)(x^{-1}, -x^{-1}mx^{-1}) = (1, mx^{-1} - mx^{-1}) = (1, 0)$ . Thus an abelian group object is the same in the category of monoids over  $X$  and of groups over  $X$ . Since multiplication by  $x \in X$  is invertible, the  $M_x$  are isomorphic, although not

canonically. However,  $M = M_1$  acquires a right  $X$ -module structure by conjugation since  $(x^{-1}, 0)(1, m)(x, 0) = (1, x^{-1}mx)$ .

Let  $Z \rightarrow X$  denote the abelian group object over  $X$  determined by this right module with identity operation on the left. That is to say that  $Z = X \times M$  as a set and the multiplication is given by  $(x, m)(x', m')(xx', x'^{-1}mx' + m')$ . We have, somewhat surprisingly:

6.1. PROPOSITION. *The map  $f : Z \rightarrow Y$  given by  $f(x, m) = (x, xm)$  is an isomorphism of groups.*

PROOF. Clearly it is invertible by the map  $g$  give by  $g(x, m) = (x, x^{-1}m)$ . To show it is a homomorphism, we compute

$$\begin{aligned} f((x, m)(x', m')) &= f(xx', x'^{-1}mx' + m') = (xx', xmx' + xx'm') \\ &= (x, xm)(x', x'm') = f(x, m)f(x', m') \end{aligned} \quad \blacksquare$$

What is interesting about this is if we begin with an  $X$ -module  $M$ , use its given structure to build  $Y$  and then let  $N = M$ , but with the right module conjugation structure and construct  $Z$ , then  $Y \cong Z$ , but  $M \not\cong N$  since the isomorphism can be constructed only with the help of  $X$ .<sup>1</sup> Since we know that for any group  $W \rightarrow X$ , the group  $\text{Hom}_X(W, Y) = \text{Der}(Y, M)$  and similarly for  $Z$ . In particular, if  $W$  is free on one generator, then  $\text{Hom}_X(W, Y) \cong M$ , while  $\text{Hom}_X(W, Z) \cong N$ . Then  $M \cong N$ , but only as abelian groups, which we already knew.

Although the following is implicit in the above, it is amusing to point out that if  $X$  is a group and  $M$  is a 2-sided  $X$ -module, then when  $N = M$ , except with the conjugate right operation, then  $\text{Der}(X, M) \cong \text{Der}(X, N)$ . In fact, let us change notation and write  $M'$  for the module  $M$  with  $m \cdot x = x^{-1}mx$  and  $x \cdot m = m$ . Then, given a derivation  $d : X \rightarrow M$ , we define  $d' : X \rightarrow M'$  by  $d'x = x^{-1}(dx)$ . Then

$$d'(xy) = y^{-1}x^{-1}d(xy) = y^{-1}x^{-1}((dx)y + x(dy)) = (d'x) \cdot y + d'y$$

so that  $d'$  is a derivation into  $M'$ . Conversely, given a derivation  $d' : X \rightarrow M'$ , define  $d : X \rightarrow M$  by  $dx = x(d'x)$ . Then

$$\begin{aligned} d(xy) &= xyd'(xy) = xy((d'x) \cdot y + d'y) = xyy^{-1}(d'x)y + xy(d'y) \\ &= x(d'x)y + x(dy) = (dx)y + x(dy) \end{aligned}$$

Clearly these operations are inverse to each other so that the groups of derivations are isomorphic, but the modules are not.

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<sup>1</sup>This argument is not complete. But, as pointed out to me by George Janelidze, the categories of right modules and 2-sided modules cannot in general be equivalent. For example, should  $X$  be a commutative group, the category of right modules is the category of modules over the integer group ring  $\mathbf{Z}[X]$ , while the category of 2-sided modules is the category of modules over  $\mathbf{Z}[X] \otimes \mathbf{Z}[X]$  and two commutative rings cannot be Morita equivalent unless they are isomorphic. For  $X = \mathbf{Z}$ , say they are not since they are polynomials in one and two variables, respectively, with different homological dimensions.

## References

- [Beck, 1967] J.M. Beck (1967, 2003), Triples, Algebras and Cohomologies. Dissertation Columbia University and TAC reprints **2**, 59 pp.
- [Frankland, 2010] M. Frankland (2010), Fibered categories of Beck modules. [http://www.home.uni-osnabrueck.de/mfrankland/Frankland\\_BeckModules\\_20100411.pdf](http://www.home.uni-osnabrueck.de/mfrankland/Frankland_BeckModules_20100411.pdf)