The regular category embedding theorem

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Abstract

I have given two apparently different the regular category embedding theorem. The first, gotten by adapting the Lubkin's argument for the abelian category, is rather opaque. The second, gotten by adapting Mitchell's proof is much more elegant. Mitchell used Grothendieck's theorem that an AB5 category with a generator has an injective cogenerator. However, the analogous result for regular categories fails. It turns out that full injectivity is not needed.

Surprisingly, it turns out that "under the hood" the two proofs are really doing much the same thing. It is using functors rather than representing diagrams that makes the difference.

Regular and exact categories

A category $\mathcal{C}$ is called **regular** if it has finite limits, coequalizers and if the regular epimorphisms are stable under pullback. It is called **exact** if, in addition, every equivalence relation is a kernel pair.

These conditions can be weakened somewhat, but it is not worth the effort to do so.

The regular category embedding theorem states that every small regular category has a full and faithful embedding into a set-valued functor category that preserves finite limits and regular epics. For exact categories, we can add the preservation of coequalizers of equivalence relations. That is an easy gloss on the regular embedding theorem so will concentrate in this talk on that result.

Some properties of regular categories

The most important property is that ever morphism $f : A \longrightarrow B$ can be factored $A \overset{g}{\twoheadrightarrow} C \overset{h}{\rightarrowtail} B$ where g is regular epic and h is monic. Among other things this implies that the composite of regular epics is regular epic, which is not true for general categories. In a regular category, a morphism is a regular epic iff it is extremal. This means that it is not possible to factor it through any proper subobject of its codomain. In general extremal epics (in a complete category) are composites, even transfinite composites, of regular epics.

Finite limit preserving functors

From now on, $\mathcal{C}$ is a small regular category, $\mathcal{F} = FL(\mathcal{C}, \mathcal{S}et)$ is the category of finite limit preserving functors from $\mathcal{C}$ to sets and $\mathcal{X} = \mathcal{F}^{\mathrm{op}}$. The functor that takes $A \in \mathcal{C}$ to $\mathrm{Hom}(A, -)$ gives a contravariant embedding of $\mathcal{C}$ into $\mathcal{F}$ and therefore a covariant embedding of $\mathcal{C}$ into $\mathcal{X}$. For the most part, we will treat $\mathcal{C}$ as a subcategory of $\mathcal{X}$ so write C instead of $\mathrm{Hom}(C, -)$. We begin with

$\mathcal{X}$ is complete and cocomplete. We know that every functor F is the colimit of all the arrows $\mathrm{Hom}(A, -) \longrightarrow F$ with $A \in \mathcal{C}$ and it is an easy exercise to show that this diagram is filtered iff F preserves finite limits. Thus in $\mathcal{X}$, every F is a filtered limit of all arrows $F \longrightarrow A$.

Proposition. *$\mathcal{X}$ is regular.*

We will need some diagrams to show the regularity.

But first, an important obeservation

Suppose $\operatorname{colim} A_i \longrightarrow A$ is an isomorphism. Then $\operatorname{Hom}(A, -) \longrightarrow \operatorname{Hom}(\operatorname{colim} A_i, -)$ and therefore $\operatorname{Hom}(A, -) \longrightarrow \lim \operatorname{Hom}(A_i, -)$ are isomorhisms in $(\mathcal{C}, \mathcal{S}et)$ and hence also in $\mathcal{F}$, the subcategory of finite limit preserving functors. But then $\operatorname{colim} \operatorname{Hom}(A_i, -) \longrightarrow \operatorname{Hom}(A, -)$ is an isomorphism in $\mathcal{X} = \mathcal{F}^{\mathrm{op}}$. This means that $\mathcal{C} \longrightarrow \mathcal{X}$ preserves colimits as well, of course, as finite limits. In particular, it preserves regular epics and the regular epic/monic factorization of morphisms, which is what we need.

Some diagrams

Consider a pullback:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & H \arrow[d] \\ G \arrow[r, two heads] & F \end{tikzcd}$$

Some diagrams

And some maps to representables:

$$\begin{tikzcd}
 & & & C \\
 & K \arrow[r] \arrow[d] & H \arrow[ur] \arrow[d] & \\
 & G \arrow[r, two heads] \arrow[dl] & F \arrow[dr] & \\
B & & & A
\end{tikzcd}$$

Some diagrams

A factorization:

$$
\begin{tikzcd}
 & & & & C \\
 & & K \arrow[r] \arrow[d] & H \arrow[ur] \arrow[d] & \\
 & & G \arrow[r, two heads] \arrow[dl] & F \arrow[ddr] & \\
 & A \times B \arrow[dl] \arrow[drrr] & & & \\
B & & & & A
\end{tikzcd}
$$

Some diagrams

$$B \twoheadrightarrow \mathrm{supp} B \rightarrowtail 1$$

$$A \times B \twoheadrightarrow A \times \mathrm{supp} B \rightarrowtail A \times 1 = A$$

$$\begin{tikzcd}
H \arrow[r, two heads] \arrow[d] & F \arrow[dd] \arrow[ddl, dashed] \\
A \times B \arrow[d] & \\
A \times \mathrm{supp} B \arrow[r, tail] & A
\end{tikzcd}$$

Some diagrams

Continuing:

$$
\begin{tikzcd}
 & & & & & C \\
 & & K \arrow[r] \arrow[d] & H \arrow[d] \arrow[rru] & & \\
 & & G \arrow[r, two heads] \arrow[dl] & F \arrow[dr] & & \\
 & A \times B \arrow[rrr, two heads] \arrow[dl] & & & A \times \mathrm{supp} B \arrow[dr] & \\
B & & & & & A
\end{tikzcd}
$$

Continuing:

$$
\begin{tikzcd}
 & & & & & C \\
 & & & & A \times \mathrm{supp} B \times C \arrow[ur] & \\
 & & K \arrow[r] \arrow[d] & H \arrow[d] \arrow[ur] & & \\
 & & G \arrow[r, two heads] \arrow[dl] & F \arrow[dr] & & \\
 & A \times B \arrow[rrr, two heads] \arrow[dl] & & & A \times \mathrm{supp} B \arrow[dr] & \\
B & & & & & A
\end{tikzcd}
$$

Some diagrams

Pullback:

$$
\begin{tikzcd}
 & & & & & C \\
 & A \times B \times C \arrow[rrr, two heads] \arrow[ddd] & & & A \times \mathrm{supp} B \times C \arrow[ur] \arrow[ddd] & \\
 & & K \arrow[ul, dashed] \arrow[r] \arrow[d] & H \arrow[ur] \arrow[d] & & \\
 & & G \arrow[r, two heads] \arrow[dl] & F \arrow[dr] & & \\
 & A \times B \arrow[rrr, two heads] \arrow[dl] & & & A \times \mathrm{supp} B \arrow[dr] & \\
B & & & & & A
\end{tikzcd}
$$

$\mathcal{X}$ is regular, finish

The sequence $A \times B \times B \times C \rightrightarrows A \times B \times C \longrightarrow A \times \mathrm{supp} B \times C$ is a coequalizer/kernel pair. As A, B, and C range over all the maps from F, G, and H, resp. to an object of $\mathcal{C}$, $A \times \mathrm{supp} B$, $A \times B$, and $A \times \mathrm{supp} B \times C$, resp., range over coinitial subsets of those cofiltered families and hence their limits are, resp. F, G, and H. Since a limit of pullback diagrams is a pullback, we conclude that $K = \lim(A \times B \times C)$ and similarly $K \times_H K$ is the colimit of the $A \times B \times B \times C$. Since filtered limits preserve finite colimits, we further conclude that $K \twoheadrightarrow H$ is a regular epic.

The argument actually shows that the opposite of the category of finite product preserving functors in $(\mathcal{C}, \mathcal{S}et)$ is also regular. In order to do this, you need to know that such a functor preserves supports, for which it suffices that it preserve the property of being a subobject of 1. But $E \longrightarrow 1$ is a monomorphism iff the diagonal $E \longrightarrow E \times E$ is an isomorphism.

Lemma

For every $F \in \mathcal{X}$, there is a regular epic $F^{\#} \twoheadrightarrow F$ such that every diagram

$$\begin{tikzcd} F^{\#} \arrow[r, two heads] \arrow[d, dashed, two heads] & F \arrow[d] \\ B \arrow[r, two heads] & A \end{tikzcd}$$

can be filled in as shown.

Proof of lemma.

Fix F and well order all possible diagrams of the form

$$\begin{tikzcd} & F \arrow[d] \\ B \arrow[r, two heads] & A \end{tikzcd}$$

Let $F_0 = F$ and having chosen F_α, choose the next $F \longrightarrow A \twoheadleftarrow B$ in the well-ordering and let

$$\begin{tikzcd} F_{\alpha+1} \arrow[r, two heads] \arrow[d] & F \arrow[d] \\ B \arrow[r, two heads] & A \end{tikzcd}$$

be a pullback. At a limit ordinal α, let $F_\alpha = \lim_{\beta<\alpha} F_\beta$.

Theorem

Every F can be covered by a projective.

Proof. Let $F^{(0)} = F$, $F^{(n+1)} = F^{(n)\#}$, and $F^* = \lim F^{(n)}$. We claim that any map $F^* \longrightarrow A$ factors through an $F^* \longrightarrow F^{(n)}$. In fact, for any functor F

$$\mathrm{Hom}_{\mathcal{X}}(F, \mathrm{Hom}(-, A)) \cong \mathrm{Hom}_{\mathcal{F}}(\mathrm{Hom}(-, A), F) \cong FA$$

(Yoneda) and then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{X}}(\mathcal{F}^*, \mathrm{Hom}(-, A)) &= \mathrm{Hom}_{\mathcal{X}}(\lim F^{(n)}, \mathrm{Hom}(-, A)) \\ &= \mathrm{Hom}_{\mathcal{F}}(\mathrm{Hom}(-, A), \mathrm{colim}\, F^{(n)}) \\ &= (\mathrm{colim}\, F^{(n)})(A) \end{aligned}$$

Proof, continued

But colimits in the functor category $\mathcal{F}$ are computed "pointwise", meaning $(\operatorname{colim} F^{(n)})(A) = \operatorname{colim}(F^{(n)}(A))$ so that each element of $(\operatorname{colim} F^{(n)})(A)$ is represented by a morphism $F^{(n)} \longrightarrow A$ in $\mathcal{X}$.
This gives

$$\begin{tikzcd}
 & F^* \arrow[dl] \arrow[d] \\
F^{(n+1)} \arrow[r] \arrow[d] & F^{(n)} \arrow[d] \\
B \arrow[r, two heads] & A
\end{tikzcd}$$

so that F^* is a $\mathcal{C}$-projective cover of F.

The subcategory $\mathcal{P}$

We let $\mathcal{P}$ denote the full subcategory of $\mathcal{X}$ consisting of projective cover P_A of every $A \in \mathcal{C}$ as well as, for each such P_A, a projective cover P'_A of the kernel pair $P_A \times_A P_A$. The result is a coequalizer diagram in $\mathcal{X}$

$$\begin{tikzcd} P'_A \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d^1"'] & P_A \arrow[r, "d"] & A \end{tikzcd}$$

for each $A \in \mathcal{C}$.

Note that the functors in $\mathcal{P}$ take regular epics in $\mathcal{C}$ to surjections in $\mathcal{S}et$.

The main theorem

The functor $\mathcal{C} \longrightarrow (\mathcal{P}^{\mathrm{op}}, \mathcal{S}et)$ that takes A to evaluation at A is full and faithful.

Proof. The embedding takes $A \in \mathcal{C}$ to $\Phi(A) = \mathrm{Hom}_{\mathcal{P}}(-, A) : \mathcal{P}^{\mathrm{op}} \longrightarrow \mathcal{S}et$. A natural transformation $\nu : \Phi(A) \longrightarrow \Phi(B)$ assigns to each $f : P \longrightarrow A$, a map $\nu(f) : P \longrightarrow B$ such that for all $g : Q \longrightarrow P$ the square

$$\begin{tikzcd}
\mathrm{Hom}(P, A) \arrow[r, "\mathrm{Hom}(g,A)"] \arrow[d, "\nu"'] & \mathrm{Hom}(Q, A) \arrow[d, "\nu"] \\
\mathrm{Hom}(P, B) \arrow[r, "\mathrm{Hom}(g,B)"'] & \mathrm{Hom}(Q, B)
\end{tikzcd}$$

which means that $\nu(fg) = \nu(f)g$.

Proof continued

Apply this to

$$\begin{tikzcd} P' \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & P \arrow[r, "d"] \arrow[dr, "\nu(d)"'] & A \arrow[d, dashed, "f"] \\ & & B \end{tikzcd}$$

From

$$\nu(d)d^0 = \nu(dd^0) = \nu(dd^1) = \nu(d)d^1$$

we see there is a unique $f : A \longrightarrow B$ such that $fd = \nu(d)$. Thus $C \longrightarrow (\mathcal{P}^{\mathrm{op}}, \mathcal{S}et)$ is full and faithful.

Brief comparison with the original proof

As I was thinking about this argument, I realized that it is not that different, after all, from the original one. The latter was based on construction of some very complicated cofiltered diagrams and the embedding was gotten my mapping diagrams to objects. But these diagrams just represented limit preserving functors. Moreover, they were constructed by adding, one at a time, indexed by ordinals, a regular to the diagram and then making it cofiltered again. This is so like the construction above that I realized it had to be essentially the same. The point is that in replacing the diagrams by the functors they represent, the whole idea becomes much more transparent.