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WEAK ADJOINTNESS IN PROOF THEORY

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0. The idea that (equivalence classes of) derivations in formal logical systems could be considered as morphisms in a category (and *vice versa*) goes back to Lambek [L1] ; the need for an equivalence relation on derivations arises partly so that one has a category, but mainly because one wishes to have the evident correspondences, for example between conjunction and product, implication and exponentiation, and so on. From the proof theorist's point of view, however, this equivalence relation originally appeared somewhat unnatural and *ad-hoc*; furthermore it became quite complicated and unwieldy as one dealt with larger fragments of first order predicate calculus, particularly when working with a sequent calculus. (See, e.g., Szabo [Sz] .) However, when formulated in natural deduction, first order logic has a canonical 2-categorical structure with derivations as morphisms, and although conjunction is not a product, it is a "weak product" in a certain sense; it is the purpose of this note to sketch the precise sense of "weak". This analysis of the 2-categorical structure also motivates and simplifies the description of the equivalence relation on derivations mentioned above: we make the 2-category an ordinary category by making all 2-cells identities, and in the process change a "weak product" into an ordinary product.

For brevity, we shall deal with the natural deduction formulation Π of intuitionistic logic given by Prawitz [P1] . For categorical reasons, it is perhaps desirable to modify Π so that it is multisorted, and so that it allows the interpretation of sorts by empty domains; we leave this and other details to the reader, (they may be found in [S2]).

1. We suppose we are given a language L containing variables, function and predicate symbols; from this we form the system Π of [P1] using the inference rules of natural deduction. There are canonical operations on derivations in Π, given by the reductions and expansions for $\&$, $\vee$, $\supset$, $\forall$, $\exists$, of [P1] . In addition, we will require the following generalisations of $\vee$E-reduction and $\exists$E-reduction:

$\vee$ -permutation

$$\dfrac{\begin{array}{c}\Sigma_0\\ A_1 \vee A_2\end{array} \quad \begin{array}{c}[A_1]\\ \Sigma_1\\ B\end{array} \quad \begin{array}{c}[A_2]\\ \Sigma_2\\ B\end{array}}{\begin{array}{c}[B]\\ \Sigma_3\\ C\end{array}} \quad \Longrightarrow \quad \dfrac{\begin{array}{c}\Sigma_0\\ A_1 \vee A_2\end{array} \quad \begin{array}{c}[A_1]\\ \Sigma_1\\ [B]\\ \Sigma_3\\ C\end{array} \quad \begin{array}{c}[A_2]\\ \Sigma_2\\ [B]\\ \Sigma_3\\ C\end{array}}{C}$$

(provided the RHS is a derivation)

$\exists$ -permutation

$$\dfrac{\begin{array}{c}\Sigma_0\\ \exists\, x\, Ax\end{array} \quad \begin{array}{c}[A(a)]\\ \Sigma_1\\ B\end{array}}{\begin{array}{c}[B]\\ \Sigma_3\\ C\end{array}} \quad \Longrightarrow \quad \dfrac{\begin{array}{c}\Sigma_0\\ \exists\, x\, Ax\end{array} \quad \begin{array}{c}[A(a)]\\ \Sigma_1\\ [B]\\ \Sigma_3\\ C\end{array}}{C}$$

(provided the RHS is a derivation)

We remark that (given the reductions) the existence of Prawitz' $\vee$-expansion and $\vee$-permutation is equivalent to the existence of the following form of $\vee$-expansion:

$$\begin{array}{c}\Sigma_0\\ [A \vee B]\\ \Sigma_1\\ C\end{array} \quad \Longrightarrow \quad \dfrac{\begin{array}{c}\Sigma_0\\ A \vee B\end{array} \quad \begin{array}{c}\dfrac{[A]}{[A \vee B]}\\ \Sigma_1\\ C\end{array} \quad \begin{array}{c}\dfrac{[B]}{[A \vee B]}\\ \Sigma_1\\ C\end{array}}{C}$$

(And analogously for $\exists$; we shall henceforth use these stronger forms of expansion for $\vee$ and $\exists$.)

It is now evident how Π has the structure of a 2-category Π : its objects are formulae, its morphisms are derivations, its 2-cells are the operations mentioned above. (We indicate domains and codomains for permutations and expansions; the rest should be obvious - just insert " $\Rightarrow$ " in the proper blank spaces in [P1] , § II.3.3.1 .) The definitions of compositions and identities are canonical.

We have lax 2-functors $\&$, $\vee$: $\Pi \times \Pi \to \Pi$, where "lax" is defined in § 2(i) . For example, $\&(A,B) = A \& B$. Also, there is a diagonal 2-functor $\Delta : \Pi \to \Pi \times \Pi$, $\Delta(A) = (A,A)$. (The quantifiers $\exists$, $\forall$ can also be considered as lax 2-functors between suitable 2-categories, and there is a suitable "diagonal" of opposite sense. Implication is best considered as a lax 2-functor $A \supset (\ )$: $\Pi \to \Pi$, for fixed A .)

The following meta-principle ("reduction" for operations) is useful:

(R) An expansion of an occurrence of a logical symbol, followed by a reduction of the same occurrence, is (provided the composite is an endooperation) the identity operation.

2. Suppose we are given the following data:

(i) $\mathcal{A}$, $\mathcal{B}$ are 2-categories,

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

are lax 2-functors, in the sense that instead of strict functoriality for morphisms, we have "comparison 2-cells":

$$\gamma^F_{gf} : FgFf \Rightarrow F(gf) \quad , \quad \iota^F_A : 1_{FA} \Rightarrow F1_A \quad ,$$

for $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. (G similarly.)

(ii) For any $A \in \mathcal{A}$, $B \in \mathcal{B}$, there are functors

$$(FA,B) \underset{\lambda_{AB}}{\overset{\kappa_{AB}}{\rightleftarrows}} (A,GB)$$

(κ , λ will be made "lax 2-natural transformations" in ways specified below.)

DEFINITION 1. Suppose, for any $A' \xrightarrow{f} A$ in $\mathcal{A}$, $B \xrightarrow{g} B'$ in $\mathcal{B}$ there are natural transformations $k_{fB} : \kappa_{A'B}(Ff,B) \Rightarrow (f,GB)\kappa_{AB}$,
$k_{A'g} : \kappa_{A'B'}(FA',g) \Rightarrow (A',Gg)\kappa_{A'B}$, $\ell_{fB} : (Ff,B)\lambda_{AB} \Rightarrow \lambda_{A'B}(f,GB)$,
$\ell_{A'g} : (FA',g)\lambda_{A'B} \Rightarrow \lambda_{A'B'}(A',Gg)$, $\alpha_{AB} : 1_{(FA,B)} \Rightarrow \lambda_{AB}\kappa_{AB}$,
$\beta_{AB} : \kappa_{AB}\lambda_{AB} \Rightarrow 1_{(A,GB)}$.

Suppose also that $(\beta_{AB} 1_{\kappa_{AB}}) \cdot (1_{\kappa_{AB}} \alpha_{AB}) = 1_{\kappa_{AB}}$, and $(1_{\lambda_{AB}} \beta_{AB}) \cdot (\alpha_{AB} 1_{\lambda_{AB}}) = 1_{\lambda_{AB}}$, ("triangle equalities"). (And if desired, suppose that the "usual coherence conditions" hold.) Then we say $F \underset{R}{\dashv} G$ is a Rax adjunction.

DEFINITION 2. If we reverse the directions of the natural transformations k_{fB} , $k_{A'g}$, ℓ_{fB} , $\ell_{A'g}$, α_{AB} , β_{AB} in Definition 1 , and replace the triangle equalities with $(\kappa\alpha)\cdot(\beta\kappa) = \kappa$, $(\alpha\lambda)\cdot(\lambda\beta) = \lambda$ then $F \underset{L}{\dashv} G$ is a Lax adjunction. (We modify the "usual coherence conditions" suitably, of course.)

PROPOSITION $\quad \amalg \times \amalg \underset{\&}{\overset{\vee}{\rightrightarrows}} \amalg$, $\quad \amalg \xrightarrow{\Delta} \amalg \times \amalg$

are lax 2-functors, as above. Furthermore,

$$\vee \underset{R}{\dashv} \Delta \quad , \quad \Delta \underset{L}{\dashv} \& \quad .$$

REMARK Analogously, $\exists$ (and equality, if we wish to include it) is also a Rax left adjoint to the suitable "diagonal", and $\forall$ is a Lax right adjoint. However, $\supset$ has some of the properties of both types of weak adjunction; this is discussed in [S2] .

The core of the proof is given by the following table:

	$\vee$	$\&$	$\supset$
ι^F	$\vee$-expansion	identity	$\&$-expansion
ι^G	identity	$\&$-expansion	$\supset$-expansion
γ^F	$\vee$-perm. + $\vee$-red.	identity	$\&$-reduction
γ^G	identity	$\&$-reduction	$\supset$-reduction
κ	$\vee$I	$\&$I	$\&$I + $\supset$I
λ	$\vee$E	$\&$E	$\&$E + $\supset$E
$k_{A'g}$	identity	$\&$-reduction	$\supset$-reduction
ℓ_{fB}	$\vee$-perm. + $\vee$-red.	identity	$\&$-reduction
$\ell_{A'g}$	$\vee$-permutation	$\&$-reduction	$\supset$-reduction
k_{fB}	$\vee$-reduction	identity	$\&$-reduction
α	$\vee$-expansion	$\&$-reduction	-
β	$\vee$-reduction	$\&$-expansion	-

The triangle equalities are precisely the principle (R) , (and the "coherence conditions" can be given by a somewhat more general version of (R)).

As mentioned earlier, considering 2-cells as identities makes & a product, v a coproduct, and so on. In this way we obtain the results of [S1] : hyperdoctrines and theories in (a modification of) Π (=) are equivalent. (And more: this equivalence restricts to one between hyperdoctrines with the Beck condition and theories which "recognise" their own pullback diagrams.)

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