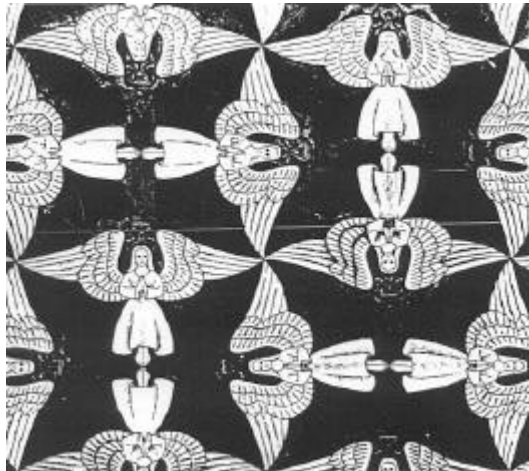


# Topics in Geometry, 189-348A

Olga Kharlampovich

## 1 Plane Ornaments

One of the most interesting instances of a deep connection between art and mathematics is provided by the surface ornaments, raised to such an admirable degree of perfection by ancient artists. The task of the artist is to find for a certain type of ornamental symmetry an elementary figure whose repetitions intertwine to give a harmonious whole. The mathematician, in turn, is interested only in the symmetry operations occurring in an ornament. Section 1 deals with the mathematical theory of plane ornaments. In addition, it provides an introduction to one of the most fundamental notions of modern mathematics, the concept of a group.



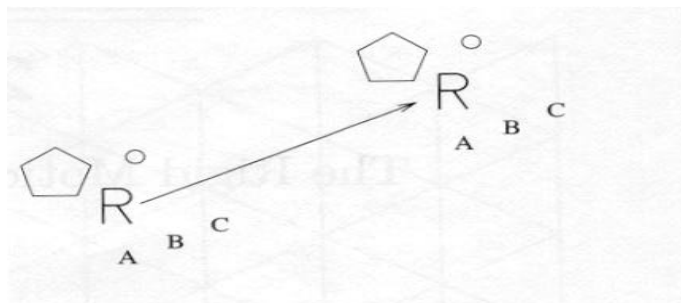
### 1.1 Isometries

**Definition 1** *An isometry is one-to-one mapping of the plane onto itself which preserves a distance between any two points.*

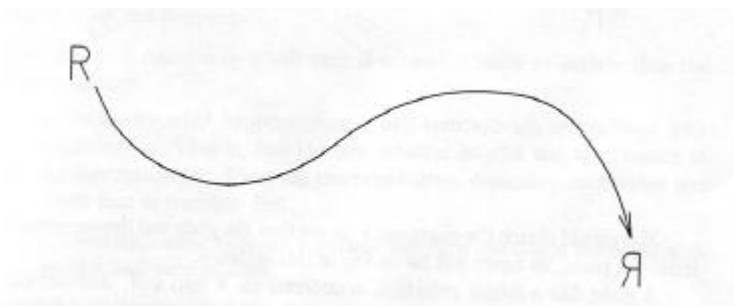
An isometry which leaves a figure invariant is called a symmetry operation. In order to classify the ornaments according to their symmetry operations we have to investigate the various isometries of the plane. In the plane, an isometry is uniquely determined by its effect on a rectangular Cartesian co-ordinate frame.

**Definition 2** An isometry is said to be direct or opposite according as it preserves or reverses the sense of the frame.

Example of direct isometry



Example of opposite isometry



A direct isometry can be achieved by a rigid motion of the plane in itself. Therefore it is often called a *proper* motion. On the other hand, an opposite isometry requires besides a proper motion a reflection in a line. Executing this reflection by half-turn about the line as axis we obtain, as a final result, a rigid motion in which, however, we must come out of the plane. Therefore an opposite isometry is also called an *improper* motion.

The simplest direct isometries are the *translations*, in which every point of the plane moves through the same distance in the same direction. A translation is uniquely determined by a directed line-segment  $AB$ , called a vector, leading from a point  $A$  to its image  $B$ . We shall often denote this translation by  $t_{AB}$ .

Another type of direct isometry is a *rotation* of the plane through a given angle about a given point. We shall show that no other proper motions exist in the plane.

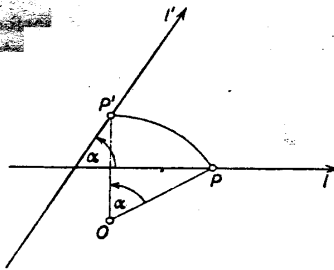


Fig. 1

We shall find it convenient to regard translations as rotations through a zero angle about an infinitely distant point. Then our statement reads as follows:

**Remark 1** *Every proper motion of the plane is a rotation.*

**Proof** To make this evident, we note that a proper motion is uniquely determined by a point  $P$ , and oriented line  $\ell$  through  $P$  and the images  $P'$  and  $\ell'$  of  $P$  and  $\ell$ . Since the cases where  $P$  and  $P'$  or the directions of  $\ell$  and  $\ell'$  coincide are trivial, we may suppose that  $P$  and  $P'$  differ and  $\ell$  and  $\ell'$  include an angle  $\alpha$  ( $0 < \alpha < 2\pi$ ). Let  $O$  be a point equidistant from  $P$  and  $P'$ , such that the rotation about  $O$  transforming  $P$  into  $P'$  has an angle equal to  $\alpha$ . This rotation transforms  $\ell$  into  $\ell'$  (Fig. 1).  $\square$

The improper motions can also be reduced to a simple type of isometry, called *glide-reflection*. A glide-reflection is the resultant of a reflection in a line and a translation in the direction of this line. Considering reflections as special cases of glide-reflections, we may assert that

**Remark 2** *Every improper motion of the plane is a glide-reflection.*

**Proof** In order to show this we notice that an improper motion is determined by the transforms  $P'$  and  $\ell'$  of a point  $P$  and an oriented line  $\ell$  through it. Consider the line parallel to the bisector of the angle between  $\ell$  and  $\ell'$  passing through the midpoint of the segment  $PP'$ . The glide-reflection in this line which carries  $P$  into  $P'$ , transfers  $\ell$  into  $\ell'$  (Fig. 2).  $\square$

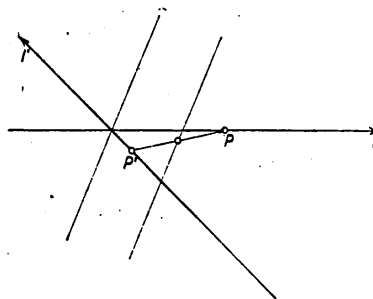


Fig. 2

## 1.2 Composition of isometries

Now we make some remarks concerning the composition of isometries. A certain analogy exists between the composition of isometries and the multiplication of numbers. Therefore we denote the transformation arising, by performing first the transformation  $U$ , then the transformation  $V$ , by  $VU$ . We call this resultant transformation the product of  $U$  and  $V$ . This kind of multiplication is associative

$$W(VU) = (WV)U,$$

so that either side may be denoted by  $WVU$ . But it is generally not commutative:  $UV \neq VU$ . If  $UV = VU$ , (as, for example, the reflection and translation in a glide-reflection) we say that  $U$  and  $V$  commute. We write  $U^2$  for  $UU$ ,  $U^3$  for  $UUU$ , and so on. Furthermore we use the symbol  $1$  for the identical transformation, which leaves all points invariant, such as, for example, the square  $m^2$  of a reflection  $m$  in a line or the cube  $r^3$  of a rotation  $r$  through  $2\pi/3$ . The identity commutes, obviously, with any transformation  $U$ :  $U1 = 1U = U$ . Again, we define  $U^{-1}$  by  $UU^{-1} = U^{-1}U = 1$ . This is the *inverse transformation* of  $U$ , which neutralizes the effect of  $U$ .

**Remark 3** *The product of two direct isometries or two opposite ones is a direct isometry, while the product of a direct and an opposite isometry (in either order) is always opposite.*

Concerning direct isometries, i.e. rotations, we have a simple rule which we shall quote as the theorem on additivity of angles of rotation.

**Theorem 1** *The product of a rotation through an angle  $\alpha$  and a rotation through an angle  $\beta$  is a rotation through the angle  $\alpha + \beta$ .*

**Proof** This becomes evident by taking into consideration the fact that the angle of rotation is given by the change of any oriented line.  $\square$

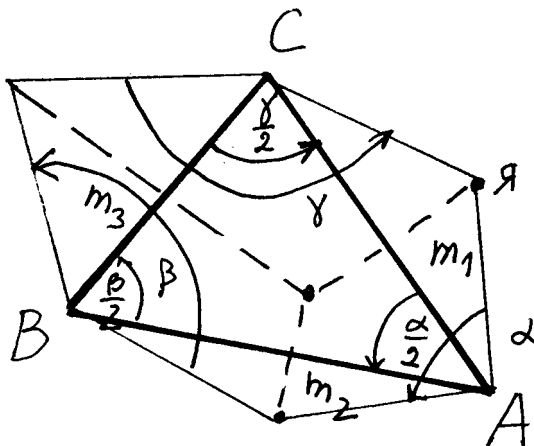
As an example we note that the product of two rotations about different centers and through equal and opposite angles is a translation, because the resultant rotation has the angle zero, but cannot be the identity since neither of the two centers of rotation remains unchanged.

The following more general rotation product theorem also gives information on the center of rotation of the product of two nondegenerate rotations.

**Theorem 2** (*Rotation product theorem*) *Let  $\alpha/2, \beta/2, \gamma/2$  be external or internal angles of a triangle  $ABC$  according as the vertices are named in the positive or negative*

sense. Then the product of rotations through angles  $\alpha$ ,  $\beta$  and  $\gamma$  about  $A$ ,  $B$  and  $C$  is the identity. The product of rotations through angles  $\alpha$  and  $\beta$  about  $A$  and  $B$  is a rotation about  $C$  through  $-\gamma = \alpha + \beta \pmod{2\pi}$ .

**Proof** Consider just the case when the vertices are named negatively.



Denote the rotations respectively by  $r_{A,\alpha}, r_{B,\beta}, r_{C,\gamma}$ . Denote by  $m_1, m_2$  and  $m_3$  reflections in lines  $AC, AB$  and  $BC$  respectively. Then  $r_{A,\alpha} = m_2 m_1$ ,  $r_{B,\beta} = m_3 m_2$ ,  $r_{C,\gamma} = m_1 m_3$ . Therefore  $r_{C,\gamma} r_{B,\beta} r_{A,\alpha} = m_1 m_3 m_3 m_2 m_2 m_1 = 1$ , because  $m_i m_i = 1$  for  $i = 1, 2, 3$ .

### 1.3 A group of isometries

**Definition 3** A group  $G$  is a set of elements together with an operation  $\circ$  (for any  $g_1, g_2 \in G$  there exists an element  $g_3 = g_2 \circ g_1 \in G$ ) satisfying the following axioms:

G1) There exists an identity element  $e$  such that for any  $g \in G$

$$e \circ g = g \circ e = g$$

G2) Associativity law: for any  $g_1, g_2, g_3 \in G$

$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3,$$

G3) For any  $g \in G$  there exists an inverse element  $g^{-1} \in G$  such that

$$g \circ g^{-1} = g^{-1} \circ g = e.$$

There is only one element satisfying G1. Indeed, suppose  $e_0 \circ g = g$  for all  $g \in G$ , then  $e_0 \circ e = e$ . But from the other hand  $e_0 \circ e = e_0$ , hence  $e = e_0$ .

For any element  $g$  there exists only one inverse element.

Examples of groups:

1. The group of all isometries of the plane.
2. The group of all proper isometries of the plane.
3. The group of symmetries of the regular  $n$ -gon.
4. Integers  $\mathbf{Z}$  with respect to the addition operation “+”.
5. Rational numbers with respect to multiplication.

A *figure* is a collection of points on the plane. Obviously, the totality of the symmetry operations of a figure constitutes a group, with respect to the composition of its transformations as a group operation. This group is called the *symmetry group of the figure*. It may happen that the symmetry group of a figure consists of the identity only. Then the figure is called *asymmetrical*. In all other cases the figure is called symmetrical.

Conversely, a given group of at least two isometries always determines a symmetrical figure consisting of the images of a certain “elementary figure” under the transformations of the group. The notion of the symmetry group will soon prove to be a powerful tool in surveying the vast family of symmetrical figures.

We will consider only *discrete* groups of isometries. A group of transformations is said to be discrete or discontinuous, if every point has a discrete set of transforms, i.e. if every point has a neighborhood containing none of its transforms except the given point itself.

## 1.4 Geometric Transformations, Klein’s Erlanger Programme

In 1872, a year after his decisive publication of the projective models for non-Euclidean geometries, Felix Klein was appointed (at age 23) to a chair at the University of Erlangen. He delivered an inaugural address proposing a new unifying principle for classifying the various geometries that were rapidly being developed, and for discovering relationships between them. This *Erlanger Programme* has had an enormous impact on all of mathematics to the present day.

The key notion, according to Klein, involves the group of all automorphisms of a mathematical structure. An automorphism is a one-to-one mapping (or transformation) of each basic set of objects in the model onto itself which preserves the basic relations among the objects.

The importance of the group of automorphisms was first recognized in connection with the problem of solving an algebraic equation by radicals. Evariste Galois (1811-

1832) showed that a solution by radicals was possible if and only if the group of automorphisms of the field extension generated by the roots of the equation is a solvable group. This implies Abel's particular discovery that the general equation of degree 5 cannot be solved by radicals. Klein later discovered a relation between the group of rotations of a dodecahedron and the roots of the quintic equation that explained why the latter can be solved by elliptic functions.

We say that a property or relation is "invariant" under a transformation or group of transformations if the property or relation still holds after the transformations are applied; a geometric figure is "invariant" if it is mapped onto itself by the transformations.

"Invariance" and "group" are the unifying concepts in Klein's Erlanger Programme. Groups of transformations had been used in geometry for many years, but Klein's originality consisted in reversing the roles, in making the group the primary object of interest and letting it operate on various geometries, looking for invariants.

Klein's idea of looking for various actions or representations of a group and their invariants has proved to be fruitful in many branches of mathematics and physics, not just in geometry.

In physics, for example, the invariance of Maxwell's equations for electromagnetism under Lorentz transformations suggested to Minkowski a new geometry of space-time whose group of automorphisms is the Lorentz group; this was the beginning of relativity theory, for which Einstein at one point considered the name "Invariantentheorie." In atomic physics, the regularities revealed in the periodic table are a direct consequence of invariance under rotations. In elementary particle physics, considerations of invariance and symmetry have led to several non-trivial predictions. E. Wigner has said that in the future we may well derive the laws of nature and try to test their validity by means of the laws of invariance rather than to try to derive the laws of invariance from what we believe to be the laws of nature."

## 1.5 Similarity transformation

We shall often make use of the product  $UVU^{-1}$ , called  $V$  transformed by  $U$ . For instance, if  $S$  is a rotation (spin) about  $A$  through the angle  $\alpha$  and  $P$  an arbitrary proper motion transforming  $A$  into  $A'$  then  $PSP^{-1}$  is a rotation about  $A'$  through  $\alpha$ . In fact, it is, as a product of direct isometries, itself direct, i.e. a rotation. By the theorem of the additivity of angles this rotation has an angle  $\alpha$ . Finally, the center of rotation must be  $A'$ , because  $P^{-1}$  transforms  $A'$  back into  $A$ . Similarly, it can be seen that, if the improper motion  $I$  transforms the center  $A$  of the rotation  $S$  of angle  $\alpha$  into  $A'$  then  $ISI^{-1}$  is a rotation about  $A'$  of angle  $-\alpha$ .

As a further example consider a translation  $T$  transformed by a rotation  $S$  of angle  $\alpha$ :  $STS^{-1}$ . This is a translation whose magnitude is the same as that of  $T$  and whose direction forms an angle  $\alpha$  with the direction of  $T$ . Indeed,  $STS^{-1}$  is a proper motion, namely a translation, its angle, by the additivity theorem, being zero. Let  $A'$  be the image of the center  $A$  of the rotation  $S$  under  $T$  and  $A''$  be the image of  $A'$  under  $S$ . All that remains to be proved is that the translation  $STS^{-1}$  moves  $A$  into  $A''$ . But this is true because  $S^{-1}$  leaves  $A$  unchanged,  $T$  transforms it into  $A'$  and  $S$  moves  $A'$  into  $A''$ .

It can also be seen that if  $G$  is a glide-reflection then  $GTG^{-1}$  is a translation of the same magnitude as  $T$ , the direction of which arises from the direction of  $T$  by reflection in the axis of  $G$ . Recapitulating the two cases: if  $T$  is a translation and  $U$  any isometry then  $UTU^{-1}$  is a translation whose vector is nothing else but the image of the vector of  $T$  under  $U$ .

## 1.6 Groups with Infinite Unit Cells. Rosette Groups

We obtain a vivid insight into a structure of a symmetry group by a connected region, called *unit cell*, or *fundamental domain*, whose transforms cover the plane without overlapping and without gaps. This notion enables us to divide the discrete groups of isometries into two classes, according as the unit cell of the group is infinite or finite.

We shall call the discrete groups of isometries of the plane ornamental groups. We start with the ornamental groups free from translations. We shall call them rosette groups. They are sometimes known as point groups, for the following reason.

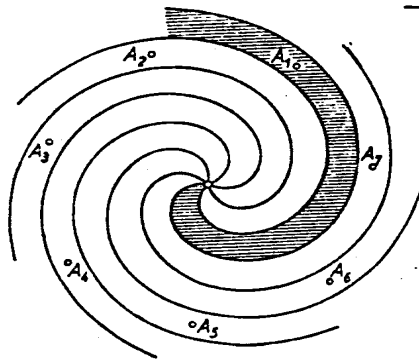
Such a group contains only rotations about a single center. For, if there were two rotations  $S_1$  and  $S_2$  with distinct centers  $O_1$  and  $O_2$ , the transformation  $S_1^{-1}S_2^{-1}S_1S_2$  of the group would be degenerate rotation, by the additivity theorem for angles of rotation. It cannot be the identity, since it displaces  $O_1$  into the image of  $O_1$  under  $S_2^{-1}S_1S_2$ , i.e. under  $S_1$  transformed by  $S_2$ . But the center of this transformed rotation is the image of  $O_1$  under  $S_2^{-1}$ , i.e. a point different from  $O_1$  and therefore  $S_2^{-1}S_1S_2$  effects a change in  $O_1$ . Hence  $S_1^{-1}S_2^{-1}S_1S_2$  would be a non-degenerate translation, contrary to our assumption.

Let us consider first the case in which the group contains direct isometries only, i.e. rotations about a single center  $O$ . Since such a rotation is uniquely given by the image of a point  $A_1$  different from  $O$ , it suffices to know the set of the transforms of  $A_1$ . These transforms lie on the circle with center  $O$  passing through  $A_1$ . Owing to the discontinuity of the group there is only a finite number,  $p$ , of points which we denote in their cyclical order by  $A_1, A_2, \dots, A_p$  into itself. Hence this rotation transforms  $A_2$  into  $A_{i+1}$  ( $A_{p+1} \equiv A_1$ ) and therefore  $\angle A_1OA_i = \angle A_iOA_{i+1} = 2\pi/p$ . This is the smallest angle of rotation in the group and the others are multiples of it. In other words, denoting the rotation about

$O$  through angle  $2\pi/p$  by  $S$ , all operations of the group are  $S, S^2, \dots, S^p = 1$ . Further powers of  $S$  are merely repetitions of the listed ones, e.g.  $S^{p+1} = S, S^{-1} = S^{p-1}$ , etc.

The center  $O$  is called a center of  $p$ -fold rotation, or a  $p$ -gonal (di-gonal, tri-gonal, etc.) center. We shall also use the terms diad, triad and so forth.

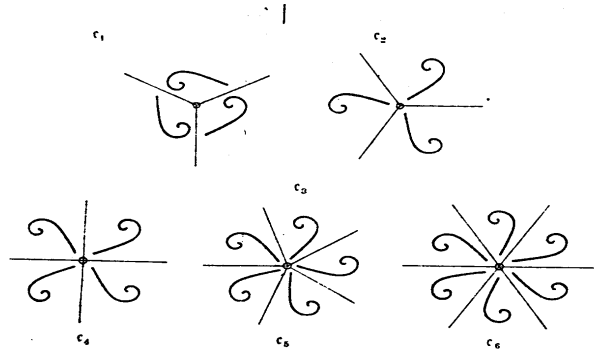
In order to construct a unit cell, let us consider a curve having exactly one point in common with each circle centered at  $O$ . The region swept over by this curve by turning it about  $O$  through  $2\pi/p$  will be a unit cell.



The simplest unit cell arises by choosing for the curve a ray emanating from  $O$ , obtaining as unit cell an angular region with vertex  $O$  and angle  $2\pi/p$ .

Usually, a rotation group of a figure is called after the figure itself. Therefore we call the group just described (which is the rotation group of a regular  $p$ -gon) the polygonal group of order  $p$ . We shall denote it by  $C_p$ . The group  $C_1$  consists of the identical transformation only; its unit cell is the whole plane.

Several of these groups are illustrated by the transforms of an asymmetrical spiral, as an elementary figure.

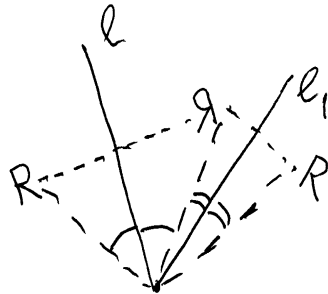


In some cases it is placed in two distinct positions in the unit cell, in order to demonstrate the different artistic effects they produce.

Let us now consider the rosette groups which also contain opposite isometries. The direct isometries contained in such a group form a subgroup  $C_p$  and the opposite isometries can only be reflections in lines passing through the center  $O$  of the rotations  $S, S^2, \dots, S^p$  of  $C_p$ . In fact, if  $U$  is any other transformation of the group then  $USU^{-1}$  is a non-degenerate rotation about the image  $O'$  of  $O$  under  $U$ . But  $O$  and  $O'$  must coincide and this proves our assertion.

This is a special case of a general fact to which we shall refer as the *principle of invariance of the centers of rotation under transformations of the group*. It, states that any transformation of an ornamental group transforms the set of the centers of rotation of the group (together with their multiplicities) into themselves. Any one axis of reflection will provide  $(n - 1)$  others. Indeed, if  $M$  is a reflection in some axis  $\ell$ , then  $SM S^{-1}$  is a reflection in an axis at  $2\pi/p$  to  $\ell$ . A reflection  $SM$  is easily seen to be a reflection in an axis at  $\pi/p$  to  $\ell$ . This gives a second set of axes bisecting the first.

Next we remind that the product of any two reflections is a rotation through twice the angle between their axes.

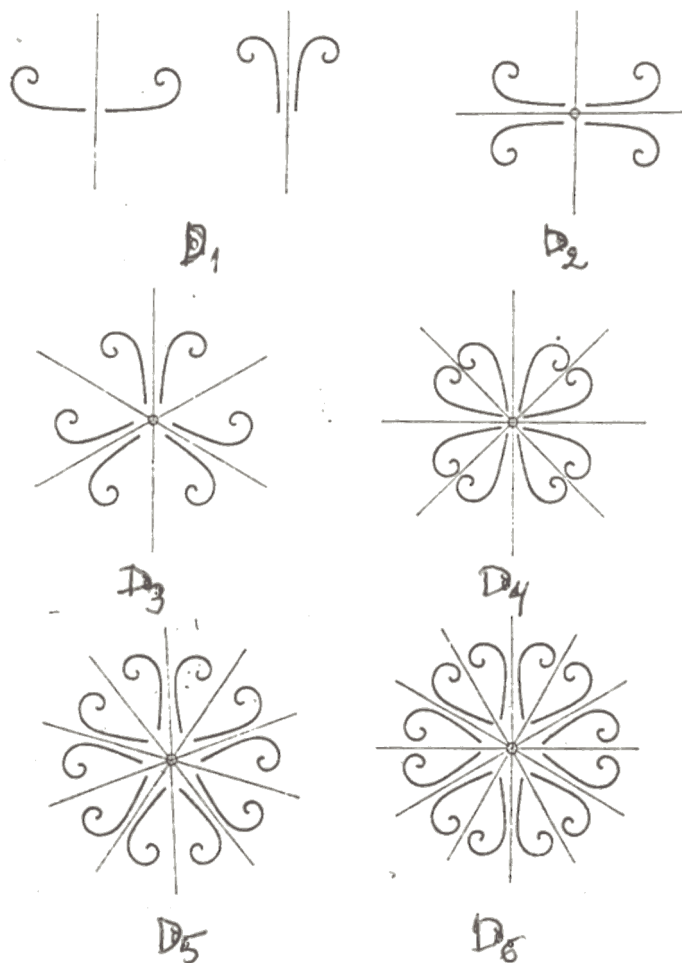


So any two axes of reflection of the group must be separated by a half a multiple of  $2\pi/p$ . This means that all such reflections have been accounted for. We have the rosette group  $D_p$  with  $2p$  axis of reflection. Leonardo da Vinci is said to have enumerated these groups.

**Definition 4** We say that a group  $G$  is generated by the set of elements  $X = \{g_1, \dots, g_n\}$  is every element of  $G$  equals to some product of elements from  $X$  and their multiples.

The group  $D_n$  is generated by two elements  $S$  (a rotation through  $2\pi/n$  and  $M$  (a

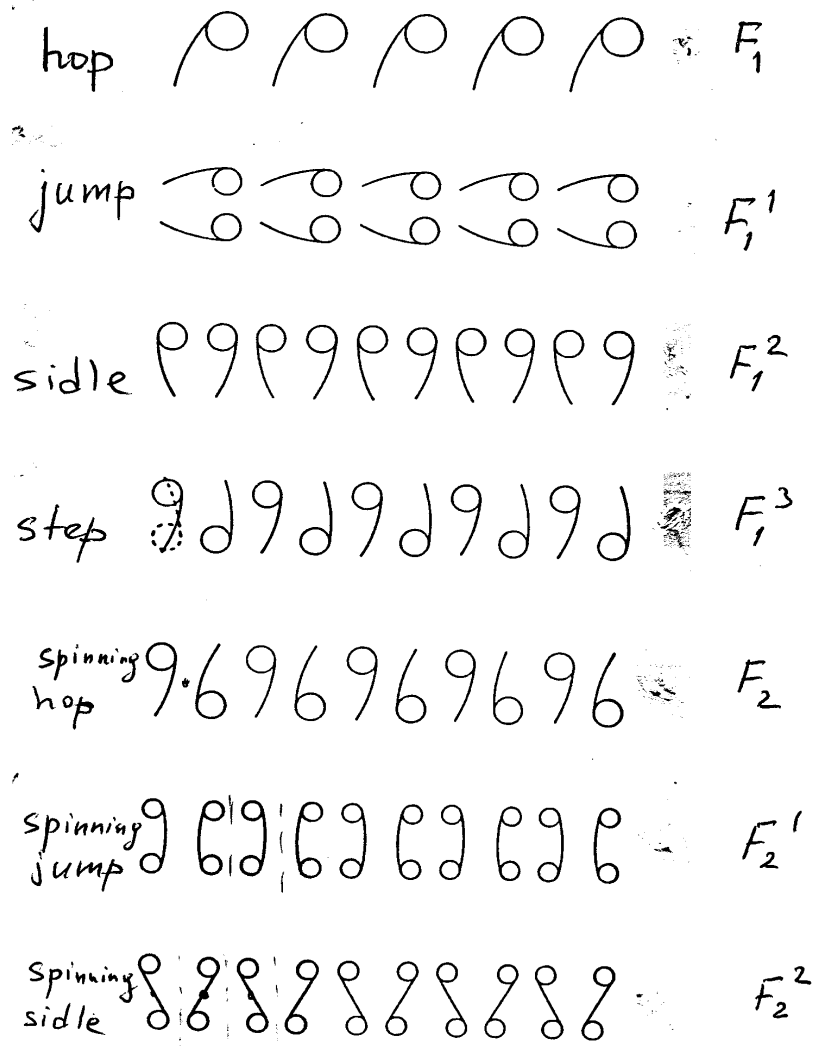
reflection in some axis through  $O$ ). These generators satisfy the following relations  $S^n = 1$ ,  $M^2 = 1$ ,  $SM = MS^{-1}$ .



## 1.7 Groups with Infinite Unit Cell. Frieze Groups

We now turn to the *frieze groups*, i.e. groups of strip patterns. These groups contain translations in only one direction. We begin with the simplest group of this type which consists entirely of translations. Then the transforms of a point must lie on one line. In view of discontinuity, these transforms divide the line into segments. These segments must be congruent. If we denote by  $T$  the translation of the group through the distance of such a segment, then all the transformations of the group are  $\dots, T^{-2}, T^{-1}, 1, T, T^2, \dots$

A simple unit sell is a strip which cuts our line in one of the above segments. We shall denote this group by  $F_1$ .



Suppose now that the group contains also rotations, but does not contain improper motions. If  $S$  is a rotation of angle  $\alpha$  then a  $STS^{-1}$  is a translation including the angle  $\alpha$  with the original one. Hence  $\alpha = \pi$  and  $S$  must be a half-turn. Thus the group admits only di-gonal centers of rotation. If  $A_0$  is a diad, then its successive transforms  $A_1, A_2, \dots$  under the smallest translation  $T$  are also diads. Consider the product  $T^{-1}S_1$ , where  $S_1$  is a half-turn about  $A_1$ . It is itself a half-turn and since it transforms  $A_0$  into

$A_1$  its center  $B_0$  must be the midpoint of the segment  $A_0A_1$ . Similarly, all midpoints  $B_i$  of the segments  $A_iA_{i+1}$  ( $i = 0, \pm 1, \pm 2, \dots$ ) must be di-gonal centers of the group. On the other hand, there are no further diads in the group. If, for example,  $S$  is any half-turn of the group having the center  $O$ , and  $S_0$  is the half-turn about  $A_0$ , then  $SS_0$  is a translation and therefore it transforms  $A_0$  into some point  $A_n$ . But,  $S_0$  leaving  $A_0$  invariant,  $S$  also must transform  $A_0$  into  $A_n$ . Therefore  $O$ , being half-way between  $A_0$  and  $A_n$ , must be one of the diads  $A_i$  or  $B_i$ . This completes the discussion of the group. As unit cell we may choose a half-strip based upon the segment  $A_0A_1$ . We denote this group by  $F_2$ .

Now we try to join to the two groups discussed above an opposite isometry. Let  $R$  be a reflection in line  $\ell$ , and  $T$  the smallest translation in a frieze group. Then  $RTR^{-1}$  is a translation, the vector of which is the image of the vector of  $t$  under  $R$ . Therefore  $\ell$  must be either parallel or orthogonal to the direction of  $T$ .

First we deal with the case when  $R$  is a reflection in a line parallel to  $T$ . Joining the products  $T^iR$  to the translations  $T^i$  ( $i = 0, \pm 1, \pm 2, \dots$ ) of  $F_1$  we obtain a group which we shall denote by  $F_1^1$ . For, in view of the relations  $RT^i = T^iR$  and  $R^2 = 1$  we have for any integers  $i$  and  $j$   $T^iRT^jR = T^{i+j}$  and  $T^iRT^j = T^{i+j}R$ , therefore the group postulates are verified.

Enlarging the group  $F_2$  in a similar manner, we find (by the principle of invariance of the centers of rotation) that the line of reflection (parallel to  $T$ ) can only be the line  $\ell$  of the diads. If  $R$  is the reflection in the line  $\ell$ , the set formed by the transformations of  $F_2$  and their products with  $r$ , constitutes a group  $F_2^1$ .

If  $S$  is a half-turn of  $F_2$  of center  $A$  then  $SR$  is a reflection in the line through  $A$  perpendicular to  $\ell$ . Therefore the group  $F_2^1$  contains besides  $R$  a set of reflections in lines passing through the diads perpendicular to  $\ell$ .

Now let  $R$  be a reflection in a line  $r$  perpendicular to the direction of  $T$ . It can be seen immediately that the transformations  $T^i, T^iR$  form a group, which we denote by  $F_1^2$ . The product  $T^iR$  is a reflection in a line parallel to  $r$  which busects the strip bounded by  $r$  and its image under  $T^i$ . So  $F_1^2$  contains, together with  $r$ , a row of parallel and equidistant lines of reflection, the distance between consecutive lines being the half distance of  $T$ .

We now attempt to enlarge the group  $F_2$  by a reflection  $R$  in a line  $r$  perpendicular to the line of the diads. If  $r$  passes through a diad the resulting group is identical with  $F_2^1$ . Thus we may suppose that  $r$  contains no centers of rotation. Then the line  $r$  must bisect the segment determined by two neighbouring diads. Under these conditions we obtain a group which we denote by  $F_2^2$ .

In this way the possibilities arising by reflections are settled. All we need is to

consider the case of glide reflections.

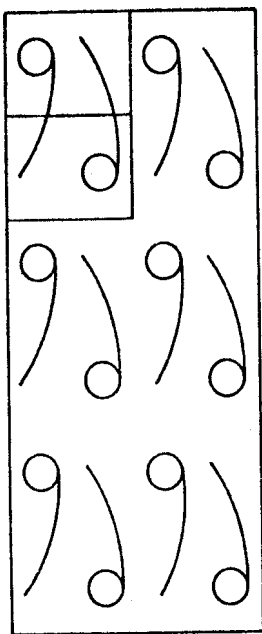
If  $G$  is a glide reflection,  $G^2$  is a translation and we have for some integer  $n \geq 1$  either  $G^2 = T^{2n}$ , or  $G^2 = T^{2n-1}$ , where  $T$  is one of the smallest translations of the group. In the first case  $T^{-n}G$  is a reflection. In the second case  $G_1 = T^{-n+1}G$  is a glide reflection such that  $G_1^2 = T$ . Then the translations  $T^i$  and glide reflections  $T^i G_1$  form a group.

Adding  $G$  to the group  $F_2$  does not give any new groups, because  $SG$  is a reflection in a line perpendicular to  $\ell$ , and we arrive at the group  $F_2^2$  considered previously.

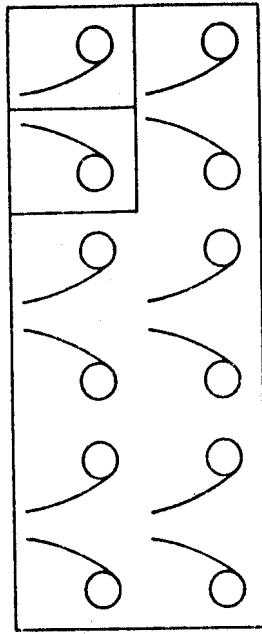
We have now constructed all seven possible frieze groups.

## 1.8 The Wallpaper groups

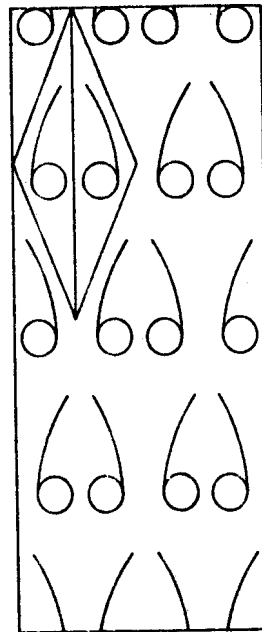
In this subsection we will describe all 17 possible wallpaper groups. Here are the correspondent patterns for them.



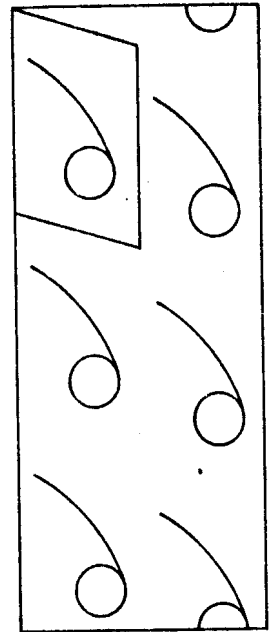
$1/m$



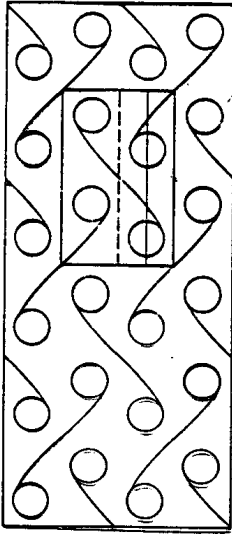
$1/m$



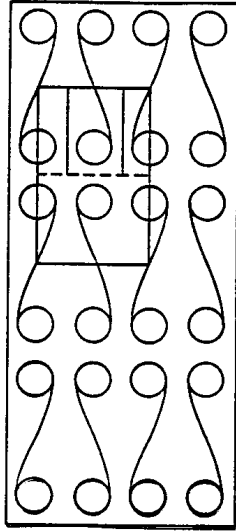
$1/m$



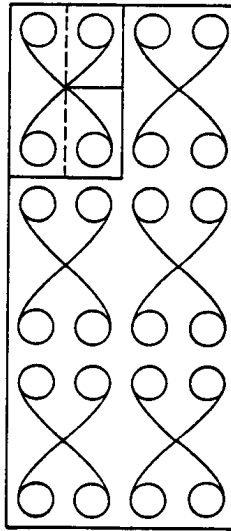
$1/m$



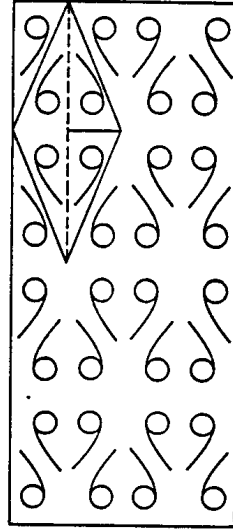
1A1



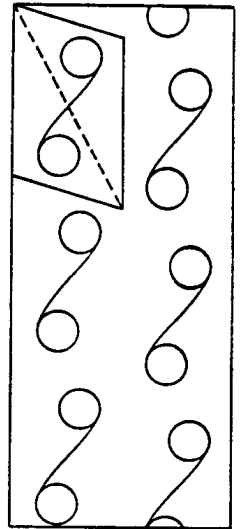
1A2



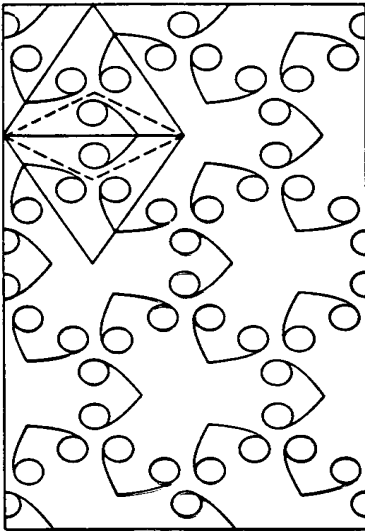
1A3



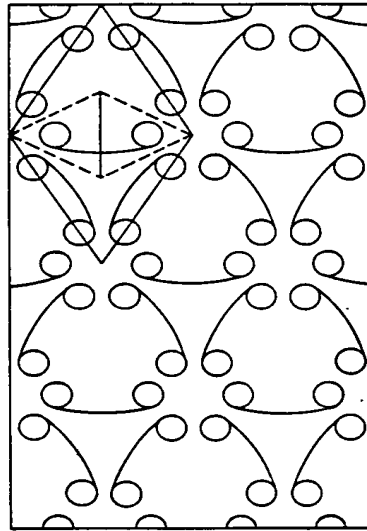
1A4



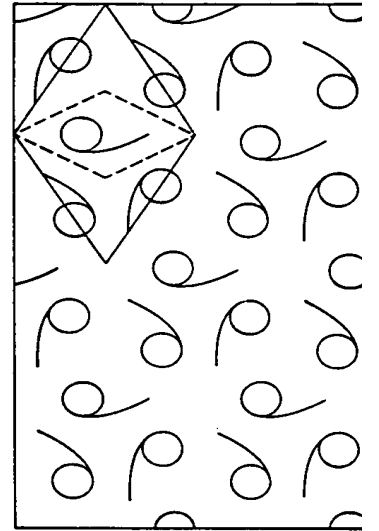
1A5



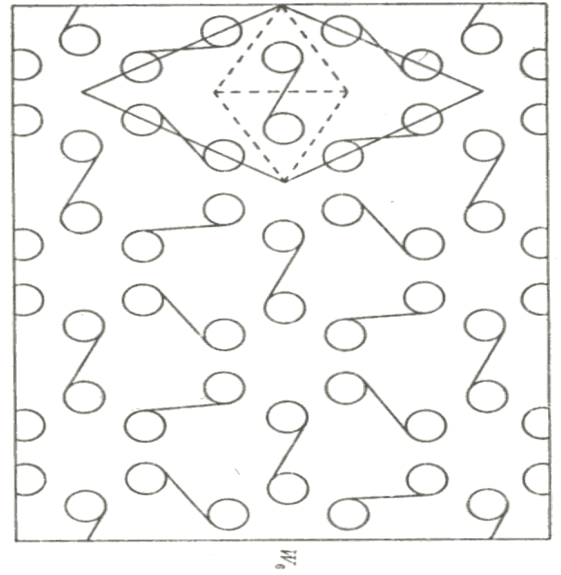
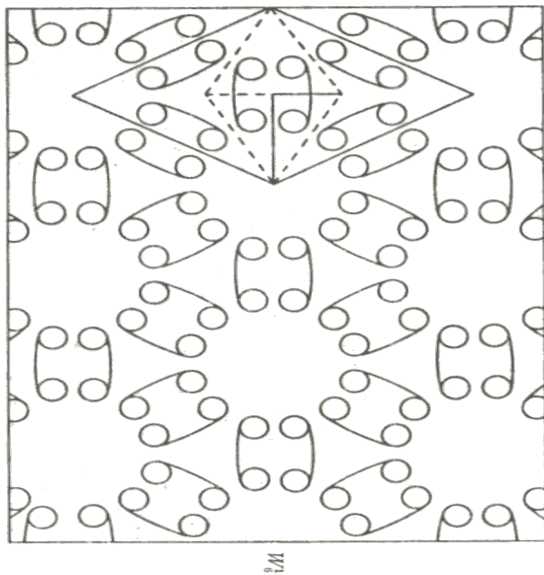
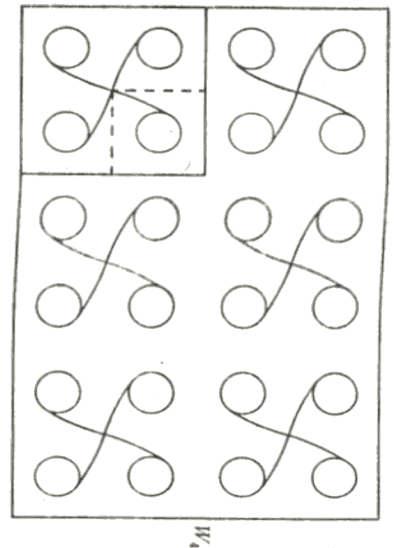
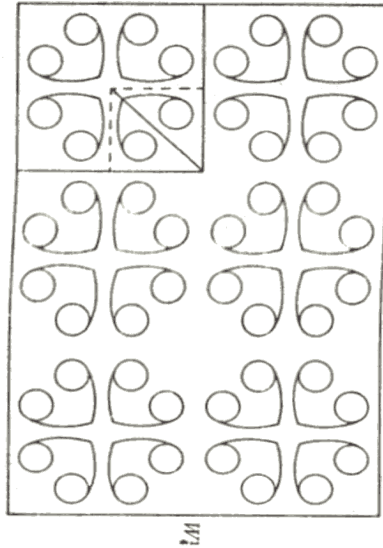
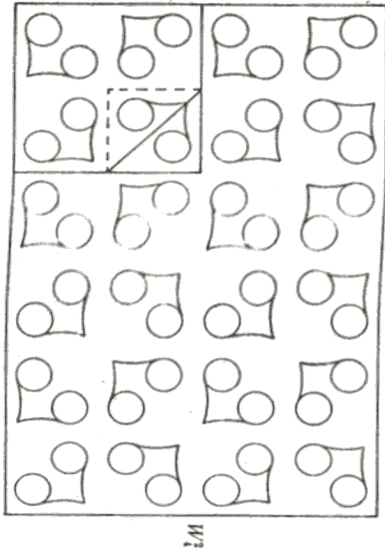
1A6



1A7



1A8



To begin with we shall construct the wallpaper groups of proper isometries.  
 Consider first the simplest case: no rotations.

Let  $S$  be a complete set of equivalent points under the symmetry group. Clearly there are in  $S$  three noncollinear points  $A, B, C$  such that the triangle  $ABC$  does not contain besides  $A, B, C$  further points of  $S$  (Fig. 1).

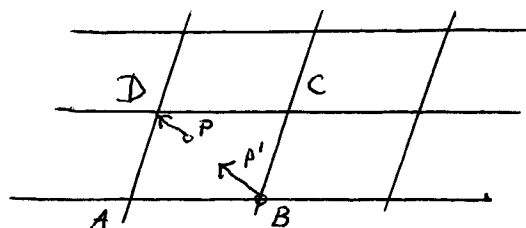


Fig 1

Let the translation  $B \rightarrow C$  moves  $A$  into  $D$ . The parallelogram  $ABCD$  generates a so-called point lattice, that is the set of all vertices of all parallelograms which arises from  $ABCD$  by the translations  $A \rightarrow B, B \rightarrow C, C \rightarrow D, D \rightarrow A$  and their repeated applications. Obviously all points of this lattice belong to  $S$ . On the other hand,  $S$  does not contain additional points. For since the above parallelograms cover the plane without intersections, such a point must lie in (or on the boundary of) a parallelogram. Then in view of the equivalence of the parallelograms,  $ABCD$  would also contain a point  $P$  of  $S$  different from the vertices  $A, B, C, D$ . Since the point  $P$  cannot lie in  $ABC$  (by the supposition made about this triangle), it must be contained in  $ACD$ . But then the translation  $P \rightarrow D$  of the group would move  $B$  into a point  $P'$  of  $ABC$  (a contradiction). Hence all transformations of the group are  $T_1^i T_2^j$  denoting by  $T_1$  and  $T_2$  the translations  $A \rightarrow B$  and  $B \rightarrow C$ . The simplest fundamental domain is a generating parallelogram, such as  $ABCD$ . We denote this group by  $W_1$ .

Now suppose the group contains a half-turn but no other kind of rotation. The subgroup of translations  $W_1$  transforms a centre of half-turn rotation (diad)  $A$  of the group into the points of a lattice; all these points are likewise diads. Then the centroid of every two lattice points must also be a diad. Indeed, if  $A_0$  is a diad, then its successive transforms  $A_1, A_2$  under the translation  $T_1$  are also diads. Consider the product  $T_1 S_1$ , where  $S_1$  is a half-turn about  $A_1$ . It is itself a half turn by the theorem about 3 rotations and, since it transforms  $A_0$  into  $A_1$ , its centre must be the midpoint  $B$  of the segment  $A_0 A_1$  (Fig. 2)

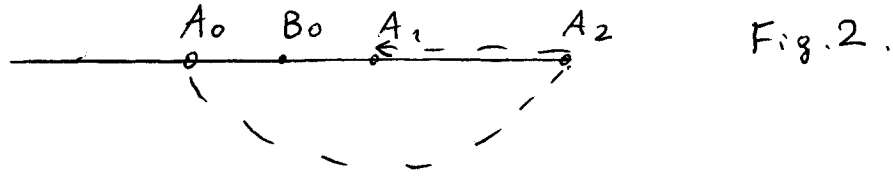


Fig. 2 .

Thus the centre, the vertices and the midpoints of the sides of a generating parallelogram provide for the totality of centres of half-turn rotations (diads). All translations and half-turn rotations form the group  $W_2$ . As unit cell we can choose one of the triangles into which a diagonal splits a generating parallelogram or another suitable half part of a generating parallelogram.

The discussion of the remaining cases follows, mainly, in a similar way. First we shall determine in each case the arrangement of the centres of rotation.

Consider the case of  $1/3$ -turn rotation. By the rotation product theorem the centres of rotation (triads) form the lattice of equilateral triangles (i.e. a point lattice having a generating parallelogram composed of two equilateral triangles). Apart from these triads the groups contains no other centres of rotation. To show this we observe first that the six triangles meeting at  $P$  do not contain any centre of more than 2-fold rotation, except the vertices. Then in view of the fact that any triangle can be transformed by a third turn about the triads into one of the six triangles, the same holds for every triangle. This means that the statement is verified for centres of more than 2-fold rotation. Diads cannot occur either, because a rotation through  $\pi$  together with a rotation through  $-2\pi/3$  would result a rotation through  $2\pi/6$ , which is impossible. The smallest translations are  $PS$  and  $PT$  (Fig. 3)

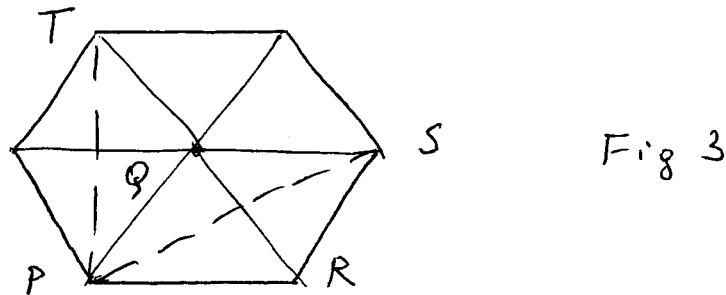


Fig 3

(not  $PR$ , because this would generate another triad in the centre of  $PQR$ ). The corresponding group is  $W_3$ . A suitable fundamental domain is a small rhombi in the Figure, the large rhombi is the generating parallelogram of the subgroup of translations.

Suppose now we have  $1/4$ -rotations or  $1/6$  rotations. The following treatment applies to both cases. Let  $P$  be a centre of  $1/p$ -rotation, there is a least distance from  $P$  at which we can find another centre  $Q$  of  $1/q$  rotation ( $q > 2$ ). We consider the product of the rotation about  $P$  through  $2\pi/p$  and the rotation about  $Q$  through  $2\pi/q$ . This is, according to the rotation product theorem, a rotation about a point  $R$  through  $-2\pi/r$  such that the triangle  $PQR$  negatively oriented has angles  $\pi/p, \pi/q, \pi/r$ . The angle sum

of a triangle being  $\pi$  we have

$$1/p + 1/q + 1/r = 1.$$

The case of  $1/4$  rotations corresponds to  $p = 4, q = 4, r = 2$ , the case of  $1/6$  rotations corresponds to  $p = 6, q = 3, r = 2$ . Successive rotations through  $2\pi/p$  about  $P$  transform  $Q$  into the vertices of a regular  $p$ -gon and successive rotations through  $2\pi/q$  about  $Q$  transform this  $p$ -gon into a set of  $q$   $p$ -gons. Further rotations lead to a so-called regular tassellation  $\{p, q\}$ , i.e. the set of regular  $p$ -gons  $q$  at each vertex. The face centres are centres of  $1/p$ -rotation, the vertices are centres of  $1/q$ -rotation, the midpoints of the sides are centres of  $1/2$ -rotation. These centres of rotation are responsible for all rotations belonging to the group. The generating parallelogram of the subgroup of translations is shown in each case as the large parallelogram (we cannot have smaller translations, because this would result additional centres of rotation). The corresponding groups are  $W_4$  and  $W_6$ .

We have now constructed all groups of proper isometries.

First of all we attempt to enlarge a group  $W_1$  by a reflection  $R$  in line  $l$ . The axis of  $R$  must divide generating parallelogram into two congruent parts. Therefore we have the generating parallelogram either a rhombus or a rectangle, the mirror line being parallel to one diagonal of the rhombus or to one side of the rectangle, respectively. This gives rise to two groups  $W_1^1$  and  $W_1^2$ , the first being generated by two non-parallel translations through equal distances and a reflection in a line which bisects the angle formed by two directions of the translations, the second one being generated by two orthogonal translations and a reflection in a line parallel to one translation (it cannot be translations in intersecting lines, because this would result a rotation).

$W_1^1$  contains also a glide-reflection which is a composition of translation and rotation in  $W_1^1$ .

We shall now show that a glide-reflection cannot be built into a lattice group other than a rhombic or rectangular one.

Let  $G$  be a glide reflection in the line contained in a wallpaper group  $W$ . We can suppose that  $G^2$  is the shortest translation parallel to  $l$ . Let  $a_0$  be a line perpendicular to  $l$  cutting it in the point  $A_0$  and let  $a_i$  and  $A_i$  be the images of  $a_0$  and  $A_0$  under  $G^i$  (Fig.4). Further, let a translation  $T$  of  $W$  displace  $A_0$  into  $B$ . Then  $GTG^{-1}$  is a translation too,

namely the translation  $A_0 \rightarrow B'$ ,  $B'$  being the image of  $B$  under the reflection in  $l$ . This shows that  $B$  can only lie on a line  $a_i$ . For, if  $B$  lay between  $a_0$  and  $a_1$ , the translations  $A_0 \rightarrow B$  and  $A_0 \rightarrow B'$  would produce a translation parallel to  $l$  smaller than  $A_0A_2$ , in contradiction to our stipulation according to which  $C^2$  is the smallest translation parallel to  $l$ . Thus we can find, in accordance with our assertion, either a rhombus of diagonal  $A_0A_2$  or a rectangle based upon  $A_0A_2$ , such that they are generating parallelograms of the lattice formed by the transforms of  $A_0$  under the translations of  $W$ .

The rhombic case having been settled, as the case of  $W_1^1$ , we obtain one further group,  $W_1^3$ , generated by a glide-reflection and a translation orthogonal to the glide axis. Thus we have a set of parallel glide axes, the distance between adjacent axes being the half distance of the generating translation.

Consider now groups in which, apart from diads, no other centres of rotation are present. To begin with, let the lattice of the diads be rhombic and suppose that the group contains a reflection in a line. Owing to the condition of invariance of the lattice under the reflection, this line must pass through a diad parallel to one of the diagonals of the generating rhombus. In this way the group,  $W_2^1$  is completely determined. It contains two sets of parallel mirror lines, both interlocked with a set of glide lines. The diagonals of a rhombus generating the subgroup of translations decompose it into four right triangles, one of them being a fundamental domain of the group. A rectangular lattice of diads gives rise to three further groups  $W_2^2, W_2^3, W_2^4$ .

We now turn to the groups arising from  $W_3$ . Since  $W_3$  contains as subgroup a rhombic lattice group, reflections and glide reflections always present themselves simultaneously (glide reflections are compositions of reflections and translations). Hence we need to consider reflections only. Let us consider a rhombus generating the lattice of the triads. We have two possibilities for enlarging  $W_3$  by a reflection, according as the mirror contains the shorter diagonal or the longer one. Accordingly, we obtain the groups  $W_3^1$  and  $W_3^2$ .

Let us now consider a generating square of the group  $W_4$ . A glide-reflection in a line parallel to one side involves, in view of presence of 1/4-rotocentres, a reflection or glide-reflection in a line parallel to one diagonal. This enables us to consider reflections only. As a result of the two possibilities depending whether the mirror lines contain or do not contain centres of rotations, we have groups  $W_4^1$  and  $W_4^2$ .  $W_4^1$  contains four families of parallel axis of reflection, but only two families of glide-reflection axes, parallel to the diagonals of the faces. Its unit cell is a right triangle having as vertices two of the nearest centres of 1/4-rotation and a diad. If the axis of reflection do not contain 1/4-rotocentres, each of them must pass through two of the nearest diads. Thus  $W_4^2$  contains two sets of axes of reflection. The right triangle, having as vertices two of the

nearest diads and 1/4-rotocentre, yields a unit cell.

$W_6$  produces one more group  $W_6^1$ . First, as above, we must not regard glide-reflections separately, and the axes of reflection are uniquely determined by the invariance of the centres of rotation.

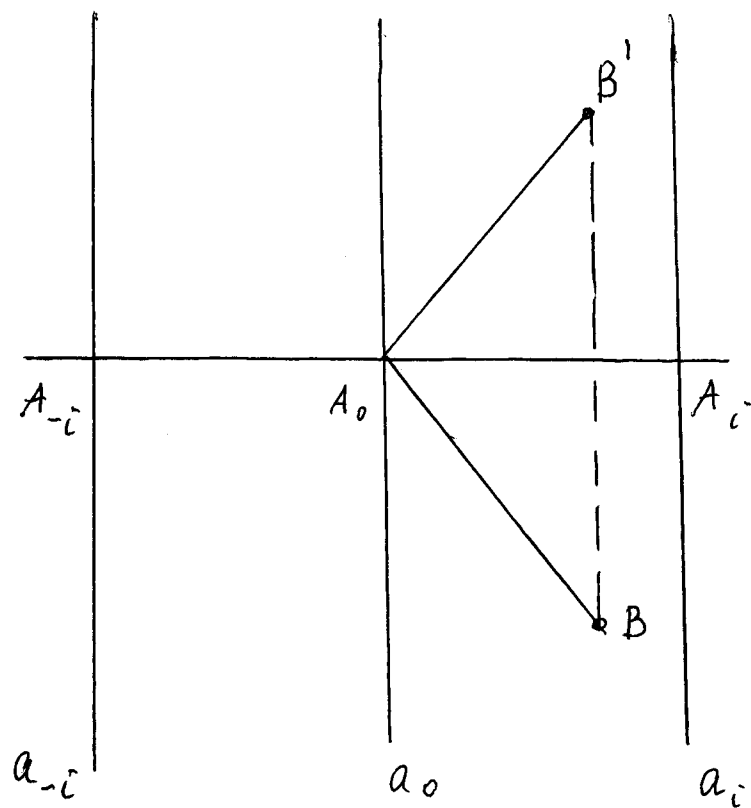


Fig. 4.

## 2 Permutation Groups

**Definition 5** If  $X$  is a nonempty set, a permutation of  $X$  is a function  $\alpha : X \rightarrow X$  which is a one-to-correspondence.

**Definition 6** If  $X$  is a nonempty set, the symmetric group on  $X$ , denoted  $S_X$ , is the group whose elements are the permutations of  $X$  and whose binary operation is composition of functions.

Of particular interest is the special case when  $X$  is finite. If  $X = \{1, 2, \dots, n\}$ , we write  $S_n$  instead of  $S_X$ , and we call  $S_n$  the symmetric group of degree  $n$ , or the symmetric group on  $n$  letters. Note that  $|S_n| = n!$ .

Let  $X$  denote the set  $\{1, 2, \dots, n\}$ . One way of denoting a permutation  $\alpha$  of  $X$  is by displaying its values:

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ \alpha(1) & \alpha(2) & \cdots & \alpha(n) \end{pmatrix}$$

Thus,  $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$  and  $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$  are permutations of  $\{1, 2, 3\}$ . Their product is  $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$  (we compute this product by first applying  $\beta$  and then  $\alpha$ ). Note that  $\beta\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ , so that  $\alpha\beta \neq \beta\alpha$ . It follows that  $S_3$ , and hence any larger symmetric group, is nonabelian.

### Cycles

We shall consider some factorizations of permutations in  $S_n$  as products of simpler permutations, called cycles.

**Definition 7** Let  $x \in X$  and  $\alpha \in S_X$ ;  $\alpha$  fixes  $x$  if  $\alpha(x) = x$ ; otherwise  $\alpha$  moves  $x$ .

**Definition 8** Let  $i_1, i_2, \dots, i_r$  be distinct integers between 1 and  $n$ . If  $\alpha \in S_n$  fixes the other integers and  $\alpha(i_1) = i_2, \alpha(i_2) = i_3, \dots, \alpha(i_{r-1}) = i_r, \alpha(i_r) = i_1$ , then  $\alpha$  is an  $r$ -cycle. We also say that  $\alpha$  is a cycle of length  $r$ .

A way to denote an  $r$ -cycle  $\alpha$ , other than the cumbersome two-rowed notation we introduced earlier, is a  $\alpha = (i_1, i_2, \dots, i_r)$ . For example, and, All 1-cycles equal the identity permutation, which we thus denote (1).

### Exercises

1. Prove that  $(123 \dots r) = (23 \dots r1) = (34 \dots r12) = \dots = (r12 \dots r-1)$ . Conclude that there are  $r$  such notations for this cycle.
2. The order of an  $r$ -cycle is  $r$ .
3. Exhibit two 2-cycles whose product is a 3-cycle. This example shows that if  $\alpha$  and  $\beta$  do not commute, nothing intelligent can be said about the order of  $\alpha\beta$  in terms of the orders of the factors.
4. Let  $\alpha$  and  $\beta$  be  $r$ -cycles in  $S_x$ . If there is an  $x_0 \in X$  such that (i) both  $\alpha$  and  $\beta$  move  $x_0$ , and (ii)  $\alpha^k(x_0) = \beta_k(x_0)$  for all integers  $k$ , then  $\alpha = \beta$ .

**Definition 9** Two permutations  $\alpha$  and  $\beta$  in  $S_x$  are disjoint in case every  $x$  moved by one is fixed by the other. In symbols, if  $\alpha(x) \neq x$ , then  $\beta(x) = x$ , and if  $\beta(y) \neq y$ , then  $\alpha(y) = y$ . A set of permutations is disjoint if each pair of them is disjoint.

**Theorem 3** Every permutation  $\alpha \neq (1)$  in  $S_n$  is the product of disjoint cycles of length  $\geq 2$ ; this factorization is unique except for the order in which the cycles are written.

**Example 1.** Write the following permutation as a product of disjoint cycles:

$$\alpha = \left( \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 4 & 7 & 2 & 5 & 1 & 8 & 9 & 3 \end{array} \right).$$

**Solution.**  $\alpha = (16)(24)(3789)$ .

**Example 2.** Calculate the product in  $S_9$  of  $\beta = (13)(245)(67)$  and  $\alpha = (19)(243)(56)$ .

**Solution.**

$$\alpha\beta = (19)(243)(56)(13)(245)(67) = (1239)(4675)(8).$$

**Definition 10** A 2-cycle is also called a transposition.

**Theorem 4** Every  $\alpha \in S_n$  is a product of transpositions.

**Proof** Every permutation is a product of disjoint cycles. It suffices to factor any  $r$ -cycle into a product of transpositions. This is done in the following way:

$$(1 \ 2 \ \dots \ r) = (1 \ r)(1 \ r-1) \dots (1 \ 2).$$

Any permutation can thus be realized as a sequence of interchanges. The factors are not uniquely determined; e.g.  $(123) = (13)(12) = (23)(13) = (13)(24)(12)(14)$ . We shall prove that the number of factors is always even or always odd.

Consider the polynomial in  $n$  variables:

$$g(x_1, \dots, x_n) = \prod_{j>i}(x_j - x_i).$$

Notice that the condition  $j > i$  implies that every pair of distinct integers between 1 and  $n$  occurs as subscripts in exactly one factor of  $g$ . If  $\alpha \in S_n$ , we form a new polynomial in the  $n$  variables:

$$\alpha g = \prod_{j>i}(x_{\alpha(j)} - x_{\alpha(i)}).$$

Since  $\alpha$  is a permutation, every pair of distinct integers between 1 and  $n$  occurs in exactly one factor of  $\alpha g$ . Therefore, except for signs, the factors of  $g$  and of  $\alpha g$  are identical. Hence  $\alpha g = \pm g$ .

**Definition 11** A permutation  $\alpha \in S_n$  is even if  $\alpha g = g$ ;  $\alpha$  is odd if  $\alpha g = -g$ .

If  $\alpha, \beta \in S_n$ , then  $(\alpha\beta)(g) = \alpha(\beta(g))$ , for both are equal to  $\prod_{j>i}(x_{\alpha\beta(j)} - x_{\alpha\beta(i)})$ . It follows that if both  $\alpha$  and  $\beta$  are even or of both are odd, then  $\alpha\beta$  is even, while if one factor is even and the other odd, then  $\alpha\beta$  is odd.

The transposition (12) is odd, for the only factor in  $g(x_1, \dots, x_n)$  that changes sign is  $x_2 - x_1$ .

**Remark 4** If  $\alpha \in S_n$ , then  $\alpha(i_1 i_2 \dots i_k) \alpha^{-1} = (\alpha(i_1) \alpha(i_2) \dots \alpha(i_k))$ .

It follows from this remark, that any transposition is an odd permutation. Hence the following theorem is true.

**Theorem 5**  $\alpha \in S_n$  is even if and only if  $\alpha$  is a product of an even number of transpositions;  $\alpha \in S_n$  is odd if and only if  $\alpha$  is a product of an odd number of transpositions.

**Definition 12** The alternating group of degree  $n$ , denoted  $A_n$ , is the set of all even permutations in  $S_n$ .

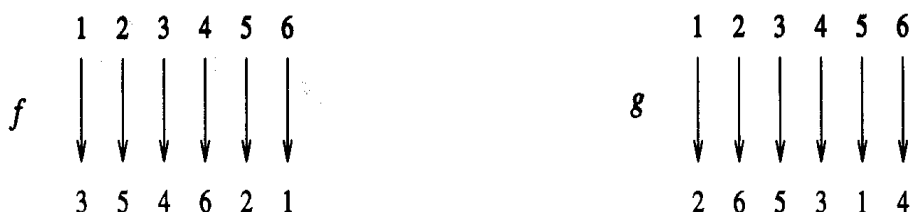
**Definition 13** A subgroup  $H$  of a group  $G$  is called a normal subgroup if for any  $g \in G$  and any  $h \in H$   $ghg^{-1} \in H$ .

**Remark 5**  $A_n$  is a normal subgroup of  $S_n$ .

## 2.1 Exercises from Farmer's book "Groups and Symmetry"

The traditional word for rearrangement is **permutation**. We will use both words in this section.

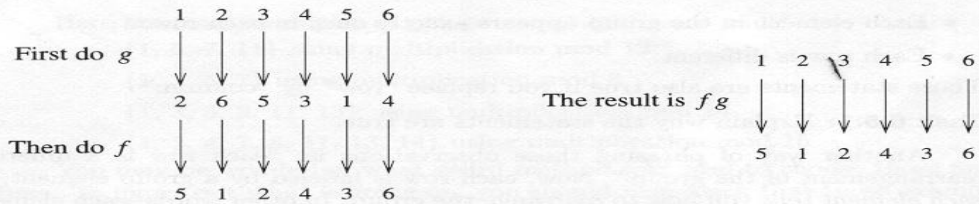
Our first examples will be based on rearranging the numbers 1 2 3 4 5 6. One way to describe a permutation is to show where each number goes. Here are two example permutations which we have named  $f$  and  $g$ :



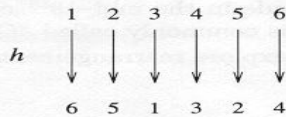
For instance,  $f$  takes the list 1 2 3 4 5 6 and rearranges it to get 3 5 4 6 2 1. We can also think of  $f$  as rearranging any list of 6 things. It takes the 1<sup>st</sup> element and puts it 3<sup>rd</sup>, it takes the 2<sup>nd</sup> element and puts it 5<sup>th</sup>, and so on.

We are interested in studying groups, so it is good news that *the collection of all permutations of 1 2 3 4 5 6 is a group*. Both  $f$  and  $g$  are elements of that group. To see that we have a group, we must check the three conditions.

- 1) Is there a do-nothing? Yes, do-nothing is the rearrangement which just leaves everything where it currently is.
- 2) Does combining two permutations give another permutation? Yes, for if we rearrange, and then rearrange again, that is the same as one big rearrangement. As an example, we can combine our sample permutations  $f$  and  $g$  to get a new permutation  $fg$ . Recall that  $fg$  means first do  $g$  then do  $f$ :



3) Does every permutation have an inverse? Yes, just “unrearrange.” For example, this permutation is the inverse of  $f$ :



To show that  $h$  is the inverse of  $f$ , we must check that  $hf$  is the same as do-nothing.



So it is correct that  $h$  is the inverse of  $f$ .

Since the set of all permutations of 1 2 3 4 5 6 satisfies the three conditions, we conclude that it is a group. This group is called  $S_6$ , pronounced “ess six.” The  $S$  stands for ‘symmetric.’ Similarly, the group of permutations of 1 2 3 4 5 is called  $S_5$ , and so on.

The notation we have been using to describe a permutation is very cumbersome. We now introduce a better method called **cycle notation**. Here is an example cycle: (5 2 1 3). This cycle is interpreted as “5 goes to 2, 2 goes to 1, 1 goes to 3, and 3 goes to 5.” We read left to right, and when we reach the end we go back to the beginning. If a cycle doesn’t mention a number, then the number does not move. For instance, the cycle just given doesn’t affect

the number 4. Since the numbers not mentioned in a cycle do not move, the “empty cycle”  $()$  is do-nothing. A cycle with just one number, such as  $(3)$ , is also do-nothing. This is because 3 goes to 3, so 3 doesn’t move, and nothing else is mentioned, so nothing else moves either.

To combine cycles, we start on the right. For example,  $(3\ 5\ 2)(6\ 3\ 1)$  means “6 goes to 3, 3 goes to 1, and 1 goes to 6, then 3 goes to 5, 5 goes to 2, and 2 goes to 3.” Note that 3 was moved more than once.

**Warning.** When combining cycles, we first do the cycle on the *right*. But within each cycle, we read left to right.

If we have to deal with several cycles, it is easiest if each number appears in only one cycle. We call this **disjoint cycles**. For example,  $(2\ 3\ 5)(7\ 4\ 6\ 1)$  is a product of disjoint cycles. The usefulness of disjoint cycles is that each number is only being moved once. An important fact is that any permutation can be written as a product of disjoint cycles. For example:

$$\begin{array}{cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
 f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 3 & 5 & 4 & 6 & 2 & 1
 \end{array}
 \qquad \text{is the same as} \qquad (2\ 5)(1\ 3\ 4\ 6)$$

$$\begin{array}{cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
 g & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 2 & 6 & 5 & 3 & 1 & 4
 \end{array}
 \qquad \text{is the same as} \qquad (1\ 2\ 6\ 4\ 3\ 5)$$

$$\begin{array}{cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
 fg & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 5 & 1 & 2 & 4 & 3 & 6
 \end{array}
 \qquad \text{is the same as} \qquad (6)(4)(1\ 5\ 3\ 2)$$

We could also represent  $fg$  as  $(1\ 5\ 3\ 2)$  because both  $(4)$  and  $(6)$  are do-nothing. The cycle representations were found by tracing through the permutation, finding in succession where each number goes.

**Task 6.6.1:** Show that the permutations in  $S_3$  can be represented by  $\{(), (1\ 2), (2\ 3), (3\ 1), (1\ 2\ 3), (1\ 3\ 2)\}$ .

If we have a product which is not disjoint, then we can turn it into a product of disjoint cycles. For example,  $(3\ 5\ 2\ 1)(6\ 2\ 1) = (6\ 1)(2\ 3\ 5)$ . This answer was found by just tracing through the cycles. Here are the steps:

First pick any number. We will start with 6. (6

First 6 goes to 2, then 2 goes to 1, so the net result is 6 goes to 1. (6 1)

Now we find where 1 goes. First 1 goes to 6, then 6 doesn't move, so the net result is 1 goes to 6. This gives one cycle. (6 1)

Now pick another number. We pick 2. (6 1)(2)

First 2 goes to 1, then 1 goes to 3, so the net result is 2 goes to 3. (6 1)(2 3)

First 3 stays where it is, then it goes to 5, so the total effect is 3 goes to 5. (6 1)(2 3 5)

First 5 stays where it is, then it goes to 2, so the total effect is 5 goes to 2. This completes another cycle. All numbers have been used, so we are done. (6 1)(2 3 5)

This method can be used to turn any product of cycles into a product of disjoint cycles. A useful exercise is to multiply the cycle representations given above for  $f$  and  $g$  to see that you get the cycle representation given for  $fg$ .

**Task 6.6.2:** Write down a product of cycles and use the above method to turn it into a product of disjoint cycles.

**Task 6.6.3:** What is the inverse of a cycle?

**Task 6.6.4:** Write down the multiplication table for  $S_3$ . Use the elements of  $S_3$  listed in Task 6.6.1.

**Task 6.6.5:** Explain why a product of disjoint cycles can be rearranged without changing the resulting permutation. For example,  $(1\ 2\ 3)(4\ 5\ 6\ 7)$  is the same as  $(4\ 5\ 6\ 7)(1\ 2\ 3)$ .

**Task 6.6.6:** Give an example to show that if you rearrange non-disjoint cycles then the result can be a different permutation.

To complete the circle of ideas in this section, we show how cycle notation can be used to describe the symmetries of a square. Here are two symmetries:



We can describe these symmetries by showing how each one rearranges the labels we put on the square:  $m = (c\ d)(b\ a)$  and  $r = (a\ b\ c\ d)$ . Any symmetry of any finite figure can be described in this way.

**Task 6.6.7:** Using the labeling above, determine which symmetries of the square correspond to  $(a\ c)(b\ d)$  and to  $(a\ c)$ .

**Task 6.6.8:** How many elements are in the group  $S_n$ ? In other words, how many ways are there to rearrange the numbers  $1\ 2\ 3\ \dots\ n$ ?

A cycle containing two elements, such as  $(3\ 5)$ , is called a **2-cycle**, or a **transposition**. It switches two elements, and everything else is unchanged. It is possible to write any permutation as a product of 2-cycles, although the 2-cycles may not be disjoint. For example,  $(1\ 2\ 3) = (2\ 3)(1\ 3)$ .

**Task 6.6.9:** Find another expression for  $(1\ 2\ 3)$  as a product of 2-cycles.

**Task 6.6.10:** Write  $(1\ 2\ 3\ 4)$  as a product of 2-cycles.

## 6.7 Notes

**Note 6.7.a:** The technical term for two groups being ‘the same’ is *isomorphic*. The matching up of the elements in the two groups is called an *isomorphism*. Those words come from the Greek roots *iso*, meaning ‘same,’ and *morph*, meaning ‘form.’ To repeat the definition from Section 6.3, an **isomorphism** between two groups is a matching of the elements of the groups which also gives a matching of the multiplication tables of the groups. If such a matching exists, then we say the two groups are **isomorphic**.

**Task 6.7.1:** Show that  $S_3$  is isomorphic to  $D_3$ . Note: you have already written down the multiplication tables for both groups, so once you decide on a matching, it will be easy to check if that matching works.

One way to see an answer to Task 6.7.1 is to label the corners of a triangle, and then see how the symmetries of the triangle rearrange those labels. It is just a coincidence that  $S_3$  is the same as  $D_3$ . For larger values of  $N$ , the groups  $S_N$  and  $D_N$  are different. However, by labeling the corners of a regular  $N$ -gon we see that  $D_N$  is a subgroup of  $S_N$  for  $N \geq 4$ .

**Note 6.7.b:** In Chapter 3 we classified finite figures into symmetry types  $C_N$  and  $D_N$ . At that time we did not actually explain the way in which figures having the same symmetry type are actually ‘the same.’ Now we can give the official definition.

**Definition.** Two finite planar figures have the **same symmetry type** if there is an isomorphism between their symmetry groups such that the isomorphism matches rotations with rotations and reflections with reflections.

The above definition does *not* merely say that the symmetry groups of the two figures must be isomorphic; there is an additional condition that there be an isomorphism which matches reflections with reflections and rotations with rotations. This distinction is relevant, as demonstrated by the following Task.

**Task 6.7.2:** Show that  $C_2$  is isomorphic to  $D_1$ . Explain why  $C_2$  and  $D_1$  are different symmetry types.