Topics in Geometry, 189-348A, Part 3

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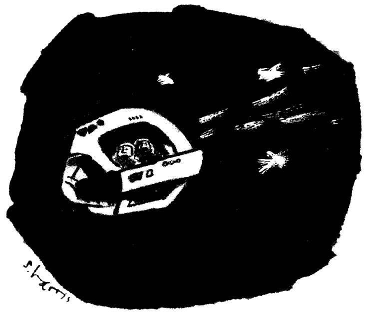
1 Introduction to Special Relativity



"IF EINSTEIN IS CORRECT, WHEN WE GET BACK, MY CAR WILL HAVE BEEN DOUBLE PARKED 320 YEARS."

One of the greatest triumphs of Maxwell's electromagnetic theory (c. 1864) was the explanation of light as an electromagnetic wave phenomenon. But waves in what? In conformity with the mechanistic view of nature then prevailing, it seemed imperative to postulate the existence of a medium - the ether - which would serve as a carrier for these waves (and for electromagnetic 'stress' in general). This led to the most urgent physical problem of the time: the detection of the earth's motion through the ether. Of the many experiments devised for this purpose, we shall mention just three. Michelson and Morley (1887), looked for a directional variation in the velocity of light on earth. Fizeau (1860), Mascart (1872), and later Lord Rayleigh (1902), looked for an expected effect of the earth's motion on the refractive index of certain dielectrics. And Trouton and Noble (1903) tried to detect an expected tendency of a charged plate condenser to face the 'ether drift'. All failed. The facile explanation that the earth might drag the ether along with it only led to other difficulties with the observed aberration of starlight, and could not resolve the problem.

In order to explain nature's apparent conspiracy to hide the ether drift, Lorentz between 1892 and 1909 developed a theory of the ether that was eventually based on two hypotheses - the longitudinal contraction of rigid bodies and the slowing down of clocks ("time- dilation") when moving through the ether at a speed v. This would so affect every apparatus designed to measure the ether drift as to neutralize all expected effects.

In 1905, in the middle of this development, Einstein proposed the principle of relativity which is now justly associated with his name. Actually Poincare had discussed essentially the same principle during the previous year, but it was Einstein who first recognized its full significance and put it to brilliant use. In it, he elevated the complete equivalence of all inertial reference frames to the status of an axiom or principle, for which no proof or explanation is to be sought. On the contrary, it explains the failure of all the ether-drift experiments, much as the principle of energy conservation explains a priori (i.e. without the need for a detailed examination of the mechanism) the failure of all attempts to construct perpetual motion machines. At first sight Einstein's relativity principle seems to be no more than a whole-hearted acceptance of the null results of all the ether-drift experiments. But by ceasing to look for special explanations of those results, and using them rather as the empirical evidence for a new principle of nature, Einstein had turned the tables: predictions could be made. The situation can be compared to that obtaining in astronomy at the time when Ptolemy's intricate geocentric system (corresponding to Lorentz's theory) gave way to the ideas of Copernicus, Galileo, and Newton. In both cases the liberation from a venerable but inconvenient reference frame ushered in a revolutionary clarification of physical thought, and consequently led to the discovery of a host of new and unexpected resuits. Soon a whole theory based on Einstein's principle (and on a "second axiom" asserting the invariance of the speed of light) was in existence, and this theory is called special relativity. Its programme was to modify all the laws of physics, where necessary, so as to make them equally valid in all inertial frames. For Einstein's principle is really a metaprinciple: it puts constraints on all the laws of physics. The modifications suggested by the theory (especially in mechanics), though highly significant in many modern applications, have negligible effect in most classical problems, which is of course why they were not discovered earlier. However, they were not exactly needed empirically in 1905 either. This is a beautiful example of the power of pure thought to leap ahead of the empirical frontier-a feature of all good physical theories, though rarely on such a heroic scale. It has led, among other things, to a new theory of space and time, and in particular to the relativity of simultaneity and the existence of a maximum speed for all particles and signais, to a new mechanics in which mass increases with speed, to the formula $E = mc^2$, and to de Broglie's association of waves with particles.

1.1 Einstein's two axioms

Special relativity is the theory of an ideal physics referred to an ideal set of infinitely extended gravity-free inertial frames.

An inertial frame is one in which spatial relations are Euclidean and in which there exists a universal time in terms of which free particles remain at rest or continue to move with constant speed along straight lines (i.e. in terms of which free patricles obey Newton's first law).

Einstein advanced the following two axioms:

1. The laws of physics are identical in all inertial frames.
2. Light signals in vacuum are propagated rectilinearly, with the same speed $c = 3 \times 10^8 ms^{-1}$, at all times, in all directions, in all inertial frames.

Time depends on speed

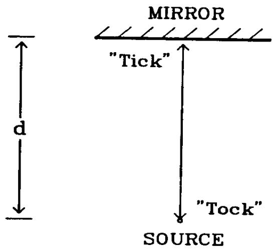


Fig. 8 A light clock at rest in frame S´

A "light clock" (Fig. 8) involves the repeated travelling of light across a constant distance. Light is emitted from the source and travels upward to a mirror. There it is reflected back to the source, where it is received and another light pulse immediately sent upward. The times taken for the light to travel upward and downward are the same.

Imagine that the light clock is at rest in an inertial frame S´, moving with a constant horizontal velocity relative to us (at rest in inertial frame S). In S´, the light is moving vertically up and down in a stationary light clock. However, according to us the light is moving horizontally as well as vertically, since the entire clock is moving along with frame S´. Thus, relative to S, the path of a light pulse would be as shown in Fig. 9. The light travels the same vertical distance as in frame S´, but travels a horizontal distance as well. Thus, the distance travelled by the light is greater in S than in S´.

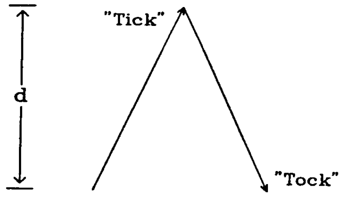


Fig. 9 The same light clock, according to observers at rest in frame S.

Now for the fun! Light travels at the same speed, c, in both frames S and S´. Since the distance traveled in S is greater than in S´, then the *time* of travel for the light must be greater in S than in S′. This is a direct consequence of the constancy of the speed of light.

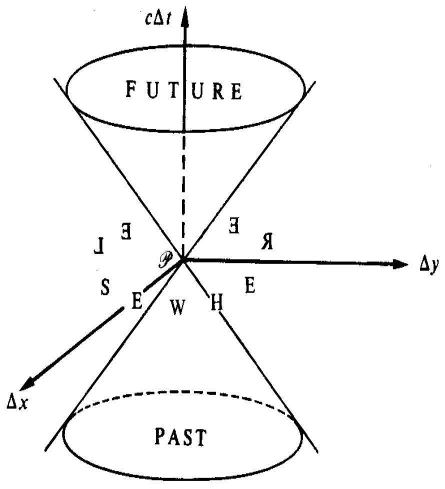
It turns out that this difference in times is valid not only for a light clock, but for any type of clock. In other words, no matter how time is measured, *the rate at which time passes depends on speed* (of an observer, a subatomic particle, etc.). This *relativity of time* will be investigated in more detail later.

In order for this effect to be measurable, the relative speed of two frames of reference must be close to the speed of light. At everyday speeds that we encounter, the effect is negligible. However, for high-speed particles that are produced, for example, by natural processes such as radioactive decay, the effect can be very large.

1.2 Minkovski's Spacetime

Minkowski's four-dimensional "spacetime" consists of events . An *event* will be specified by four coordinates, one of time and three of position, e.g. (t, x, y, z). An interval Δs between events is measured as $(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$.

SPACETIME



Each material particle in the course of its history appears as a line, that line is called its “worldline”. Displacement vectors for which $\Delta s^2 > 0$ are said to be timelike. Since a particle must always move with speed less than c, its worldline must alvays have slope less than unity (relative to the time axis). Hence no portion of such a worldline through $\mathcal{P}$ can lie outside of the cone.

Lorentz transformation

Suppose that inertial frame S' moves in the direction of the positive x-axis of S with constant velocity v. Suppose that the event $\mathcal{P}$ has coordinates (t, x, y, z) in S and coordinates (t', x', y', z') in S'.

Then the standard Galilean transformation of the coordinates would be $t' = t,\ x' = x - vt,\ y' = y,\ z' = z$. The Lorentz transformation is the following:

$$t' = \gamma(t - vx/c^2),\ x' = \gamma(x - vt),\ y' = y,\ z' = z,$$

where $\gamma = \gamma(v) = 1/(\sqrt{1 - v^2/c^2})$ is called a Lorentz factor.

Properties of Lorenz transformation:

1) The most striking feature of the Lorentz transformation is the transformation of time, which exhibits the relativity of simaltaneity: events with equal t do not necessarily correspond to events with equal t'.

2)the transformation is symmetric in $x, y, z, T = ct$.

3) The Lorentz transformation replaces the old Galilean transformatiom, to which it nevertheless approximates when $\frac{v}{c}$ is small.

4) Since the Lorentz transformation is linear and homogeneous, the coordinate differences satisfy the same transformation equations:

$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2),\ \Delta x' = \gamma(\Delta x - v\Delta t),\ \Delta y' = \Delta y,\ \Delta z' = \Delta z.$$

5)$\gamma > 1$.

The length contraction

Consider two events occuring simaltaneously at the end points of the train moving with high speed. Let the inertial frame of the train be $S' : (t', x', y', z')$ and the frame of an observer (outside of the trane) be $S : (t, x, y, z)$. So the time difference in S between the events is 0. $\Delta t = 0$.

Using the above formulas we have $\Delta x' = \gamma \Delta x$. Hence $\Delta x' > \Delta x$, and the length of the train in S is smaller than in S'.

The length of a body in the direction of its motion is reduced by a Lorentz factor.

10. The length contraction paradox

Consider the admittedly unrealistic situation of a man carrying horizontally a 20-foot pole and wanting to get it into a 10-foot garage. He will run at speed $v = 0.866\,c$ to make $\gamma = 2$, so that the pole contracts to 10 feet. It will be well to insist on having a sufficiently massive block of concrete at the back of the garage, so that there is no question of whether the pole finally stops in the inertial frame of the garage, or vice versa. So the man runs with his (now contracted) pole into the garage and a friend quickly closes the door. In principle we do not doubt the feasibility of this experiment, i.e. the reality of length contraction. When the pole stops in the rest frame of the garage, it is, in fact, being 'rotated in spacetime' and will tend to assume, if it can, its original length relative to the garage. Thus, if it survived the impact, it must now either bend, or burst the door.

At this point a 'paradox' might occur to the reader:[1] what about the symmetry of the phenomenon? Relative to the runner, won't the garage be only 5 feet long? Yes, indeed. Then how can the 20-foot pole get into the 5-foot garage? Very well, let us consider what happens in the rest frame of the pole. The open garage now comes towards the stationary pole. Because of the concrete block, it keeps on going even after the impact, taking the front end of the pole with it (see Fig. 7). But the back end of the pole is still at rest: it cannot yet 'know' that the front end has been struck, because of the finite speed of propagation of *all* signals. Even if the 'signal' (in this case the elastic shock wave) travels along the pole with the speed of light, that signal has 20 feet to travel against the garage front's 15 feet, before reaching the back end of the pole. This race would be a dead heat if v were $0.75\,c$. But v is $0.866\,c$! So the pole *more* than just gets in. (It could even get into a garage whose length is as little as 5.4 feet at rest and thus 2.7 feet in motion: the garage front would then have to travel 17.3 feet against the shock wave's 20 feet, requiring speeds in the ratio 17.3 to 20, i.e. 0.865 to 1 for a dead heat.)

There is one important moral to this story: whatever result we get by correct reasoning in any one frame, must be true; in particular, it must be true when viewed from any other frame. As long as the set of physical laws we are using is self-consistent and Lorentz-invariant, there *must* be an explanation of the result in every other frame, although it may be quite a different explanation from that in the first frame. For example, as we shall see, the magnetic force experienced by an electron traversing the field of a permanent magnet is felt as a purely electric force in the rest frame of the electron.

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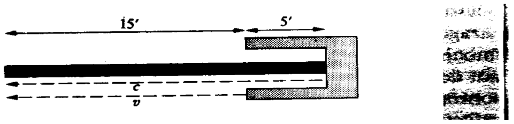
[1] It is perhaps surprising that no such paradox seems to have been encountered before 1960. See: Rindler, W. (1960) *Special Relativity*, Oliver and Boyd, Edinburgh p. 37; see also Rindler, W. (1961) *American Journal of Physics*, **29**, 365.



The time dilation

A clock moving with velocity v through an inertial frame S goes slow by a Lorentz factor relative to the standard clocks at rest in S.

Let S' be a frame connected with the moving clock. Then $\Delta x' = 0$ By symmetry of Lorentz transformation we have $\Delta t = \gamma(\Delta t' + \frac{v}{c}\Delta x')$ ot $\Delta t = \gamma \Delta t'$.

The Twin Paradox

One of the most famous discussions arising from time dilation is the "twin paradox." This puzzle concerns two twins, one who stays on Earth, and one who makes a high-speed trip in a spaceship and eventually returns to Earth (Fig. 14). The question is whether one twin will be younger than the other when the travelling twin returns.

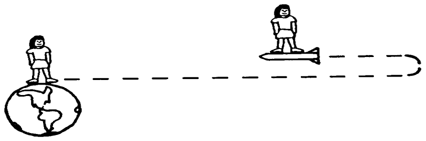


Fig. 14 Which twin will be younger when they are re-united?

One way to answer the question is the following. If we first consider the outward trip, the travelling twin is always at the same position in his/her frame, and since one-position time is less than two-position time, the time interval for the travelling twin will be less than that for the stay-at-home twin. Similarly, for the return trip the travelling twin is always at the same position in his/her frame, and the time interval for the travelling twin will be less. Thus, over the entire trip, the travelling twin will have experienced a smaller time interval than the stay-at-home twin, and the travelling twin will be younger.

However, here is the paradox. During both the outward trip and the return trip, the stay-at-home twin is also always at the same position in his/her frame. Using reasoning similar to that in the previous paragraph, we can conclude that the stay-at-home twin will have experienced a smaller time interval than the travelling twin. This seems to be quite a dilemma. The situation appears to be completely symmetrical — each twin sees the other moving away and then returning.

We can resolve this paradox by recognizing that *the travelling twin is in two different inertial frames of reference during the trip*. For the outward journey this twin is at rest in a frame moving away from Earth, and for the return journey in a frame moving toward Earth. At the turnaround point, the rocketship slows down, then speeds up in the opposite direction. During this acceleration phase, objects that are not tied down in the spacecraft are flung against the walls, and the travelling twin experiences that acceleration.

Thus, the travelling twin knows that he/she is the one who is indeed changing inertial frames at turnaround. To consider the stay-at-home twin to be travelling and *turning around* is not physically correct, because *this twin experiences no acceleration at "turnaround."* Therefore, the correct analysis is the one in which we consider the time intervals for both the outward and return journeys to be one-position time for the travelling twin. Hence, *the travelling twin is younger*.

A detailed mathematical analysis leads to the same conclusion. This was confirmed by an experiment in 1971 in which atomic clocks were flown on commercial jet flights around the world. Within experimenta error, the observed time intervals were those predicted by relativity.