1 Euclidean and non-Euclidean geometries

1.1 Euclid’s first four postulates

Euclid based his geometry on five fundamental assumptions, called axioms or postulates.

Euclid postulate 1. For every point \( P \) and for every point \( Q \) not equal to \( P \) there exists a unique line \( \ell \) that passes through \( P \) and \( Q \).

We will denote this unique line by \( PQ \).

To state the second postulate we must make the first definition.

Definition 1. Given two points \( A \) and \( B \). The segment \( AB \) is the set whose members are the points \( A \) and \( B \) and all points that lie on the line \( AB \) and are between \( A \) and \( B \). The two given points \( A \) and \( B \) are called the endpoints of the segment \( AB \).

Euclid’s postulate 2. For every segment \( AB \) and for every segment \( CD \) there exists a unique point \( E \) such that \( B \) is between \( A \) and \( E \) and segment \( CD \) is congruent to segment \( BE \).

Euclid postulate 3. For every point \( O \) and every point \( A \) not equal to \( O \) there exists a circle with center \( O \) and radius \( OA \).
Definition 2 The ray \( \overrightarrow{AB} \) is the following set of points lying on the line \( \overline{AB} \): those points that belong to the segment \( AB \) and all points \( C \) such that \( B \) is between \( A \) and \( C \). The ray \( \overrightarrow{AB} \) is said to emanate from \( A \) and to be part of line \( \overline{AB} \).

Definition 3 Rays \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) are opposite if they are distinct, if they emanate from the same point \( A \), and if they are part of the same line \( \overline{AB} = \overline{AC} \).

Definition 4 An angle with vertex \( A \) is a point \( A \) together with two nonopposite rays \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) (called the sides of the angle) emanating from \( A \).

Definition 5 If two angles \( \angle BAD \) and \( \angle CAD \) have a common side \( \overrightarrow{AD} \) and the other two sides \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) form opposite rays, the angles are supplements of each other, or supplementary angles.
**Definition 6** An angle $\angle BAD$ is a right angle if it has a supplementary angle to which it is congruent.

Euclid postulate 4. All right angles are congruent to each other.

This postulate expresses a sort of homogeneity: two right angles may be very far away from each other, they nevertheless have the same size. The postulate therefore provides a natural standard of measurement for angles.

**Definition 7** Two lines $\ell$ and $m$ are parallel if they do not intersect, i.e., if no point lies on both of them. We denote this by $\ell \parallel m$.

The Euclidean parallel postulate For every line $\ell$ and every point $p$ that does not lie on $\ell$ there exists a unique line through $P$ that is parallel to $\ell$. 
1.2 The discovery of non-Euclidean geometry (From Martin Jay Greenberg’s book)

János Bolyai

It is remarkable that sometimes when the time is right for a new idea to come forth, the idea occurs to several people more or less simultaneously. Thus it was in the eighteenth century with the discovery of the calculus by Newton in England and Leibniz in Germany, and in the nineteenth century with the discovery of non-Euclidean geometry. When János Bolyai (1802–1860) announced privately his discoveries in non-Euclidean geometry, his father Wolfgang admonished him:

It seems to me advisable, if you have actually succeeded in obtaining a solution of the problem, that, for a two-fold reason, its publication be hastened: first, because ideas easily pass from one to another who, in that case, can publish them; secondly, because it seems to be true that many things have, as it were, an epoch in which they are discovered in several places simultaneously, just as the violets appear on all sides in springtime.*

János Bolyai did publish his discoveries, as a 26-page appendix to a book by his father (the Tentamen, 1831). His father eagerly sent a copy of this book to his friend, the German mathematician Carl Friedrich Gauss (1777–1855), undisputedly the foremost mathematician of his time. Wolfgang Bolyai had become close friends with Gauss 35 years earlier, when they were both students in Göttingen. After Wolfgang returned to Hungary, they maintained an intimate correspondence† and when Wolfgang sent Gauss his own attempt to prove the parallel postulate, Gauss tactfully pointed out the fatal flaw.
János was 13 years old when he mastered the differential and integral calculus. His father wrote to Gauss begging him to take the young prodigy into his household as an apprentice mathematician. Gauss never replied to this request (perhaps because he was having enough trouble with his own son Eugene, who had run away from home). Fifteen years later, when Wolfgang mailed the *Tentamen* to Gauss, he certainly must have felt that his son had vindicated his belief in him, and János must have expected Gauss to publicize his achievement. One can therefore imagine the disappointment János must have felt when he read the following letter to his father from Gauss:

If I begin with the statement that I dare not praise such a work, you will of course be startled for a moment; but I cannot do otherwise; to praise it would amount to praising myself; for the entire content of the work, the path which your son has taken, the results to which he is led, coincide almost exactly with my own meditations which have occupied my mind for from thirty to thirty-five years. On this account I find myself surprised to the extreme.

My intention was, in regard to my own work, of which very little up to the present has been published, not to allow it to become known during my lifetime. Most people have not the insight to understand our conclusions and I have encountered only a few who received with any particular interest what I communicated to them. In order to understand these things, one must first have a keen perception of what is needed, and upon this point the majority are quite confused. On the other hand, it was my plan to put all down on paper eventually, so that at least it would not finally perish with me.

So I am greatly surprised to be spared this effort, and am overjoyed that it happens to be the son of my old friend who outstrips me in such a remarkable way. 

Despite the compliment in Gauss' last sentence, János was bitterly disappointed with the great mathematician's reply; he even imagined that his father had secretly informed Gauss of his results and that Gauss was now trying to appropriate them as his own. A man of fiery temperament, who had fought and won thirteen successive duels (unlike Galois, who was killed in a duel at age 20), János fell into deep mental depression and never again published his research. A translation of his immortal "appendix" can be found in R. Bonola's *Non-Euclidean Geometry* (1955).
In 1851, he wrote:

In my opinion, and as I am persuaded, in the opinion of anyone judging without prejudice, all the reasons brought up by Gauss to explain why he would not publish anything in his life on this subject are powerless and void; for in science, as in common life, it is necessary to clarify things of public interest which are still vague, and to awaken, to strengthen and to promote the lacking or dormant sense for the true and right. Alas, to the great detriment and disadvantage of mankind, only very few people have a sense for mathematics; and for such a reason and pretence Gauss, in order to remain consistent, should have kept a great part of his excellent work to himself. It is a fact that, among mathematicians, and even among celebrated ones, there are, unfortunately, many superficial people, but this should not give a sensible man a reason for writing only superficial and mediocre things and for leaving science lethargically in its inherited state. Such a supposition may be said to be unnatural and sheer folly; therefore I take it rightly amiss that Gauss, instead of acknowledging honestly, definitely and frankly the great worth of the Appendix and the Tentamen, and instead of expressing his great joy and interest and trying to prepare an appropriate reception for the good cause, avoiding all these, he rested content with pious wishes and complaints about the lack of adequate civilization. Verily, it is not this attitude we call life, work and merit.*

Gauss

There is evidence that Gauss had anticipated some of J. Bolyai’s discoveries, in fact, that Gauss had been working on non-Euclidean geometry since the age of 15, i.e., since 1792 (see Bonola, Chapter 3). In 1817, Gauss wrote to W. Olbers: “I am becoming more and more convinced that the necessity of our [Euclidean] geometry cannot be proved, at least not by human reason nor for human reason. Perhaps in another life we will be able to obtain insight into the nature of space, which is now attainable.” In 1824, Gauss answered F. A. Taurinus, who had attempted to investigate the theory of parallels:

In regard to your attempt, I have nothing (or not much) to say except that it is incomplete. It is true that your demonstration of the proof that the sum of the three angles of a plane triangle cannot be greater than 180° is somewhat lacking in geometrical rigor. But this in itself can easily be remedied, and there is no doubt that the impossibility can be proved most rigorously. But the situation is quite different in the second part, that the sum of the

angles cannot be less than 180°; this is the critical point, the reef on which all
the wrecks occur. I imagine that this problem has not engaged you very long.
I have pondered it for over thirty years, and I do not believe that anyone can
have given more thought to this second part than I, though I have never
published anything on it.

The assumption that the sum of the three angles is less than 180° leads to
a curious geometry, quite different from ours [the Euclidean], but thoroughly
consistent, which I have developed to my entire satisfaction, so that I can solve
every problem in it with the exception of the determination of a constant,
which cannot be designated a priori. The greater one takes this constant, the
nearer one comes to Euclidean geometry, and when it is chosen infinitely
large the two coincide. The theorems of this geometry appear to be paradoxical
and, to the uninitiated, absurd; but calm, steady reflection reveals that they
contain nothing at all impossible. For example, the three angles of a triangle
become as small as one wishes, if only the sides are taken large enough; yet
the area of the triangle can never exceed a definite limit, regardless of how
great the sides are taken, nor indeed can it ever reach it.

All my efforts to discover a contradiction, an inconsistency, in this non-
Euclidean geometry have been without success, and the one thing in it which is
opposed to our conceptions is that, if it were true, there must exist in space a
linear magnitude, determined for itself (but unknown to us). But it seems to
me that we know, despite the say-nothing word-wisdom of the metaphysicians,
too little, or too nearly nothing at all, about the true nature of space, to
consider as absolutely impossible that which appears to us unnatural. If this
non-Euclidean geometry were true, and it were possible to compare that
constant with such magnitudes as we encounter in our measurements on the
earth and in the heavens, it could then be determined a posteriori. Consequently,
in jest I have sometimes expressed the wish that the Euclidean
geometry were not true, since then we would have a priori an absolute
standard of measure.

I do not fear that any man who has shown that he possesses a thoughtful
mathematical mind will misunderstand what has been said above, but in any
case consider it a private communication of which no public use or use leading
in any way to publicity is to be made. Perhaps I shall myself, if I have at some
future time more leisure than in my present circumstances, make public my
investigations.*

It is amazing that, despite his great reputation, Gauss was actually
afraid to make public his discoveries in non-Euclidean geometry. He
wrote to F. W. Bessel in 1829 that he feared “the howl from the Boeotians”

* Wolfe; op. cit., pp. 46–47.
if he were to publish his revolutionary discoveries.* He told H. C. Schumacher that he had “a great antipathy against being drawn into any sort of polemic.”

The “metaphysicians” referred to by Gauss in his letter to Taurinus were followers of Immanuel Kant, the supreme European philosopher in the late eighteenth century and much of the nineteenth century. Gauss’ discovery of non-Euclidean geometry refuted Kant’s position that Euclidean space is inherent in the structure of our mind. In his Critique of Pure Reason (1781) Kant declared that “the concept of [Euclidean] space is by no means of empirical origin, but is an inevitable necessity of thought.”

Another reason that Gauss withheld his discoveries was that he was a perfectionist, one who published only completed works of art. His devotion to perfected work was expressed by the motto on his seal, pauc sed matura (“few but ripe”). There is a story that the distinguished mathematician K. G. J. Jacobi often came to Gauss to relate new discoveries, only to have Gauss pull out some papers from his desk drawer that contained the very same discoveries. Perhaps it is because Gauss was so preoccupied with original work in many branches of mathematics, as well as in astronomy, geodesy, and physics (he co-invented an improved telegraph with W. Weber), that he did not have the opportunity to put his results on non-Euclidean geometry into polished form. These results were revealed among his private notes only after his death.

It would be impossible in this short book to present the tremendous range of Gauss’ work. Gauss was called “the prince of mathematicians” and only Archimedes and Newton might be considered his equal. (For more detail on his life and work, see the biographies by Bell, Dunnington, and Hall.)

**Lobachevsky**

Another actor in this historical drama came along to steal the limelight from both J. Bolyai and Gauss: the Russian mathematician Nikolai Ivanovich Lobachevsky (1792–1856). He was the first to actually publish

*An allusion to dull, obtuse individuals. Actually, the “Bocotian” critics of non-Euclidean geometry—conceited people who claimed to have proved that Gauss, Riemann, and Helmholtz were blockheads—did not show up before the middle of the 1870s. “If you witnessed the struggle against Einstein in the Twenties, you may have some idea of [the] amusing kind of literature produced by these critics.” Frege, rebuking Hilbert like a schoolboy, also joined the Bocotians. “Your system of axioms, he said to Hilbert, is like a system of equations you cannot solve.” (Freudenthal, 1962)
an account of non-Euclidean geometry (1829). His work attracted little attention when it appeared, largely because it was published in Russian and the Russians who read it were severely critical. In 1840 he published a treatise in German, which came to the attention of Gauss (who praised it in a letter to Schumacher, at the same time reiterating his own priority in the field). At first Lobachevsky called his geometry “imaginary geometry,” then later “pangeometry.” In his published works he developed the subject quite thoroughly.

Lobachevsky openly challenged the Kantian doctrine of space as a subjective intuition. In 1835 he wrote: “The fruitlessness of the attempts made since Euclid’s time... aroused in me the suspicion that the truth... was not contained in the data themselves; that to establish it the aid of experiment would be needed, for example, of astronomical observations, as in the case of other laws of nature.”

Lobachevsky has been called “the great emancipator” by Eric Temple Bell; his name, said Bell, should be as familiar to every schoolboy as that of Michelangelo or Napoleon.† Unfortunately, Lobachevsky was not so appreciated in his lifetime; in fact, in 1846 he was fired from the University of Kazan, despite twenty years of outstanding service as a teacher and administrator. He had to dictate his last book in the year before his death, for by then he was blind.

It was not until after Gauss’ death in 1855, when his correspondence was published, that the mathematical world began to take non-Euclidean ideas seriously. (Yet, as late as 1888 Lewis Carroll was poking fun at non-Euclidean geometry.) Some of the best mathematicians (Beltrami, Klein, Poincaré, and Riemann) took up the subject, extending it, clarifying it, and applying it to other branches of mathematics, notably complex function theory. In 1868 the Italian mathematician Beltrami settled once and for all the question of a proof for the parallel postulate. He proved that no proof was possible! He did this by proving that non-Euclidean geometry is just as consistent as Euclidean geometry. (We will discuss his proof in the next chapter.)
1.3 Plane Euclidean geometry

Each ordered pair \((p_1, p_2)\) of real numbers determines exactly one point \(P\) of the plane. The point \((0, 0)\) is called the \textit{origin}. The ordered pair \((p_1, p_2)\) is also called the \textit{coordinate vector of} \(P\). We may think of the vector \((p_1, p_2)\) as the line segment beginning at the origin and ending at \(P\). We shall consider words “point” and “vector” as synonyms.

If \(x = (x_1, x_2)\) and \(y = (y_1, y_2)\), then we define

\[
x + y = (x_1 + y_1, x_2 + y_2).
\]

If \(c\) is a real number and \(x\) is a vector, then

\[
(cx_1, cx_2).
\]

These operations are called vector addition and scalar multiplication. The vector \((0, 0)\) is called the zero vector. These operations have the following properties:

**Theorem 1**

1. \((x + y) + z = x + (y + z)\).
2. \(x + y = y + x\).
3. \(x + 0 = x\).
4. \(x + (-x) = 0\).
5. \(1x = x\).
6. \(c(x + y) = cx + cy\).
7. \((c + d)x = cx + dx\).
8. \(c(dx) = (cd)x\).

The set of all vectors \((x_1, x_2)\) with vector addition and scalar multiplication satisfying the above properties is called the vector space \(\mathbb{R}^2\).

Given two vectors \(x\) and \(y\), we define

\[
<x, y> = x_1y_1 + x_2y_2.
\]

The number \(<x, y>\) is called the \textit{inner product} of \(x\) and \(y\). The following identities may be verified directly.
Theorem 2
1. \(<x, y + z> = <x, y> + <x, z>\) for all \(x, y, z \in \mathbb{R}^2\).
2. \(<x, cy> = c <x, y>\) for all \(x, y \in \mathbb{R}^2\) and all \(c \in \mathbb{R}\).
3. \(<x, y> = <y, x>\) for all \(x, y \in \mathbb{R}^2\).
4. If \(<x, y> = 0\) for all \(x \in \mathbb{R}^2\), then \(y\) must be the zero vector.

This theorem says that the inner product is bilinear, symmetric, and nondegenerate.

For any vector \(x \in \mathbb{R}^2\) we define the length of \(x\) to be \(|x| = \sqrt{x_1^2 + x_2^2}\). Note that
\[ |x|^2 = <x, x>. \]

Theorem 3
The length function has the following properties:
1. \(|x| \geq 0\) for all \(x \in \mathbb{R}^2\).
2. If \(|x| = 0\), then \(x = 0\) (the zero vector).
3. \(|cx| = |c||x|\) for all \(x \in \mathbb{R}^2\) and all \(c \in \mathbb{R}\).

Theorem 4 (Cauchy-Schwarz inequality). For two vectors \(x\) and \(y\) in \(\mathbb{R}^2\) we have
\[ |<x, y>| \leq |x||y|. \]
Equality holds if and only if \(x\) and \(y\) are proportional.

Proof If one of the vectors is zero, the assertion is obviously true. Suppose both \(x\) and \(y\) are nonzero vectors. Consider the function defined by
\[ f(t) = |x + ty|^2 \text{ for } t \in \mathbb{R}. \]
It follows from the property of length that \(f(y)\) is nonnegative for all \(t\) and that \(f(t)\) can become zero if and only if \(x\) is a multiple of \(y\). On the other hand, \(f\) is a polynomial of degree 2. Indeed,
\[ f(t) = |x|^2 + 2t <x, y> + t^2|y|^2. \]
As a polynomial of degree 2 \(f(t)\) remains nonnegative only if the discriminant in nonpositive, hence
\[ 4 <x, y>^2 - 4|x|^2|y|^2 \leq 0 \text{ and then } |<x, y>| \leq |x||y|. \]
In addition, \(f(t)\) assumes the zero value only if \(|<x, y>| = |x||y|\). Thus, \(|<x, y>| = |x||y|\) if and only if \(x\) and \(y\) are proportional.
**Corollary 1** For \( x, y \in \mathbb{R}^2 \),

\[ |x + y| \leq |x| + |y|. \]

Equality holds if and only if \( x \) and \( y \) are proportional with a nonnegative proportionality factor.

**The Euclidean plane \( \mathbb{E}^2 \).** The Euclidean plane \( \mathbb{E}^2 \) is \( \mathbb{R}^2 \) plus the notion of distance between points.

\[ d(P, Q) = |Q - P|. \]

**Theorem 5** Let \( P, Q, \) and \( R \) be points of \( \mathbb{E}^2 \). Then

1. \( d(P, Q) \geq 0 \).
2. \( d(P, Q) = 0 \) if and only if \( P = Q \).
3. \( d(P, Q) = d(Q, P) \).
4. \( d(P, Q) + d(Q, R) \geq d(P, R) \) (the triangle inequality).

### 1.4 Lines

A line is characterized by the property that the vectors joining pairs of points are proportional. We define a *direction* to be the set of all vectors proportional to a given nonzero vector.

For a given vector \( v \) let

\[ [v] = \{tv | t \in \mathbb{R}\}. \]

If \( P \) is any point and \( v \) is a nonzero vector, then

\[ \ell = \{X | X - P \in [v]\} \]

is called the *line* through \( P \) with direction \([v]\). We also write it in the form

\[ \ell = P + [v]. \]

When \( \ell = P + [v] \) is a line, we say that \( v \) is a direction vector of \( \ell \).
Theorem 6 Let $P$ and $Q$ be distinct points of $\mathbb{E}^2$. Then there is a unique line containing $P$ and $Q$, which we denote by $PQ$.

The equation of this line is $\ell = P + [Q - P]$. This a typical point $X$ on the line $\ell = PQ$ is written

$$\alpha(t) = P + t(Q - P) = (1 - t)P + tQ.$$ 

This equation may be regarded as a parametric representation of the line. The parameter $t$ is related to the distance along $\ell$ by the formula

$$d(\alpha(t_1), \alpha(t_2)) = |t_2 - t_1||Q - P|.$$ 

If $X = (1 - t)P + tQ$, where $0 < t < 1$, we say that $X$ is between $P$ and $Q$.

Theorem 7 Let $P, Q$ and $X$ be distinct points of $\mathbb{E}^2$. Then $X$ is between $P$ and $Q$ if and only if

$$d(P, X) + d(X, Q) = d(P, Q).$$

Example The equation of the line $\ell$ through $P = (1, -3)$ in the direction $[v]$, where $v = (3, 4)$ is $X = P + tv$ or

$$x_1 = 1 + 3t$$
$$x_2 = -3 + 4t.$$ 

Let $P, Q$ be distinct points. The set consisting of $P, Q$ and all points between them is called a segment and is denoted by $PQ$. The points $P, Q$ are the end points of the segment. All other points of the segment are called interior points. If $M$ is a point satisfying

$$d(P, M) = d(M, Q) = 1/2d(P, Q),$$

then $M$ is a midpoint of $PQ$, $M = 1/2(P + Q)$.

Theorem 8 Two distinct lines have at most one point of intersection.

If three or more points lie on some line, the points are said to be collinear.

1.5 Orthonormal pairs

Two vectors $v$ and $w$ are said to be orthogonal if $<v, w> = 0$. If $v = (v_1, v_2)$, we define $v^\perp = (-v_2, v_1)$. Clearly, $v$ and $v^\perp$ are orthogonal and have the same length. We also see that

$$v^{\perp\perp} = v.$$ 

A vector of length 1 is said to be a unit vector. A pair $\{v, w\}$ of unit orthogonal vectors is called an orthonormal pair.
Definition 8 A set of vectors \( \{v_1, \ldots, v_n\} \) is linearly independent if the equality
\[
0 = \mu_1 v_1 + \ldots + \mu_n v_n
\]
implies that all \( \mu_1, \ldots, \mu_n \) are zeros.

Definition 9 A set of vectors \( \{v, w\} \) in \( \mathbb{R}^2 \) is a basis if for every vector \( x \in \mathbb{R}^2 \) there exists unique numbers \( \lambda, \mu \) such that
\[
x = \lambda v + \mu w,
\]
that is \( x \) can be expressed uniquely as a linear combination of \( v \) and \( w \).

Any two linearly independent vectors in \( \mathbb{R}^2 \) form a basis.

Theorem 9 Let \( \{v, w\} \) be an orthonormal pair of vectors in \( \mathbb{R}^2 \). Then for all \( x \in \mathbb{R}^2 \),
\[
x = \langle x, v \rangle v + \langle x, w \rangle w.
\]

Proof Because \( v \) and \( w \) are linearly independent, they form a basis for \( \mathbb{R}^2 \). Thus every \( x \in \mathbb{R}^2 \) can be uniquely expressed as \( x = \lambda v + \mu w \). Then
\[
\langle x, v \rangle = \lambda \langle v, v \rangle + \mu \langle w, v \rangle = \lambda
\]
and
\[
\langle x, w \rangle = \lambda \langle v, w \rangle + \mu \langle w, w \rangle = \mu.
\]

If \( \ell \) is a line with direction vector \( v \), the vector \( v^\perp \) is called a normal vector to \( \ell \).

Theorem 10 Let \( P \) be any point and let \( \{v, N\} \) be an orthonormal pair of vectors. Then \( P + [v] = \{X \mid \langle X - P, N \rangle = 0\} \).

If \( N \) is any nonzero vector, \( \{X \mid \langle X - P, N \rangle = 0\} \) is the line through \( P \) with normal vector \( N \) and, hence, direction vector \( N^\perp \).

Proof We have the identity
\[
X - P = \langle X - P, v \rangle v + \langle X - P, N \rangle N
\]
for any \( X \in \mathbb{R}^2 \). It is obvious that \( X = P + tv \) if and only if \( \langle X - P, N \rangle = 0 \).

Example The equation of the line through \( P = (2, 1) \) with normal vector \( N = (3, 4) \) is \( \langle X - P, N \rangle = (x_1 - 2)3 + (x_2 - 1)4 = 0 \) or \( 3x_1 + 4x_2 - 6 - 4 = 0 \) or \( 3x_1 + 4x_2 - 10 = 0 \).

Rk Let \( a, b, c \) be real numbers. Then \( \{(x_1, x_2) \mid ax_1 + bx_2 + c = 0\} \) is
i) the empty set if \( a = b = 0 \) and \( c \neq 0 \),
ii) the whole plane if \( a = b = c = 0 \),
iii) a line with normal vector \((a, b)\) otherwise.
1.6 Perpendicular lines

Two lines $\ell$ and $m$ are perpendicular if they have orthogonal direction vectors. We write $\ell \perp m$.

**Theorem 11** (Pythagoras) Let $P, Q, R \in \mathbb{E}^2$ be distinct points. Then $|R - P|^2 = |Q - P|^2 + |R - Q|^2$ if and only if the lines $\overline{QP}$ and $\overline{RQ}$ are perpendicular.

**Proof** Put $x = Q - P$, $y = R - Q$. Then $x + y = R - P$. We have

$$|x + y|^2 = |x|^2 + 2 <x, y> + |y|^2,$$

hence $|x + y|^2 = |x|^2 + |y|^2$ if and only if $<x, y> = 0$.

**Theorem 12** 1) If $\ell \perp m$ then they have a unique point in common.

2) Let $X$ be a point and $\ell$ be a line, then there is a unique line through $X$ perpendicular to $\ell$. This line has equation $m = X + [N]$, where $N$ is a unit normal vector to $\ell$.

3) $\ell$ and $m$ intersect in the point $F = X - <X - P, N > N$, where $P$ is any point on $\ell$. This point $F$ is called the foot of the perpendicular.

4) $d(X, F) = |<X - P, N > |$, and $F$ is the point of $\ell$ nearest to $X$.

**Proof** Let $\ell = P + [v]$ and $m = Q + [N]$, so that $\{v, N\}$ is an orthonormal set. We write

$$P - Q = <P - Q, v > v + <P - Q, N > N,$$

and, hence,

$$P - <P - Q, v > v = Q + <P - Q, N > N.$$

We set

$$F = P - <P - Q, v > v = Q + <P - Q, N > N.$$

This point belongs to both $\ell$ and $m$. $F$ is the only common point because if there were two, the lines would have to coincide.

To prove 2)–4) take $Q = X$, then $F = X + <P - X, N > N = X - <X - P, N > N$. $d(X, F) = |F - X| = |<X - N, N > N| = |<X - N, N > ||N| = |<X - N, N > |$.

**Definition 10** The number $d(X, F)$ is called the distance from the point $X$ to the line $\ell$ and is written $d(X, \ell)$. 

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Theorem 13 1) Two distinct lines are parallel if and only if they have the same direction \([v]\).
2) If \(\ell \parallel m\), and \(m \parallel n\) then either \(\ell = n\) or \(\ell \parallel n\).
3) If \(\ell \parallel m\), and \(m \perp n\), then \(\ell \perp n\).
4) If \(\ell \perp m\), and \(m \perp n\), then \(\ell \parallel m\) or \(\ell = m\).
5) If \(\ell \parallel m\), then there is a unique number \(d(\ell, m)\) such that
\[
d(X, \ell) = d(Y, m) = d(\ell, m)
\]
for all \(X \in m\) and all \(Y \in \ell\). If \(N\) is a unit normal vector to \(\ell\) and \(m\), then \(d(\ell, m) = | <X - Y, N >|\).

1.7 Reflections, translations, rotations

Let \(\ell\) be a line passing through a point \(P\) and having unit normal \(N\). Two points \(X\) and \(X'\) are symmetrical about \(\ell\) if the midpoint of the segment \(XX'\) is the foot \(F\) of the perpendicular from \(X\) to \(\ell\).

In other words, \(1/2(X + X') = F\). By theorem 12 this means that
\[
1/2X + 1/2X' = X - <X - P, N > N,
\]
\[
1/2X' = 1/2X - <X - P, N > N,
\]
\[
X' = X - 2 <X - P, N > N.
\]
Then the reflection \(M_\ell\) in line \(\ell\) is given by the formula
\[
M_\ell(X) = X - 2 <X - P, N > N.
\]

Exercise. Verify that 1) \(d(M_\ell(X), M_\ell(Y)) = d(X, Y)\) for all points \(X, Y \in E^2\).
2) $M_\ell^2(X) = X$
3) $M_\ell(X) = X$ if and only if $X \in \ell$.
4) If $\ell \parallel m$, then $M_\ell M_m$ is a translation.

To prove 4) choose $P$ arbitrary on $m$ and $Q$ to be a foot of the perpendicular from $P$ to $\ell$.

$$M_\ell M_m(x) = M_m(x) - 2 < M_m(x) - P, N > N =$$

$$X - 2 < X - Q, N > N - 2 < X - 2 < X - Q, N > N - P, N > N =$$

$$X - 2 < X - Q, N > N - 2 < X - P, N > N + 4 < X - Q, N > N > N =$$

$$X + 2 < P - Q, N > N.$$

Since $P - Q \parallel N$ we have

$$X + 2 < P - Q, N > N = X + 2(P - Q).$$

**Theorem 14** Let $T$ be a nontrivial translation along $\ell$. Then $\ell$ has a direction vector $v$ such that $T(X) = X + v$. 
We define rotation as a product of reflections in two intersecting lines.

Let \( \ell \) be a line with unit direction vector \( v \). There is a unique real number \( \eta \in (-\pi, \pi] \) such that

\[
v = (\cos \eta, \sin \eta).
\]

The unit normal \( v^\perp \) is written as

\[
N = (-\sin \eta, \cos \eta).
\]

We now try to express \( N_\ell \) in terms of \( \eta \). First note that

\[
N_\ell(X) = X - 2 < X - P, N > N
\]

\[
N_\ell - P = X - P - 2 < X - P, N > N.
\]

Let \( \ell_0 \) be the line through 0 with direction \([v]\). Then

\[
N_{\ell_0}(X) = X - 2 < X, N > N.
\]

Thus,

\[
N_\ell(X) = N_{\ell_0}(X - P) + P.
\]

In other words, if we denote by \( T_P \) the translation by \( P \), then

\[
N_\ell = T_P N_{\ell_0} T_{-P}.
\]

Hence we can deal with \( N_{\ell_0} \) and use this equality to return to the original situation.

For any \( X \) note that

\[
<X, N> = -x_1 \sin \eta + x_2 \cos \eta.
\]

Thus writing \( X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) and \( N = \begin{bmatrix} -\sin \eta \\ \cos \eta \end{bmatrix} \), we get

\[
N_{\ell_0} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2(-x_1 \sin \eta + x_2 \cos \eta) \begin{bmatrix} -\sin \eta \\ \cos \eta \end{bmatrix}
\]

\[
= \begin{bmatrix} \cos 2\eta & \sin 2\eta \\ \sin 2\eta & -\cos 2\eta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\]
In other words, \( N_{10} \) is a linear mapping \( \mathbb{R}^2 \to \mathbb{R}^2 \). We denote its matrix by the symbol \( \text{ref}\eta \). This matrix represents reflection in the line through the origin whose direction vector is \((\cos \eta, \sin \eta)\):

\[
\text{ref}\eta = \begin{bmatrix}
    \cos 2\eta & \sin 2\eta \\
    \sin 2\eta & -\cos 2\eta
\end{bmatrix}.
\]

Consider now another line \( m \) through \( P \) and the associated line \( m_0 \). Then if \((\cos \phi, \sin \phi)\) is a direction vector of \( m \),

\[
\text{ref}\eta \text{ref}\phi = \begin{bmatrix}
    \cos 2\eta & \sin 2\eta \\
    \sin 2\eta & -\cos 2\eta
\end{bmatrix} \begin{bmatrix}
    \cos 2\phi & \sin 2\phi \\
    \sin 2\phi & -\cos 2\phi
\end{bmatrix} = \begin{bmatrix}
    \cos 2(\eta - \phi) - \sin 2(\eta - \phi) \\
    \sin 2(\eta - \phi) \cos 2(\eta - \phi)
\end{bmatrix}.
\]

Hence, the product of two reflections in intersecting axes is a rotation through twice the angle between the axes. There is a special symbol, \( \text{rot}\eta \) for a matrix of the form:

\[
\text{rot}\eta = \begin{bmatrix}
    \cos \eta & -\sin \eta \\
    \sin \eta & \cos \eta
\end{bmatrix}.
\]

**Definition 11**  An angle \( \phi \) between two directions \([v], [w] \) is defined as

\[
\phi = \arccos \frac{\langle v, w \rangle}{|v||w|}.
\]

Hence \( \phi \in [0, \pi] \).

Indeed, the absolute value of the fraction \( \frac{\langle v, w \rangle}{|v||w|} \) is not more than 1 by Cauchy-Schwarz inequality hence there exist a unique angle \( \phi \) in the interval \([0, \pi] \) such that \( \cos \phi \) equals this ratio.

## 2 Geometry on the sphere

### 2.1 Preliminaries from \( E^3 \)

We introduce the coordinate three-space \( \mathbb{R}^3 \), an inner product, and the concept of length of a vector. In particular, if \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \), then

\[
x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3),
\]

\[
(cx, cx_2, cx_3),
\]

\[
\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3,
\]

\[
|x| = \sqrt{\langle x, x \rangle}.
\]

Theorems 1–7 apply equally well in this setting. The theorem of Pythagoras has the same proof in \( E^3 \). The definition of \( v^\perp \) is however peculiar to \( E^3 \). Instead we have the cross product.
2.2 Determinants

We define determinants of $2 \times 2$ and $3 \times 3$ matrices.

\[
\begin{vmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]

\[
\begin{vmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{23}a_{32} - a_{11}a_{23}a_{32}
\]

To avoid memorizing these unwieldy expressions we suggest using the mnemonic devices described in Figure 2. The first formula in Example 7 is obtained from Figure 2a by multiplying the entries on the rightward arrow and subtracting the product of the entries on the leftward arrow. The second formula in Example 7 is obtained by recopying the first and second columns as shown in Figure 2b. The determinant is then computed by summing the products on the rightward arrows and subtracting the products on the leftward arrows.

We can see that

\[
    \det \begin{bmatrix}
        a_{11} & a_{12} & a_{13} \\
        a_{21} & a_{22} & a_{23} \\
        a_{31} & a_{32} & a_{33}
    \end{bmatrix}
    = a_{11}\det \begin{bmatrix}
        a_{22} & a_{23} \\
        a_{32} & a_{33}
    \end{bmatrix} - a_{12}\det \begin{bmatrix}
        a_{21} & a_{23} \\
        a_{31} & a_{33}
    \end{bmatrix} + a_{13}\det \begin{bmatrix}
        a_{21} & a_{22} \\
        a_{31} & a_{32}
    \end{bmatrix}.
\]

Properties of determinant.
1. If $A$ has a row of zeros or a column of zeros, then $\det(A) = 0$.
2. If $A$ has two proportional rows (or columns), then $\det(A) = 0$. 

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3.

The cross product

The problem of finding a vector perpendicular to two given vectors is solved as follows:

**Definition 12** Let $u$ and $v$ be vectors in $\mathbb{R}^3$. Then $u \times v$ is the unique vector $z$ such that, for all $x \in \mathbb{R}^3$,

$$<z, x> = \det(x, u, v) = x_1 \det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix} - x_2 \det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix} + x_3 \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}.$$ 

Hence $u \times v = (\det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}, -\det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}, \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}).$

**Theorem 15**

i. $u \times v$ is well defined.

ii. $<u \times v, u> = <u \times v, v> = 0.$
iii. \( u \times v = -v \times u \).

iv. \( <u \times v, w> = <u, v \times w> \).

v. \((u \times v) \times w = <u, w > v - <v, w > u \).

**Proof:** We first recall a result from linear algebra; namely, that every linear function from \( \mathbb{R}^3 \) to \( \mathbb{R} \) can be expressed in the form

\[ x \rightarrow \langle x, z \rangle \]

for some fixed vector \( z \) (Theorem 8D). As we know, the function

\[ x \rightarrow \det(x, u, v) \]

is linear for each fixed choice of \( u \) and \( v \). This proves (i). Identities (ii)–(iv) can be easily deduced from the properties of determinants. On the other hand, (v) (often called the vector triple product formula) is rather complicated, but detailed computation can be avoided by exploiting the linearity. First, observe that

\[ \varepsilon_1 \times \varepsilon_2 = \varepsilon_3, \quad \varepsilon_2 \times \varepsilon_3 = \varepsilon_1, \quad \text{and} \quad \varepsilon_3 \times \varepsilon_1 = \varepsilon_2. \]

Thus,

\[ (\varepsilon_1 \times \varepsilon_2) \times \varepsilon_3 = 0 = (\varepsilon_1, \varepsilon_3)\varepsilon_2 - (\varepsilon_2, \varepsilon_3)\varepsilon_1, \]

\[ (\varepsilon_2 \times \varepsilon_3) \times \varepsilon_3 = -\varepsilon_2 = (\varepsilon_2, \varepsilon_3)\varepsilon_3 - (\varepsilon_3, \varepsilon_3)\varepsilon_2, \]

\[ (\varepsilon_3 \times \varepsilon_1) \times \varepsilon_3 = \varepsilon_1 = (\varepsilon_3, \varepsilon_3)\varepsilon_1 - (\varepsilon_1, \varepsilon_3)\varepsilon_3. \]
By linearity we have
\[(u \times v) \times e_3 = \langle u, e_3 \rangle v - \langle v, e_3 \rangle u.\]

By symmetry the analogous identity is true when $e_3$ is replaced by $e_1$ or $e_2$. Finally, by linearity
\[(u \times v) \times w = \langle u, w \rangle v - \langle v, w \rangle u.\]

**Corollary.**

i. $u \times v = 0$ if and only if $u$ and $v$ are proportional.

ii. If $u \times v \neq 0$, then $\{u, v, u \times v\}$ is a basis for $\mathbb{R}^3$.

iii. $\langle u \times v, w \times z \rangle = \langle u, w \rangle \langle v, z \rangle - \langle v, w \rangle \langle u, z \rangle$.

iv. $|u \times v|^2 = |u|^2 |v|^2 - \langle u, v \rangle^2$.

This last statement is known as the Lagrange identity. Note that it yields another proof of the Cauchy–Schwarz inequality.

**Proof:** (i), (iii), and (iv) can be easily deduced from the results of the theorem. (See Exercise 1.) For (ii) we will show that the set of vectors in question is linearly independent. Then general results from linear algebra (Appendix D) can be applied.

Now if there exist numbers $\lambda, \mu, \nu$ with
\[\lambda u + \mu v + \nu(u \times v) = 0,\]
we can take inner product with $u \times v$ to obtain
\[\nu |u \times v|^2 = 0,\]
and, hence, $\nu = 0$. Further, taking cross products with $v$ and $u$, respectively, yields
\[\lambda(u \times v) = 0, \quad \mu(v \times u) = 0,\]
so that $\lambda = \mu = 0$.

**Orthonormal bases**

A triple $\{u, v, w\}$ of mutually orthogonal unit vectors is called an orthonormal triple.

**Theorem 2.** If $\{u, v, w\}$ is an orthonormal triple, then for all $x \in \mathbb{R}^3$,
\[x = \langle x, u \rangle u + \langle x, v \rangle v + \langle x, w \rangle w.\]
Theorem 3. If \( u \) is any unit vector, there exist vectors \( v \) and \( w \) so that \( \{u, v, w\} \) is an orthonormal basis.

Proof: Let \( \xi \) be any unit vector other than \( \pm u \). Then let \( v = u \times \xi \) divided by its length, and \( w = u \times v \). Noting that \( |u \times v|^2 = |u|^2|v|^2 - \langle u, v \rangle^2 = 1 \), we see that \( \{u, v, w\} \) is orthonormal. \( \square \)

Planes

A plane is a set \( \Pi \) of points of \( \mathbb{E}^3 \) with the following properties:

i. \( \Pi \) is not contained in any line.
ii. The line joining any two points of \( \Pi \) lies in \( \Pi \).
iii. Not every point of \( \mathbb{E}^3 \) is in \( \Pi \).

Theorem 4.

i. If \( v \) and \( w \) are not proportional, and \( P \) is any point, then \( P + [v, w] \) is a plane. We speak of the plane through \( P \) spanned by \( \{v, w\} \).
ii. If \( P, Q, \) and \( R \) are noncollinear points, there is a unique plane \( \Pi \) containing them. In this case we speak of the plane \( PQR \).
iii. If \( N \) is a unit vector and \( P \) is a point, then \( \langle X - P, N \rangle = 0 \) is a plane. We speak of the plane through \( P \) with unit normal \( N \). See Figure 4.2.

Notation: \( [v, w] = \{tv + sw|t, s \in \mathbb{R}\} \) is called the span of \( \{v, w\} \).

Proof:

i. Suppose that \( \alpha = P + [v, w] \) is a set as described in (i). We show that \( \alpha \) is a plane. First of all, let \( Q = P + v \) and \( R = P + w \). Then, because \( Q - P \) and \( R - P \) are not proportional, the points \( P, Q, \) and \( R \) are not collinear and \( \alpha \) is not contained in any line. Secondly, if \( X = P + v \times w \), we see that \( X \notin \alpha \) because \( \{v, w, v \times w\} \) is a linearly independent set. Thus, not every point of \( \mathbb{E}^3 \) is in \( \alpha \). Thirdly, let

\[
X = P + x_1v + x_2w, \quad Y = P + y_1v + y_2w
\]

be points of \( \alpha \), and let \( t \) be any real number. Then

\[
(1 - t)X + tY = (1 - t)P + tP + ((1 - t)x_1 + ty_1)v + ((1 - t)x_2 + ty_2)w = P + ((1 - t)x_1 + ty_1)v + ((1 - t)x_2 + ty_2)w.
\]

This exhibits a typical point of \( \overrightarrow{XY} \) as a member of \( \alpha \) and concludes the proof that \( \alpha \) is a plane.
2.3 Incidence geometry of the sphere

The sphere $S^2$ is determined by the condition

$$S^2 = \{ x \in E^2 \mid |x| = 1 \}.$$

**Definition 13** Let $\eta$ be a unit vector. Then

$$\ell = \{ x \in S^2 \mid \langle \eta, x \rangle = 0 \}$$

is called the line with pole $\eta$. We also call $\ell$ the polar line of $\eta$.

Two points $P$ and $Q$ are said to be *antipodal* if $P = -Q$.

**Theorem 16** If $\eta$ is a pole of $\ell$, so is its antipode $-\eta$.
If $P$ lies on $\ell$, so does its antipode $-P$.

**Theorem 17** Let $P$ and $Q$ be distinct points of $S^2$ that are not antipodal. Then

there is a unique line containing $P$ and $Q$, which we denote by $PQ$.

**Proof.** A pole of a line containing both points must be orthogonal to $P$ and $Q$. Because

$P$ and $Q$ are not antipodal, we may choose $\eta$ equal to $P \times Q/|P \times Q|$.

Now consider uniqueness. If $\eta$ is a pole of any line through $P$ and $Q$, we must have

$$\langle \eta, P \rangle = 0, \quad \langle \eta, Q \rangle = 0.$$

By the triple product formula

$$(P \times Q) \times \eta = \langle P, \eta \rangle \times Q - \langle Q, \eta \rangle \times P = 0.$$

Hence $\eta$ is a multiple of $P \times Q$. □

**Theorem 18** Let $\ell_\eta$ and $\ell_\xi$ be distinct lines of $S^2$. Then they have exactly two points of intersection

$$\pm (\eta \times \xi)/|\eta \times \xi|.$$
2.4 Distance and triangle inequality

The distance between two points $P$ and $Q$ of $S^2$ is defined by the equation

$$d(P, Q) = \arccos <P, Q>.$$  

Remind that $x = \arccos y$ if and only if $y = \cos x$ and $x \in [0, \pi]$. The graph of $\arccos$ function:

\[\text{Theorem 19} \quad \text{If } P, Q \text{ and } R \text{ are points of } S^2, \text{ then}
\]
1. $d(P, Q) \geq 0$.
2. $d(P, Q) = 0$ if and only if $P = Q$.
3. $d(P, Q) = d(Q, P)$.
4. $d(P, Q) + d(Q, R) \geq d(P, R)$ (the triangle inequality). If equality holds then $P, Q, R$ are collinear (belong to the same line).

Properties 1–3 follow from the properties of the arccos function and the Cauchy-Schwarz inequality. We will prove Property 4.

Let $r = d(P, Q)$, $p = d(Q, R)$, $q = d(P, R)$. We have to prove $r + p \geq q$. If $r \geq q$ or $p \geq q$ we are done. Hence we can suppose $q > r, p$. By the Cauchy-Schwarz inequality we have

$$< P \times R, Q \times R >^2 \leq |P \times R|^2 |Q \times R|^2.$$  

By Corollary to Theorem 15, $< P \times R, Q \times R > = < P, Q > < R, R > - < P, R > < Q, R > = \cos r - \cos q \cos p$. By the same corollary, $|P \times R|^2 = 1 - < P, R >^2 = 1 - \cos^2 q = \sin^2 q$ and $|Q \times R|^2 = 1 - < Q, R >^2 = 1 - \cos^2 p = \sin^2 p$. Hence,

$$(\cos r - \cos p \cos q)^2 \leq \sin^2 p \sin^2 q.$$  

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Notice that $\sin p, \sin q \geq 0$, because $p, q \in [0, \pi]$. The cos is a decreasing function on the interval $[0, \pi]$, hence $(\cos r - \cos p \cos q) \geq 0$, and we can write

$$(\cos r - \cos p \cos q) \leq \sin p \sin q$$

and

$$\cos r \leq \cos p \cos q + \sin p \sin q = \cos(q - p),$$

where $q - p > 0$. Because the cos function is decreasing on $[0, \pi]$ this implies $r \geq q - p$ and $r + p \geq q$.

The case $r = q - p$ implies that the Cauchy-Schwarz inequality is an equality. Thus, $P \times R$ and $Q \times R$ are proportional. Assuming that $P \times R \neq 0$ (otherwise $P, Q$ and $R$ are automatically collinear), we see that the pole of the line $\overline{PR}$ is proportional to $P \times R$ and hence to $Q \times R$. This shows that $Q$ lies on $\overline{PR}$. □

**Definition 14** The angle between $\ell_\eta$ and $\ell_\xi$ is defined as the angle between $\eta$ and $\xi$. The radian measure of an angle $\angle PQR$ is

$$\arccos \left( \frac{Q \times P}{|Q \times P|^2} \cdot \frac{Q \times R}{|Q \times R|^2} \right).$$

The lines are perpendicular if their poles are orthogonal.

**Theorem 20** Let $\ell$ and $m$ be distinct lines of $S^2$. Then there is a unique line $n$ such that $\ell \perp n$ and $m \perp n$. The intersection points of $\ell$ and $m$ are the poles of $n$.

**Theorem 21** Let $\ell$ be a line of $S^2$, and $P$ be a point. If $P$ is not a pole of $\ell$, there is a unique line $m$ through $P$ perpendicular to $\ell$.

---

**Motions of $S^2$**
Definition 15 For any line $\ell$ the reflection in $\ell$ is the mapping $-\iota$ given by

$$-\iota X = X - 2 < X, \eta > \eta,$$

where $\eta$ is a pole of $\ell$.

We can see that the reflection in $\ell_\eta$ is the restriction on $S^2$ of the reflection of $E^3$ with respect to the plane $\pi$ through the origin with a normal vector $\eta$.

Theorem 22 Let $<\eta, \eta > = 1$ and define $T : R^3 \rightarrow R^3$ by

$$T(X) = X - 2 < X, \eta > \eta.$$

Then

1. $T$ is linear.
2. $< T(X), T(Y) > = < X, Y >$ for all $X, Y \in R^3$.

Proof $< TX, TY > = < X - 2 < X, \eta > \eta, Y - 2 < Y, \eta > \eta >$

$= < X, Y > - 2 < X, \eta > < \eta, Y > - 2 < X, \eta > < Y, \eta > + 4 < X, \eta > < Y, \eta > < \eta, \eta > = < X, Y >$.

Theorem 23

1. $d(-\iota X, -\iota Y) = d(X, Y)$ for all points $X, Y$ in $S^2$.
2. $-\iota -\iota (X) = X$ for all points $X$ in $S^2$.
3. $-\iota : S^2 \rightarrow S^2$ is a bijection.
4. $-\iota (X) = X$ if and only if $X \in \ell$.

Definition 16 If $\alpha$ and $\beta$ are lines passing through a point $P$, then the isometry $-\alpha -\beta$ is called a rotation about $P$. The special case $\alpha = \beta$ determines the identity, a trivial rotation. Every rotation in $S^2$ is also a translation.

Parametric representation of lines

Suppose that $\ell_\eta$ is a line with a pole $\eta$. Let $P$ and $Q$ be chosen so that $\{\eta, P, Q\}$ is orthonormal. Then set

$$\alpha(t) = (\cos t)P + (\sin t)Q.$$

Theorem 24

1. $\ell = \{\alpha(t)|t \in R\}$
2. Each point of $\ell$ occurs exactly once as a value of $\alpha(t)$ while $t$ ranges through the interval $[0, 2\pi)$.
3. $d(\alpha(t_1), \alpha(t_2)) = |t_1 - t_2|$ if $0 \leq |t_1 - t_2| \leq \pi$. 

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3 The Hyperbolic Plane

Introduction

The projective plane provides one alternative to Euclidean geometry. A second alternative is explored in this chapter.

The three geometries are contrasted in the following example: Take a segment $P_1P_2$ as shown in Figure 7.1. Erect equal segments $P_1Q_1$ and $P_2Q_2$ perpendicular to $P_1P_2$.

In $\mathbb{E}^2$ the segment $Q_1Q_2$ will have length equal to that of $P_1P_2$. However, in $\mathbb{P}^2$, the length of $Q_1Q_2$ will be less than that of $P_1P_2$. In $\mathbb{H}^2$ we shall see that $Q_1Q_2$ will be longer than $P_1P_2$.

This construction is also related to the question of parallelism. Let $\ell_0$ be a line, and let $P$ be a point not on $\ell_0$. Drop a perpendicular $PP_0$ from $P$ to $\ell_0$, and let $\ell$ be the line through $P$ perpendicular to $PP_0$. (See Figure 7.2.)

In $\mathbb{E}^2$, $\ell$ will be parallel to $\ell_0$. In $\mathbb{P}^2$, $\ell$ will meet $\ell_0$. In $\mathbb{H}^2$ it will turn out that $\ell$ does not meet $\ell_0$.

We will now proceed to construct the geometry $\mathbb{H}^2$. It will again consist of "points" and "lines" with a "distance" function defined for each pair of points. As in the case of $\mathbb{E}^2$ and $\mathbb{P}^2$, we find that isometries of $\mathbb{H}^2$ are generated by reflections and satisfy the three reflections theorems.

Algebraic preliminaries

Our model of spherical geometry was a certain subset of $\mathbb{R}^3$, and the usual inner product of $\mathbb{R}^3$ played an important role. Our model of hyperbolic geometry will also be a subset of $\mathbb{R}^3$. However, the bilinear form on which hyperbolic geometry is based is defined by

$$b(x, y) = x_1y_1 + x_2y_2 - x_3y_3$$

(see also Chapter 6). A function of this type is used in Einstein's special theory of relativity. (See Frankel [15] or Taylor–Wheeler [29].) This explains some of the terms used in discussing its properties.
Properties of cross product

$\mathbf{u} \times \mathbf{v}$ is a unique vector such that for any $\mathbf{x}$ in $\mathbb{R}^3$

$$b(\mathbf{u} \times \mathbf{v}, \mathbf{x}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\mathbf{u} \times \mathbf{v} = \left( \begin{array}{c} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{array} \right)$$

1. $\mathbf{u} \times \mathbf{0} = \mathbf{0}$
2. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
3. $b(\mathbf{u} \times \mathbf{v}, \mathbf{u}) = 0$, $b(\mathbf{u} \times \mathbf{v}, \mathbf{v}) = 0$
   
   In $(*)$ take $\mathbf{w} = \mathbf{x}$
4. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = -b(\mathbf{u}, \mathbf{w}) \mathbf{v} + b(\mathbf{v}, \mathbf{w}) \mathbf{u}$

Verify 4 for vectors

$\mathbf{e}_1 = (1, 0, 0)$
$\mathbf{e}_2 = (0, 1, 0)$
$\mathbf{e}_3 = (0, 0, 1)$

5. $b(\mathbf{u} \times \mathbf{v}, \mathbf{w} \times \mathbf{z}) = -b(\mathbf{u}, \mathbf{w}) b(\mathbf{v}, \mathbf{z}) + b(\mathbf{v}, \mathbf{w}) b(\mathbf{u}, \mathbf{z})$

For comparison in $\mathbb{R}^2$

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \times \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle \langle \mathbf{v}, \mathbf{z} \rangle - \langle \mathbf{u}, \mathbf{z} \rangle \langle \mathbf{v}, \mathbf{w} \rangle$$
Definition. A nonzero vector $v \in \mathbb{R}^3$ is said to be

i. spacelike if $b(v, v) > 0$. If $b(v, v) = 1$, it is a unit spacelike vector. An example is $e_1$.

ii. timelike if $b(v, v) < 0$. If $b(v, v) = -1$, it is a unit timelike vector. An example is $e_3$.

iii. lightlike if $b(v, v) = 0$. An example is $e_1 - e_3$.

We use the notation $|v|$ for the “length” of a vector $v$ (i.e., $|v| = |b(v, v)|^{1/2}$). Unit vectors satisfy $|v| = 1$.

In this chapter we use the term “orthonormal” to mean orthonormal with respect to $b$. Note that $\{e_1, e_2, e_3\}$ is orthonormal.

Theorem 1.

i. Every orthonormal set of three vectors is a basis for $\mathbb{R}^3$.

ii. Every orthonormal basis has two spacelike vectors and one timelike vector.

iii. For every orthonormal pair $\{u, v\}$ of vectors, $\{u, v, u \times v\}$ is an orthonormal basis. (The cross product is taken with respect to $b$.)

iv. For every unit spacelike or unit timelike vector $v$, there is an orthonormal basis containing $v$.

Proof:

i. We need only show that an orthonormal set is linearly independent. If an equation of the form

$$0 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$$

holds, where $\{e_1, e_2, e_3\}$ is orthonormal, then for each $i$,

$$0 = b(0, e_i) = \lambda_i b(e_i, e_i)$$

implies that $\lambda_i = 0$.

ii. First note that all three vectors cannot be spacelike. In fact, if all $b(e_i, e_i)$ are equal and

$$x = \sum_{i=1}^{3} x_i e_i,$$

we have

$$b(x, x) = \sum_{i=1}^{3} x_i^2 b(e_i, e_i).$$

This would imply that all vectors are spacelike. Similarly, if all the $e_i$ were timelike, every vector in $\mathbb{R}^3$ would be timelike. We conclude that any orthonormal basis has at least one spacelike vector and one timelike vector.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis. Suppose that $e_1$ is spacelike.
and \( e_3 \) is timelike. Then \((e_1 \times e_3) \times e_2 = 0\), so that \(e_2\) is a multiple of \(e_1 \times e_3\). Further,
\[
b(e_1 \times e_3, e_1 \times e_3) = -b(e_1, e_1) b(e_3, e_3) = 1,
\]
so that \(e_1 \times e_3\) (and hence \(e_2\)) is spacelike.

iii. We note that
\[
b(u \times v, u \times v) = -b(u, u) b(v, v) = \pm 1,
\]
and, hence, \(\{u, v, u \times v\}\) is orthonormal.

iv. Suppose that \(v\) is spacelike. Let \(w\) be any unit timelike vector (e.g., \(e_3 = (0, 0, 1)\)). If \(b(v, w) = 0\), we can use \(\{v, w, v \times w\}\) as our basis. If not, choose \(\tilde{u} = v + \lambda w\), where \(\lambda = -1/b(v, w)\). Then
\[
b(\tilde{u}, \tilde{u}) = 1 + 2\lambda b(v, w) - \lambda^2 = 1 - 2 - \lambda^2 = -(1 + \lambda^2) .
\]

If we set
\[
u = \frac{v + \lambda w}{\sqrt{1 + \lambda^2}},
\]
then \(\{u, v, u \times v\}\) is an orthonormal basis.

Suppose now that \(v\) is timelike. A similar construction, using a unit spacelike vector \(w\), leads to an orthonormal basis \(\{u, v, u \times v\}\), where \(u = (v + \lambda w)/\sqrt{1 + \lambda^2}\) and \(\lambda = 1/b(v, w)\).}

\[\square\]

**Theorem 2.**

i. For any \(x \in \mathbb{R}^3\),
\[
x = \sum_{i=1}^{3} b(x, e_i) b(e_i, e_i) e_i .
\]

if \(\{e_1, e_2, e_3\}\) is an orthonormal basis.

ii. Let \(v\) be a timelike vector. Suppose that \(w \times v \neq 0\) and \(b(v, w) = 0\). Then \(w\) is spacelike.

The Cauchy–Schwarz inequality played an important role in \(\mathbb{E}^3\) and \(\mathbb{S}^3\). Here is the hyperbolic version.

**Theorem 3.** Let \(\xi\) and \(\eta\) be spacelike vectors in \(\mathbb{R}^3\) such that \(\xi \times \eta\) is timelike. Then
\[
b(\xi, \eta)^2 < b(\xi, \xi) b(\eta, \eta)
\]

**Proof:** Let \(P\) be a unit timelike vector in the direction \([\xi \times \eta]\). As in the proof of Theorem 1.4, we consider the function
\[
f(t) = b(\xi + t \eta, \xi + t \eta).
\]
Because $b(\xi + t\eta, P) = 0$ for all real values of $t$ and $P \times (\xi + t\eta) \neq 0$, Theorem 2 applies, and $\xi + t\eta$ is spacelike. In other words, $f(t) > 0$ for all $t$ and

$$b(\xi, \eta)^2 < b(\xi, \xi)b(\eta, \eta).$$

**Remark:** If we weaken the hypothesis to $b(\xi \times \eta, \xi \times \eta) \leq 0$, the conclusion becomes

$$b(\xi, \eta)^2 \leq b(\xi, \xi)b(\eta, \eta).$$

However, equality can occur even if $\xi$ and $\eta$ are not proportional. (See Exercise 2.)

There is a similar result for timelike vectors.

**Theorem 4.** Let $v$ and $w$ be timelike vectors. Then

$$b(v, w)^2 \geq b(v, v)b(w, w).$$

(7.3)

**Proof:** By Theorem 2, $v \times w$ is spacelike or zero. Thus

$$b(v \times w, v \times w) \geq 0.$$  

In other words,

$$b(v, v)b(w, w) - b(v, w)^2 \leq 0$$

with equality holding if and only if $v$ and $w$ are proportional.

**Corollary.** If $v$ and $w$ are unit timelike vectors, then $|b(v, w)| \geq 1$. The "inner product" $b(v, w)$ is positive if and only if $b(v, e_3)$ and $b(w, e_3)$ have opposite signs.

**Proof:** The first statement is immediate from the theorem. To prove the second, we introduce the following notation. Let $v = (p_1, p_2, r)$ and $w = (q_1, q_2, s)$. Consider $p = (p_1, p_2)$ and $q = (q_1, q_2)$ as vectors in $\mathbb{R}^2$. Then

$$b(v, w) = \langle p, q \rangle - rs.$$

Because $(|p| + |q|)^2 \geq 0$ with equality if and only if $p = q = 0$, we have

$$|p|^2 + |q|^2 \geq -2|p||q|.$$  

Adding $1 + |p|^2$ to each side yields

$$1 + |p|^2 (1 + |q|^2) \geq (|p||q| - 1)^2.$$  

But $|p|^2 = r^2 = -1$ and $|q|^2 = s^2 = -1$, so that
\[(|p||q| - 1)^2 \leq r^2 s^2.\]  
\hfill (7.4)

Suppose now that \(r\) and \(s\) are both positive but \(b(v, w)\) is also positive. Then \(\langle p, q \rangle \geq 1 + rs\); that is, \(\langle p, q \rangle - 1 \geq rs\). By the Cauchy–Schwarz inequality for \(\mathbb{R}^2\), we get

\[|p||q| - 1 \geq rs,
\]
which is incompatible with (7.4). We conclude that \(b(v, w)\) must be negative when \(r\) and \(s\) are positive. The conclusion now follows from the linearity of the function \(b\).

\[\Box\]

**Incidence geometry of \(\mathbb{H}^2\)**

The hyperbolic plane \(\mathbb{H}^2\) is defined as follows:

\[\mathbb{H}^2 = \{x \in \mathbb{R}^3 | x_3 > 0 \text{ and } b(x, x) = -1\}.
\]

Thus, as a set, \(\mathbb{H}^2\) is just the upper half of a hyperboloid of two sheets.

**Definition.** Let \(\xi\) be a unit spacelike vector. Then

\[\xi = \{x \in \mathbb{H}^2 | b(\xi, x) = 0\}
\]

is called the line with unit normal (or pole) \(\xi\).

**Remark:** Like the situation in spherical geometry, a line of \(\mathbb{H}^2\) is the intersection with \(\mathbb{H}^2\) of a plane through the origin of \(\mathbb{R}^3\). Not all planes through the origin meet \(\mathbb{H}^2\). However, if \(\xi\) is timelike, it can be completed to a basis orthonormal with respect to \(b\) (Theorem 1). In particular, there are points \(x \in \mathbb{H}^2\) such that \(b(\xi, x) = 0\). We will now proceed to a detailed study of lines in hyperbolic geometry.

**Theorem 5.** Let \(P\) and \(Q\) be distinct points of \(\mathbb{H}^2\). Then there is a unique line containing \(P\) and \(Q\), which we denote by \(\overline{PQ}\).

**Proof:** Apply Theorem 2(ii) with \(v = P\) and \(w = P \times Q\). The triple product formula shows that \(P \times (P \times Q) \neq 0\) and hence, that \(P \times Q\) is spacelike. Let \(\xi\) be a unit vector in the direction \([P \times Q]\). Then the line whose unit normal is \(\xi\) must pass through \(P\) and \(Q\). This is the only line through \(P\) and \(Q\) because the unit normal to any such line must be orthogonal to \(P\) and \(Q\) (with respect to \(b\)) and hence, must be a multiple of \(P \times Q\). \(\Box\)

Just as in spherical geometry, the cross product is used to find the point of intersection of a pair of lines. However, if \(\xi\) and \(\eta\) are spacelike unit vectors, \(\xi \times \eta\) need not be timelike, and therefore the lines may not
intersect in $H^2$. In fact, all three possibilities for $\xi \times \eta$ can occur. This is what makes $H^2$ a richer incidence geometry than any we have studied previously.

**Definition.** Let $\ell$ and $m$ be two lines with respective unit normals $\xi$ and $\eta$. We say that $\ell$ and $m$ are

i. intersecting lines if $\xi \times \eta$ is timelike,

ii. parallel lines if $\xi \times \eta$ is lightlike,

iii. ultraparallel lines if $\xi \times \eta$ is spacelike.

**Theorem 6.** Intersecting lines have exactly one point in common. This point is the unique point of $H^2$ that is a multiple of $\xi \times \eta$.

**Proof:** Clearly, the point in question lies on both lines. If $P$ is any other point that lies on both lines, then

$$ P \times (\xi \times \eta) = -b(P, \eta)\xi + b(P, \xi)\eta = 0, $$

so that $P$ is a multiple of $\xi \times \eta$ as required. $\square$

**Remark:** Neither parallel nor ultraparallel lines intersect.

**Perpendicular lines**

**Definition.** Two lines with unit normals $\xi$ and $\eta$ are said to be perpendicular if $b(\xi, \eta) = 0$.

**Theorem 7.** If two lines are ultraparallel, there is a unique line $\gamma$ that is perpendicular to both of them. Conversely, if two lines have a common perpendicular, they must be ultraparallel.

**Proof:** Let $\xi$ and $\eta$ be unit normals of two ultraparallel lines. Let $\xi$ be the unit (spacelike) vector that is a multiple of $\xi \times \eta$. Then $b(\xi, \xi) = b(\eta, \eta) = 0$, so the line with unit normal $\xi$ is a common perpendicular to the two lines.

Conversely, if the two lines have a common perpendicular, its unit normal $\xi$ is a spacelike vector satisfying $\xi \times (\xi \times \eta) = 0$ and, thus, is a multiple of $\xi \times \eta$. This means that $\xi \times \eta$ is spacelike, and the lines are ultraparallel. $\square$

**Theorem 8.**

i. If $\ell$ and $m$ are perpendicular lines of $H^2$, then $\ell$ intersects $m$.

ii. Let $X$ be a point of $H^2$ and $\ell$ a line of $H^2$. Then there is a unique line through $X$ perpendicular to $\ell$.  

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Proof:

i. Let $\xi$ and $\eta$ be unit normals to $\ell$ and $m$, respectively. Then $(\xi, \eta, \xi \times \eta)$ is an orthonormal basis by Theorem 1. Hence, $\xi \times \eta$ is timelike.

ii. Let $\xi$ be a unit normal to $\ell$. Let $\eta$ be a unit vector proportional to $\xi \times X$. This is possible because $\xi \times X$, being a nonzero vector orthogonal to $X$, must be spacelike.

The line $m$ whose unit normal is $\eta$ clearly passes through $X$ but is perpendicular to $\ell$. There is only one line with this property, because a unit normal to such a line must be orthogonal to $\xi$ and $X$ and, therefore, a multiple of their cross product.

\[ \square \]

Definition. The point $F$ where $m$ intersects $\ell$ is called the foot of the perpendicular from $X$ to $\ell$ (provided $X$ is not on $\ell$).

Remark: In the next section we define distance between two points of $H^2$. As in $E^2$ we can use this to define

\[ d(X, \ell) = d(X, F), \]

where $F$ is the foot of the perpendicular from $X$ to $\ell$.

Pencils

Definition. Let $\ell$ and $m$ be a pair of distinct lines with respective unit normals $\xi$ and $\eta$. Then the set $\mathcal{P}$ of lines whose unit normals $\xi$ are orthogonal to $\xi \times \eta$ is called a pencil of lines. $\mathcal{P}$ is called a pencil of intersecting lines, a pencil of parallels, or a pencil of ultraparallels according to whether $\xi \times \eta$ is timelike, lightlike, or spacelike.

Remark: At the moment this definition may look somewhat strange. Clearly, if $\xi \times \eta$ is timelike, then lines with unit normal $\xi$ will be the lines passing through the point of intersection, as expected. If $\xi \times \eta$ is spacelike, the pencil will consist of all lines perpendicular to a certain line. However, it is not yet evident what the pencil looks like when $\xi \times \eta$ is lightlike. When we look at $H^2$ as a subset of $P^2$, we will get a more concrete interpretation for $\xi \times \eta$ and the associated pencils.

Remark:

i. The set of all lines of $H^2$ perpendicular to a certain line of $H^2$ is a pencil of ultraparallels.

ii. Any two lines of $H^2$ determine a unique pencil.
Distance in $H^2$

We parametrize lines of $H^2$ much as we did in $S^2$. Let $e_3$ be an arbitrary point of $H^2$. Let $e_1$ and $e_2$ be vectors of $R^3$ such that $(e_1, e_2, e_3)$ is an orthonormal basis.

A typical point on the plane through the origin spanned by $(e_1, e_3)$ is $\lambda e_3 + \mu e_1$. This point is on $H^2$ if and only if $\lambda > 0$ and

$$b(\lambda e_3 + \mu e_1, \lambda e_3 + \mu e_1) = -1;$$

that is,

$$\lambda^2 = 1 + \mu^2.$$

Using Theorem 3F, we may call $\lambda = \cosh t$ and $\mu = \sinh t$. Then as $t$ ranges through all real numbers, $(\cosh t)e_3 + (\sinh t)e_1$ runs through all the points of the line. We define distance in such a way that $t$ measures distance along the line.

Definition. For $x, y$ in $H^2$ define

$$d(x, y) = \cosh^{-1}(-b(x, y)).$$

Remark: This definition is possible because $b(x, y) \leq -1$, as was shown in the corollary to Theorem 4.

Theorem 9. Let $\alpha(t) = (\cosh t)e_3 + (\sinh t)e_1$. Then

$$d(\alpha(t_1), \alpha(t_2)) = |t_1 - t_2|.$$

Proof: Exercise 6. $\square$

Definition. If $t_1 < t < t_2$, then $\alpha(t)$ is between $\alpha(t_1)$ and $\alpha(t_2)$.

Now that we have defined distance between two points in the hyperbolic plane, it is necessary to determine which of the properties of Euclidean distance carry over to the hyperbolic case. The following is immediate from the definition.

Theorem 10. If $P$ and $Q$ are points of $H^2$, then

1. $d(P, Q) \geq 0$.
2. $d(P, Q) = 0$ if and only if $P = Q$.
3. $d(P, Q) = d(Q, P)$.

We now address ourselves to the triangle inequality. Our proof of the triangle inequality in the spherical case relied on the cross product operation of $E^3$. Here we use the hyperbolic cross product.
Theorem 11 (Triangle Inequality). Let $P$, $Q$, and $R$ be points of $\mathbb{H}^{2}$. Then $d(P, Q) + d(P, R) \equiv d(Q, R)$ with equality if and only if $P$, $Q$, $R$ are collinear and $P$ lies between $Q$ and $R$.

Proof: If $P$, $Q$, and $R$ are not collinear, then $P \times Q$ and $R \times Q$ will not be proportional. Thus, $(P \times Q) \times (R \times Q) = b(P \times Q, R \times Q)$ is timelike. We may apply the hyperbolic Cauchy–Schwarz inequality (Theorem 3) to get

$$b(P \times Q, R \times Q)^{2} \leq b(P \times Q, P \times Q) b(R \times Q, R \times Q). \quad (7.5)$$

But

$$b(P \times Q, R \times Q) = b((P \times Q) \times R, Q)$$
$$= -b(P, R)b(Q, Q) + b(Q, R)b(P, Q)$$
$$= b(P, R) + b(Q, R)b(P, Q)$$

because $b(Q, Q) = -1$. Let $d(Q, R) = p$, $d(P, R) = q$, $d(P, Q) = r$.

Then

$$\cosh p = -b(Q, R), \quad \cosh r = -b(Q, P), \quad \cosh q = -b(P, R).$$

Thus,

$$b(P \times Q, R \times Q) = \cosh p \cosh r - \cosh q.$$ 

Also

$$b(P \times Q, P \times Q) = -b(P, P)b(Q, Q) + b(R, Q)^{2}$$
$$= -1 + \cosh^{2} r = \sinh^{2} r.$$ 

and, similarly, $b(R \times Q, R \times Q) = \sinh^{2} p$. Equation (7.5) now becomes

$$(\cosh p \cosh r - \cosh q)^{2} \leq \sinh^{2} r \sinh^{2} p.$$ 

Hence,

$$\cosh p \cosh r - \cosh q \leq \sinh r \sinh p,$$

$$\cosh q \geq \cosh (p - r),$$

$$q \geq p - r,$$

$$p \leq q + r.$$ 

This is what we wanted to prove. Now if $p = q + r$, we have equality in (7.5). From Theorem 3 this means that $(P \times Q) \times (R \times Q)$ is not timelike, and, hence, $b(P \times Q, R) = 0$; that is, $R$ lies on $\overrightarrow{PQ}$. The fact that $P$ lies between $Q$ and $R$ can be deduced easily from Theorem 9 and is left as an exercise (Exercise 7).

Remark: The properties of the hyperbolic functions used in this section may be found in Appendix F.
Exercise. Let $\ell_\eta$ be defined by $\eta = (1, 0, 0)$, $P = (1, 0, \sqrt{2})$. Find all lines through $P$ which
1) intersect $\ell_\eta$,
2) parallel $\ell_\eta$,
3) ultraparallel to $\ell_\eta$.

Solution. We have to find vectors $\psi$ such that $P \in \ell_\psi$, and
1) $b(\psi \times \tau, \psi \times \tau) < 0$,
2) $b(\psi \times \tau, \psi \times \tau) = 0$,
3) $b(\psi \times \tau, \psi \times \tau) > 0$.

The third coordinate of $\psi$ is either 0 or nonzero. If it is zero, we will look for $\psi$ in the form $\psi = (a, c, 0)$. If it is nonzero, we can take $\psi$ in the form $\psi = (a, c, 1)$.

Consider first the case $\psi = (a, c, 1)$. $b(P, \psi) = a - \sqrt{2} = 0$, hence $a = \sqrt{2}$, and $\psi = (\sqrt{2}, c, 1)$.

- $\psi \times \eta = (0, 1, c)$, and $b(\psi \times \eta, \psi \times \eta) = 1 - c^2$.
- If $|c| > 1$ then $\ell_\eta$ intersects $\ell_\psi$.
- If $c = \pm 1$ then $\ell_\eta$ is parallel to $\ell_\psi$.
- If $|c| < 1$ then $\ell_\eta$ is ultraparallel to $\ell_\psi$.

The second case, $\psi = (a, c, 0)$ gives $b(P, \psi) = a = 0$, and $\psi \times \eta = (0, 0, c)$ timelike, hence for $\psi(0, 1, 0)$ the lines $\ell_\psi$ and $\ell_\eta$ are intersecting.

**Definition 17** The angle between two lines $\ell_\eta$ and $\ell_\psi$ is defined as the angle between their poles. If we have a triangle $PQR$ then

$$\cos \angle PQR = b\left(\frac{Q \times P}{|Q \times P|}, \frac{Q \times R}{|Q \times R|}\right).$$
Parallel and ultraparallel lines in $H^2$

In $\mathbb{R}^3$, $b(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3$

Divide $\mathbb{R}^3$ into timelike, spacelike and lightlike vectors.

- $b(x, x) < 0$ timelike
- $b(x, x) > 0$ spacelike
- $b(x, x) = 0$ lightlike

Draw the surface defined by the equation $b(x, x) = 0$.

$x_1^2 + x_2^2 - x_3^2 = 0$

Take the intersection of this surface with the plane $x_1 = 0$.

The intersection consists of two lines:

- $x_2 = x_3$, $x_2 = -x_3$

![Diagram](image_url)
In $\mathbb{R}^3$ with $L(x,y) = x_1y_1 + x_2y_2 - x_3y_3$, the equation of the cone $L(x,y) = 0$.

The lightlike points (vectors) lie on the cone, timelike points lie inside and spacelike points lie outside the cone.

$$H^2 = \{ x \in \mathbb{R}^3 | L(x,y) = -1, \quad x_3 > 0 \}$$

$$H^2 \cap \{ x \in \mathbb{R}^3 | x_1 = 0 \}$$

has the equation

$$x_2^2 - x_3^2 = 1$$

$D^2$ will denote the interior of this unit disc.

$D^2$ denotes the interior of $H^2$ with the unit normal vector $\mathbf{n}$ (with respect to the bilinear form $L$).
Let $\mathbf{II}_3$ denote the plane in $\mathbb{R}^3$ with normal vector $\mathbf{z}$. If $\mathbf{z}$ is spacelike,

$$\mathbf{l}_3 = \{ \mathbf{x} \in \mathbb{H}^2 \mid b(x, \mathbf{z}) = 0 \mathbf{z} \}$$

$$\mathbf{II}_3 = \{ \mathbf{x} \in \mathbb{R}^3 \mid b(x, \mathbf{z}) = 0 \mathbf{z} \}$$

$$\mathbf{l}_3 = \mathbf{II}_3 \cap \mathbb{H}^2$$

$$\mathbf{II}_3 \cap \mathbb{H}^2$$ is the line in $\mathbb{R}^3$ in the direction $\mathbf{z} \times \mathbf{z}$.

Fact: $\mathbf{II}_3$ intersects $\mathbb{H}^2$ (determine the line $\mathbf{l}_3$) iff $\mathbf{II}_3$ intersects $\mathbb{D}^2$. The line of the intersection is an open chord.

1. Intersecting lines (chords intersect inside circle)

2. Parallel lines in $\mathbb{H}^2$ (chords intersect at boundary)

3. Ultraparallel lines in $\mathbb{H}^2$ (chords intersect outside circle)
Theorem. Let $ACD$ be a triangle in $H^2$.

$$\text{Then } \cos A = \frac{\cosh d \cosh c - \cosh a}{\sinh d \sinh c}$$

Proof. $\cos A = \frac{b(A \times D, A \times C)}{\sqrt{b(A \times C, A \times A) b(A \times D, A \times D)}}$

Because $A, C$ are unit timelines:

$$b(A \times D, A \times C) = b(D, C), b(A, C) b(A, D) = -\cosh a + \cosh d \cosh c$$

$$b(A \times C, A \times A) = -1, b(D, C) = -1 + \cosh^2 d = \sinh^2 d$$

$$\cos A = \frac{\cosh d \cosh c - \cosh a}{\sinh d \sinh c}.$$  Notice that $\cosh c, \cosh d, \cosh a < 1$

Theorem. The angle sum of a triangle in $H^2$ is less than $\pi$.

Proof. We will prove the theorem for a triangle with $\angle D = \frac{\pi}{2}$. Then $\cosh a \cosh c = \cosh d$.

We will show that $A + C < \frac{\pi}{2}$ or $\cos(A + C) > 0$.

$$\cos (A + C) = \cos A \cos C - \sin A \sin C,$$

hence it is enough to prove that $\cos A \cos C > \sin A \sin C$, or $\cosh A \cosh C > 1$.

$$\cos A = \frac{\cosh a (\cosh c - 1)}{\sinh d \sinh c} = \frac{\cosh a \sinh c}{\sinh d}.$$  The inequality $\cos^2 A + \cos^2 C > 1$ which we want to prove is equivalent to

$$\cosh^2 a (1 - \cosh^2 c) + \cosh^2 c (1 - \cosh^2 a) > \sinh^2 d = \cosh^2 d - 1 = \cosh^2 a \sinh^2 c - 1$$

This is equivalent to

$$\cosh^2 a \cosh^2 c - \cosh^2 a - \cosh^2 c > -1$$

$$\cosh^2 a (\cosh^2 c - 1) > \cosh^2 c - 1$$

and $\cosh^2 c > 1$ which is true.
The Poincaré Models

A disk model due to Henri Poincaré (1854–1912)* also represents points of the hyperbolic plane by the points interior to a Euclidean circle γ, but lines are represented differently. First, all open chords that pass through the center O of γ (i.e., all open diameters l of γ) represent lines. The other lines are represented by open arcs of circles orthogonal to γ. More precisely, let δ be a circle orthogonal to γ (at each point of intersection of γ and

![Figure 7.6](image)

δ the radii of γ and δ through that point are perpendicular). Then intersecting δ with the interior of γ gives an open arc m, which by definition represents a hyperbolic line in the Poincaré model. So we will call Poincaré line, or “P-line," either an open diameter l of γ or an open circular arc m orthogonal to γ (see Figure 7.6).

A point interior to γ “lies on” a Poincaré line if it lies on it in the Euclidean sense. Similarly, “between” has its usual Euclidean interpretation (for A, B, and C on an open arc coming from an orthogonal circle δ with center P, B is between A and C if P B is between P A and P C).

![Figure 7.7](image)

* Poincaré was the cousin of the president of France. Like Gauss, Poincaré made profound discoveries in many branches of mathematics and physics; he even started a new branch of mathematics, algebraic topology. He used his models of hyperbolic geometry to discover new theorems about automorphic functions of a complex variable. Poincaré is also important as a philosopher of science (see Chapter 8).
The interpretation of congruence for segments in the Poincaré model is complicated, being based on a way of measuring length that is different from the usual Euclidean way, just as in the Klein model (see p. 199). Congruence for angles has the usual Euclidean meaning, however, and this is the main advantage of the Poincaré model over the Klein model.* Specifically, if two directed circular arcs intersect at a point A, the number of degrees in the angle they make is by definition the number of degrees in the angle between their tangent rays at A (see Figure 7.7). Or, if one directed circular arc intersects an ordinary ray at A, the number of degrees in the angle they make is by definition the number of degrees in the angle between the tangent ray and the ordinary ray at A.

![Figure 7.8](image)

Having interpreted all the undefined terms of hyperbolic geometry in the Poincaré model, we get (by substitution) interpretations of all the defined terms. For example, two Poincaré lines are parallel if and only if they have no point in common. Then all the axioms of hyperbolic geometry get translated into statements in Euclidean geometry, and it will be shown in the section after next (Inversion in Circles) that these interpretations are theorems in Euclidean geometry. Hence, the Poincaré model furnishes another proof that if Euclidean geometry is consistent, so is hyperbolic geometry.

The limiting parallel rays in the Poincaré model are illustrated in Figure 7.9.

![Figure 7.9](image)

* Technically, we say that the Poincaré model is conformal—it represents angles accurately, while the Klein model is not. Another example of a conformal model is Mercator's map of the surface of the earth.
Here we have chosen \( l \) to be an open diameter \( \overline{AB} \); the rays are circular arcs that meet \( \overline{AB} \) at \( A \) and \( B \) and are tangent to this line at those points. You can see how these rays approach \( l \) asymptotically as you move out toward the ideal points represented by \( A \) and \( B \).

Figure 7.10 illustrates two parallel Poincaré lines with a common perpendicular. The diagram shows how \( m \) diverges from \( l \) on either side of the common perpendicular \( PO \).

To define congruence of segments in the disk model, we introduce the following definition of length:

**DEFINITION.** Let \( A \) and \( B \) be points inside \( \gamma \), and let \( P \) and \( Q \) be the ends of the \( P \)-line through \( A \) and \( B \). We define the *cross-ratio* \( (AB,PQ) \) by

\[
(AB,PQ) = \frac{(AP)(BQ)}{(BP)(AQ)}
\]

(where, e.g., \( AP \) is the Euclidean length of the Euclidean segment \( AP \)). We then define the Poincaré length \( d(AB) \) by

\[
d(AB) = |\log(AB,PQ)|.
\]
Notice first of all that this length does not depend on the order in which we write P and Q. For if \((AB, PQ) = x\), then \((AB, QP) = 1/x\), and 
\[|\log(1/x)| = |\log x| = |\log x|\]. Moreover, since \((AB, PQ) = (BA, QP)\), we see that \(d(AB)\) also does not depend on the order in which we write A and B.

We may therefore interpret the Poincaré segments AB and CD to be Poincaré-congruent if \(d(AB) = d(CD)\). With this interpretation, Axiom C-2 is immediately verified.

Suppose we fix the point A on the P-line from P to Q and let point B move continuously from A to P, where Q * A * B * P, as in Figure 7.26. The cross-ratio \((AB, PQ)\) will increase continuously from one to \(\infty\), since \((AP)/(AQ)\) is constant, BP approaches zero, and BQ approaches PQ. If we fix B and let A move continuously from B to Q, we get the same result. It follows immediately that for any Poincaré ray \(\overline{CD}\), there is a unique point E on \(\overline{CD}\) such that \(d(CE) = d(AB)\), where A and B are given in advance. This verifies Axiom C-1.

We next verify Axiom C-3. This will follow immediately from the additivity of the Poincaré length, which asserts that if A * B * C in the sense of the disk model, then \(d(AC) + d(CB) = d(AB)\). To prove this additivity, label the ends so that Q * A * B * P. Then the cross-ratios \((AB, PQ)\), \((AC, PQ)\), and \((CB, PQ)\) are all greater than one (because \(AP > BP\), \(BQ > AQ\), etc.); their logs are thus positive and we can drop the absolute value signs. We have

\[
d(AC) + d(CB) = \log(AC, PQ) + \log(CB, PQ) \\
= \log[(AC, PQ)(CB, PQ)].
\]

but \((AC, PQ)(CB, PQ) = (AB, PQ)\), as can be seen by cancelling terms.
In the other Poincaré model mentioned here, the points of the hyperbolic plane are represented by the points of one of the Euclidean half-planes determined by a fixed Euclidean line. If we use the Cartesian model for the Euclidean plane, it is customary to make the \( x \) axis the fixed line and then to use for our model the upper half-plane consisting of all points \((x, y)\) with \(y > 0\). Hyperbolic lines are represented in two ways:

1. as rays emanating from points on the \( x \) axis and perpendicular to the \( x \) axis;
2. as semicircles in the upper half-plane whose center lies on the \( x \) axis (see Figure 7.14).

Incidence and betweenness have the usual Euclidean interpretation. This model is conformal also (degrees of angles are measured in the Euclidean way). Measurement of lengths will be discussed later.
3.1 The projective plane $P^2$

Definition 18 The projective plane $P^2$ is the set of all pairs \( \{x, -x\} \) of antipodal points of $S^2$.

Two alternative definitions are:
1) The set of all lines through the origin in $E^3$.
2) The set of all equivalence classes of ordered triplets \((x_1, x_2, x_3)\) of numbers (i.e. vectors in $E^3$) not all zero, where two vectors are equivalent if they are proportional. Every representative of an equivalence class $P$ is called a homogeneous coordinate vector of $P$.

Two lines in $P^2$ have exactly one point of intersection. Two points of $P^2$ lie on exactly one line.

One of the reasons for the invention of $P^2$ was to simplify the incidence geometry of $E^2$. To illustrate this, consider the following picture in $E^3$. We regard the plane $x_3 = 1$ as a model of $E^2$. Every line through the origin of $E^3$ that is not parallel to $E^2$ meets $E^2$ in a unique point. If \((x_1, x_2, x_3)\) are homogeneous coordinates for such a point of $P^2$, then \((x_1/x_3, x_2/x_3, 1)\) is the corresponding point of $E^2$. Conversely, each point of $E^2$ determines a unique point of $P^2$.

Every line in $E^2$ determines a unique plane through the origin in $E^3$ and, hence, a unique line in $P^2$. Denote by $\ell_\infty$ the exceptional line in $P^2$, which is the plane through the origin in $E^3$ parallel to $E^2$.

Let $T : E^2 \to P^2$ be the map we have been discussing.

Theorem 25 1) $T$ maps $E^2$ bijectively to $P^2 - \ell_\infty$.
2) Let $P, Q$ be points of $E^2$. Then $T(P), T(Q)$ determine a line $\ell'$ of $P^2$, and $T$ maps $\ell = PQ$ bijectively to $\ell' - \ell_\infty$.
3) Let $\ell, m$ be lines of $E^2$. If $\ell \cap m = P$, then $\ell' \cap m' = T(P)$. If $\ell \parallel m$, then $\ell' \cap m'$ lies on $\ell_\infty$. 

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\[ T : \mathbb{E}^2 \rightarrow \mathbb{P}^2 - \mathbb{C}_\infty \]

If \( e \parallel m = P \), then \( e' \parallel m' = T(P) \).

If \( \ell, \ell', m, m' \in \mathbb{E}^2 \), then \( \ell \parallel m \parallel \ell' \parallel m' \parallel \mathbb{C}_\infty \in \mathbb{P}^2 \).