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Elementary theory of free non-abelian groups

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Abstract

We prove that any two non-abelian free groups have the same elementary theory and that this theory is decidable. These results solve two questions that were raised by Tarski in 1945.

Keywords: Free group; Elementary theory

Contents

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0. Introduction

0.1. Tarski's conjectures

Around 1945 Tarski formulated two conjectures about the elementary theory of a free group. The first of them states that the elementary theory of non-abelian free groups of different ranks coincide. The second one states that the elementary theory of a free group is decidable. We recall that the *Elementary theory* $Th(G)$ of a group G is the set of all first order sentences in the language of group theory which are true in G. A discussion of these conjectures can be found in several textbooks on model theory (see, for example, Chang and Keisler [3] or Ershov and Palutin [6]) as well as in several textbooks on group theory (see, for example, Lyndon and Schupp [22]). We announced a proof of the Tarski conjectures and gave a short survey of related results in [11] which can be considered as the introduction to this paper. Our work on the Tarski conjectures consisted of five papers: [12–16]. The last three papers circulated as preprints since 1999. The results of [15] were later partly included in [14] and partly into this paper. There were some inaccuracies in [16], and we are grateful to all the people who noticed them. This paper is a revised version of it. Some results on algorithmic problems for fully residually free groups have been taken out from the original version of [16] and published separately in [17].

We prove the following two theorems.

Theorem 1. *The elementary theory of all finitely generated non-abelian free groups coincide.*

Theorem 2. *The elementary theory of a free group is decidable.*

While this paper was going through the refereeing and revision, another proof of the first theorem, due to Sela, appeared in a series of preprints [28–33].

Our proof of Tarski's conjectures takes on the strongest possible, positive form, namely: *the free group* $F(a_1,\dots,a_n)$ *freely generated by* $a_1,\dots,a_n$ *is an elementary subgroup of* $F(a_1,\dots,a_n,\dots,a_{n+p})$ *for every* $n \geqslant 2$ *and* $p \geqslant 0$. Moreover, we prove also that: *the elementary theory* $Th(F)$ *of a free group* F *even with constants from* F *in the language is decidable.*

We will also describe in Theorem 41 finitely generated groups that are elementary equivalent to a non-abelian free group.

0.2. Prerequisites

The present paper is a direct continuation of [14]. The basic prerequisites are presented there as well as the necessary preliminary material. Some of the results of [14] are improved in [19] and [20]. It is assumed that the reader is familiar with [14] or with [19] and [20]. We refer to the books [34] and [4] for the results about Bass–Serre theory, and to [27] for the results about JSJ decompositions.

0.3. Our approach

We will give here a concise description of the main components of our approach.

(1) *An extended use of algebraic geometry* that provides the necessary topological machinery as well as a method for transcribing geometric notions into the language of pure group theory. Analogs of the introductory notions of algebraic geometry over groups has been developed by Baumslag, Myasnikov and Remeslennikov in [1]. The area took a further step in the work by Kharlampovich and Myasnikov [12]. In these two works standard algebraic geometry notions were developed such as algebraic sets, the Zariski topology, Noetherian domains, irreducible varieties, radicals and coordinate groups.

(2) *A simple algebraic description of finitely generated fully residually free groups* that is given in the papers by the authors [12,13]. It is amazing that these groups, that have been widely studied before, turned out to be the central object in the work on the theory of a free group. First of all, they are precisely coordinate groups of irreducible algebraic varieties. Secondly, they form precisely the class of finitely generated groups that are universally equivalent to free groups [26]. By giving the algebraic description of these groups we completely described irreducible algebraic varieties over a free group. Our description implies numerous nice properties of finitely generated fully residually free groups: finite presentability, free action on Z^n-trees, existence of non-trivial cyclic splittings, etc.

(3) *Embedding of finitely generated fully residually free groups into coordinate groups of NTQ systems* [13,19]; NTQ is an abbreviation for "non-degenerate triangular quasi-quadratic." These systems are constructed inductively from quadratic equations, and one can apply the techniques developed for quadratic equations to the study of these systems.

(4) *Implicit function theorems for algebraic varieties corresponding to regular quadratic and NTQ systems over free groups* [14]. Coordinate groups of regular NTQ systems turn out to form the class of finitely generated groups that are elementary equivalent to a free group.

(5) *Generalized equations and the elimination process* which is a variation of Makanin–Razborov process for solving equations and description of a solution set in a free group as a diagram of homomorphisms. This process is a symbolic rewriting process of a certain type. It is also a tool to obtain effectively a so-called JSJ decomposition of a finitely generated fully residually free group.

(6) *Free Lyndon's length functions from finitely generated fully residually free group into $\mathbb{Z}^n$.* This allows to use the technique of generalized equations, cancellation trees, and elimination process not only for equations in a free group but also for equations in a finitely generated fully residually free group.

(7) *Techniques of infinite words combined with the ideas of Stallings foldings* that are used to solve most of the algorithmic problems in finitely generated fully residually free groups (see [17,25]).

1. Elements of algebraic geometry over groups

1.1. Equations and algebraic sets

We recall here the definitions of some basic objects (see [1] for details). Let G be a group generated by a finite set A, $F(X)$ be a free group with basis $X = \{x_1, x_2, \ldots, x_n\}$, $G[X] = G * F(X)$ be a free product of G and $F(X)$. If $S \subset G[X]$ then the expression $S = 1$ is called *a system of equations* over G. As an element of the free product, the left side of every equation in

$S = 1$ can be written as a product of some elements from $X \cup X^{-1}$ (which are called *variables*) and some elements from A (*constants*).

A *solution* of the system $S(X) = 1$ over a group G is a tuple of elements $g_1, \dots, g_n \in G$ such that after replacement of each x_i by g_i the left-hand side of every equation in $S = 1$ turns into the trivial element of G. Equivalently, a solution of the system $S = 1$ over G can be described as a G-homomorphism $\phi : G[X] \to G$ such that $\phi(S) = 1$. Denote by $ncl(S)$ the normal closure of S in $G[X]$, and by G_S the quotient group $G[X]/ncl(S)$. Then every solution of $S(X) = 1$ in G gives rise to a G-homomorphism $G_S \to G$, and vice versa. By $V_G(S)$ we denote the set of all solutions in G of the system $S = 1$, it is called the *algebraic set defined by* S. This algebraic set $V_G(S)$ uniquely corresponds to the normal subgroup

$$R(S) = \{T(x) \in G[X] \mid \forall A \in G^n \ (S(A) = 1 \to T(A) = 1)\}$$

of the group $G[X]$. Notice that if $V_G(S) = \emptyset$, then $R(S) = G[X]$. The subgroup $R(S)$ contains S, and it is called the *radical of* S. The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the *coordinate group* of the algebraic set $V(S)$. Again, every solution of $S(X) = 1$ in G can be described as a G-homomorphism $G_{R(S)} \to G$.

A group G is *fully residually free* if it is discriminated by a free group. Recall that a group G is discriminated by a free group if for every finite family of non-trivial elements in G there exists a homomorphism from G to a free group such that the images of these elements are non-trivial (see [14, Section 2.6] for an introduction into the theory of fully residually free groups).

We define a Zariski topology on G^n by taking algebraic sets in G^n as a sub-basis for the closed sets of this topology. If G is a non-abelian freely discriminated group, then the union of two algebraic sets is again algebraic. Therefore the closed sets in the Zariski topology over G are precisely the algebraic sets. The Zariski topology over G^n is *Noetherian* for every n, i.e., every proper descending chain of closed sets in G^n is finite. This implies that every algebraic set V in G^n is a finite union of irreducible subsets (they are called *irreducible components* of V), and such decomposition of V is unique. Recall that a closed subset V is *irreducible* if it is not a union of two proper closed (in the induced topology) subsets.

1.2. Rational equivalence

Recall, that two systems of equations $S(X) = 1$ and $T(Y) = 1$ with coefficients from F are *rationally equivalent* if there are polynomial maps $Y = P_X(X)$ and $X = P_Y(Y)$ such that the restriction P_S of the map $P_X : F^{|X|} \to F^{|Y|}$ onto the algebraic set $V_F(S)$ gives a bijection $P_S : V_F(S) \to V_F(T)$, and the restriction P_T of the map P_Y onto the algebraic set $V_F(T)$ gives the inverse of P_S. Sometimes we refer to the maps P_X and P_Y as to change of coordinates. It was shown in (see [1]) that systems $S(X) = 1$ and $T(Y) = 1$ are rationally equivalent if and only if their coordinate groups $F_{R(S)}$ and $F_{R(T)}$ are isomorphic as F-groups. Observe that if $f : F_{R(S)} \to F_{R(T)}$ is an F-isomorphism, then $X = f(X)$ and $Y = f^{-1}(Y)$ are corresponding change of coordinates for the systems $S(X) = 1$ and $T(Y) = 1$.

Definition 1. A system $S(X) = 1$ with coefficients in a group G *splits* if there exists a non-trivial partition of X into $k > 1$ disjoint subsets $X = X_1 \cup \dots \cup X_k$, and the elements $S_i(X_i) \in G[X_i]$ such that $S(X) = 1$ is the union of the systems $S_i(X_i) = 1$, $i = 1, \dots, k$.

We say that a system $S(X) = 1$ with coefficients in G *splits up to the rational equivalence* or *rationally splits* if some system $T(Y) = 1$ which is rationally equivalent to $S(X) = 1$ splits.

Notice that $S(X) = 1$ rationally splits if and only if its coordinate group $G_{R(S)}$ is a non-trivial free product of the coordinate groups $G_{R(S_i)}$ with the group of constants G amalgamated. In the case when the systems $S_i(X_i) = 1$ are coefficient free for $i \neq 1$, one has $G_{R(S)} \simeq G_1 * \cdots * G_n$.

2. Quadratic equations and NTQ systems

2.1. *Quadratic equations over freely discriminated groups*

In this section we collect some results about quadratic equations over fully residually free groups, which will be in use throughout this paper.

Let $S \subset G[X]$. Denote by $var(S)$ the set of variables that occur in S.

Definition 2. A set $S \subset G[X]$ is called quadratic if every variable from $var(S)$ occurs in S not more then twice. The set S is strictly quadratic if every letter from $var(S)$ occurs in S exactly twice.

A system $S = 1$ over G is *quadratic* [*strictly quadratic*] if the corresponding set S is quadratic [strictly quadratic].

Definition 3. A standard quadratic equation over the group G is an equation of the one of the following forms (below d, c_i are non-trivial elements from G):

$$\prod_{i=1}^{n} [x_i, y_i] = 1, \quad n > 0, \tag{1}$$

$$\prod_{i=1}^{n} [x_i, y_i] \prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n, m \geqslant 0,\ m + n \geqslant 1, \tag{2}$$

$$\prod_{i=1}^{n} x_i^2 = 1, \quad n > 0, \tag{3}$$

$$\prod_{i=1}^{n} x_i^2 \prod_{i=1}^{m} z_i^{-1} c_i z_i d = 1, \quad n, m \geqslant 0,\ n + m \geqslant 1. \tag{4}$$

Equations (1), (2) are called *orientable* of genus n, Eqs. (3), (4) are called *non-orientable* of genus n.

The proof of the following fact can be found in [22].

Lemma 1. *Let W be a strictly quadratic word over a group G. Then there is a G-automorphism $f \in Aut_G(G[X])$ such that W^f is a standard quadratic word over G.*

Definition 4. Strictly quadratic words of the type $[x, y]$, x^2, $z^{-1}cz$, where $c \in G$, are called *atomic quadratic words* or simply *atoms*.

By definition a standard quadratic equation $S=1$ over G has the form

$$r_1 r_2 \cdots r_k d = 1,$$

where r_i are atoms, $d \in G$. This number k is called the *atomic rank* of this equation, we denote it by $r(S)$. The *size* of a quadratic equation $S=1$ is a pair $(g(S), r(S))$, where $g(S)$ is the genus of $S=1$ and $r(S)$ is an atomic rank of S. We compare sizes lexicographically from the left. In Section 1.1 we defined the notion of the coordinate group $G_{R(S)}$. Every solution of the system $S=1$ is a homomorphism $\phi: G_{R(S)} \to G$.

Theorem 3. [8] *Let $S(X)=1$ be a quadratic equation over a free non-abelian group $F=F(A)$. Then one can effectively find a finite set of solutions $\Phi = \{\phi_i\colon F_{R(S)} \to F(A \cup Y)\}$ of $S(X)=1$ in a free group $F(A \cup Y)$ such that every solution of $S=1$ in F can be represented as a composition $\sigma\phi\tau$, where σ is an F-automorphism of $F_{R(S)}$, $\phi \in \Phi$, and τ is an arbitrary F-homomorphism (specialization) from $F(A \cup Y)$ into F.*

2.2. Regular quadratic equations over freely discriminated groups

Definition 5. Let $S=1$ be a standard quadratic equation written in the atomic form $r_1 r_2 \cdots r_k d = 1$ with $k \geqslant 2$. A solution $\phi: G_{R(S)} \to G$ of $S=1$ is called:

(1) degenerate, if $r_i^{\phi} = 1$ for some i, and non-degenerate otherwise;
(2) commutative, if $[r_i^{\phi}, r_{i+1}^{\phi}] = 1$ for all $i = 1, \ldots, k-1$, and non-commutative otherwise;
(3) in a general position, if $[r_i^{\phi}, r_{i+1}^{\phi}] \neq 1$ for all $i = 1, \ldots, k-1$.

Theorem 4. [12] *Let G be a freely discriminated group and $S=1$ a standard quadratic equation over G. In the following cases $S=1$ always has a solution in G in a general position:*

(1) *$S=1$ is of the form (1), from Definition 3, $n>2$;*
(2) *$S=1$ is of the form (2), $n>0$, $n+m>1$;*
(3) *$S=1$ is of the form (3), $n>3$;*
(4) *$S=1$ is of the form (4), $n>2$;*
(5) *$r(S) \geqslant 2$ and $S=1$ has a non-commutative solution.*

The following theorem describes the radical $R(S)$ of a standard quadratic equation $S=1$ which has at least one solution in a freely discriminated group G.

Theorem 5. [12] *Let G be a freely discriminated group and $S=1$ a standard quadratic equation over G which has a solution in G. Then*

(1) *if $S=[x,y]d$ or $S=[x_1,y_1][x_2,y_2]$, then $R(S)=ncl(S)$;*
(2) *if $S=x^2 d$, then $R(S)=ncl(xb)$ where $b^2=d$;*
(3) *if $S=c^z d$, then $R(S)=ncl([zb^{-1},c])$ where $d^{-1}=c^b$;*
(4) *if $S=x_1^2 x_2^2$, then $R(S)=ncl([x_1,x_2])$;*
(5) *if $S=x_1^2 x_2^2 x_3^2$, then $R(S)=ncl([x_1,x_2],[x_1,x_3],[x_2,x_3])$;*
(6) *if $r(S) \geqslant 2$ and $S=1$ has a non-commutative solution, then $R(S)=ncl(S)$;*

(7) *if $S = 1$ is of the type (4) and all solutions of $S = 1$ are commutative, then $R(S)$ is the normal closure of the following system*:

$$\{x_1 \cdots x_n = s_1 \cdots s_n,\ [x_k, x_l] = 1,\ [a_i^{-1} z_i, x_k] = 1,\ [x_k, C] = 1,\ [a_i^{-1} z_i, C] = 1,$$
$$[a_i^{-1} z_i, a_j^{-1} z_j] = 1\ (k, l = 1, \dots, n;\ i, j = 1, \dots, m)\},$$

where $x_k \to s_k, z_i \to a_i$ is a solution of $S = 1$ and $C = C_G(c_1^{a_1}, \dots, c_m^{a_m}, s_1, \dots, s_n)$ is the corresponding centralizer. The group $G_{R(S)}$ is an extension of the centralizer C.

Put

$$\kappa(S) = |X| + \varepsilon(S),$$

where $\varepsilon(S) = 1$ if S is of the type (2) or (4), and $\varepsilon(S) = 0$ otherwise.

Definition 6. Let G be a freely discriminated group and $S = 1$ a standard quadratic equation over G which has a solution in G. The equation $S(X) = 1$ is *regular* if $\kappa(S) \geqslant 4$ and there is a non-commutative solution of $S(X) = 1$ in G, or it is an equation of the type $[x, y]d = 1$.

Notice, that if $S(X) = 1$ has a solution in G, $\kappa(S) \geqslant 4$, and $n > 0$ in the orientable case ($n > 2$ in the non-orientable case), then the equation $S = 1$ has a non-commutative solution, hence regular. Theorems 4 and 5 imply the following corollary.

Corollary 1. *Let G be a freely discriminated group and $S = 1$ a standard quadratic equation over G which has a solution in G.*

(1) *If $S(X) = 1$ has positive genus then it is regular, unless it is the equation $[x, y] = 1$.*
(2) *If $S(X) = 1$ is non-orientable equation of positive genus then it is regular, unless it is an equation of the type $x^2 c^z = a^2 c$, $x_1^2 x_2^2 = a_1^2 a_2^2$, $x_1^2 x_2^2 x_3^2 = 1$, or $S(X) = 1$ can be transformed to the form $[\bar{z}_i, \bar{z}_j] = [\bar{z}_i, a] = 1$, $i, j = 1, \dots, m$, by changing variables.*
(3) *If $S(X) = 1$ has genus 0 then it is regular unless either it is an equation of the type $c_1^{z_1} = d$, $c_1^{z_1} c_2^{z_2} = c_1 c_2$, or $S(X) = 1$ can be transformed to the form $[\bar{z}_i, \bar{z}_j] = [\bar{z}_i, a] = 1$, $i, j = 1, \dots, m$, by changing variables.*

2.3. NTQ systems and NTQ groups

We recall now the definition of a NTQ group from [14] and [13].

Let G be a group with a generating set A. A system of equations $S = 1$ is called *triangular quasi-quadratic* (shortly, TQ) over G if it can be partitioned into the following subsystems:

$$S_1(X_1, X_2, \dots, X_n, A) = 1,$$
$$S_2(X_2, \dots, X_n, A) = 1,$$
$$\vdots$$
$$S_n(X_n, A) = 1,$$

where for each i one of the following holds:

(1) S_i is quadratic in variables X_i;
(2) $S_i = \{[y,z]=1,\ [y,u]=1 \mid y,z \in X_i\}$ where u is a group word in $X_{i+1} \cup \cdots \cup X_n \cup A$. In this case we say that $S_i = 1$ corresponds to an extension of a centralizer;
(3) $S_i = \{[y,z]=1 \mid y,z \in X_i\}$;
(4) S_i is the empty equation.

Sometimes, we join several consecutive subsystems $S_i = 1, S_{i+1} = 1, \ldots, S_{i+j} = 1$ of a TQ system $S = 1$ into one block, thus partitioning the system $S = 1$ into new blocks. It is convenient to call a new system also a triangular quasi-quadratic system.

In the notations above define $G_i = G_{R(S_i,\ldots,S_n)}$ for $i = 1,\ldots,n$ and put $G_{n+1} = G$. The TQ system $S = 1$ is called *non-degenerate* (shortly, NTQ) if the following conditions hold:

(5) each system $S_i = 1$, where $X_{i+1},\ldots,X_n$ are viewed as the corresponding constants from G_{i+1} (under the canonical maps $X_j \to G_{i+1}$, $j = i+1,\ldots,n$) has a solution in G_{i+1};
(6) the element in G_{i+1} represented by the word u from (2) is not a proper power in G_{i+1}.

The coordinate group of an NTQ system is called an *NTQ group*.

An NTQ system $S = 1$ is called *regular* if each non-empty quadratic equation in S_i is regular (see Definition 6).

We say that an NTQ system $S(X) = 1$ is in the *standard form* if all quadratic equations in (1) are in the standard form. By Lemma 1 every NTQ system $S(X) = 1$ is automorphically (hence rationally) equivalent to a unique NTQ system in the standard form, i.e., there exists an automorphism $\phi \in Aut\, G[X]$ such that $S^\phi = 1$ is a standard NTQ system.

In the sequent we always assume (if not said otherwise) that NTQ systems are given in the standard form.

Let $S = 1$ be an (standard) NTQ system. Denote by S_R the set of all regular quadratic equations in $S = 1$. Let $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$ be a sequence of sizes of equations from S_R in the decreasing (left lexicographical) order. Then the tuple $size(S) = (n_1,\ldots,n_k)$ is called the *regular size* of the system $S = 1$. We compare regular sizes of systems lexicographically from the left.

For the NTQ system $S = 1$ denote by $\pi_i : G_i \to G_{i+1}$ a G_{i+1}-homomorphism (a solution of $S_i = 1$ in G_{i+1}) which exists by the condition (5). The set $\mathcal{V}$ of all G-homomorphisms $\phi : F_{R(S)} \to G$ of the type

$$\phi = \sigma_1\pi_1 \cdots \sigma_n\pi_n, \tag{5}$$

where σ_i is a G_{i+1}-automorphism of G_i, is called the *fundamental sequence* associated with the NTQ systems $S = 1$ with respect to the tuple $(\pi_1,\ldots,\pi_n)$. Sometimes, in the decomposition (5) the automorphisms σ_i run over a designated subgroup of $Aut\, G_i$, in this case we refer to the set $\mathcal{V}$ as to a *restricted fundamental sequence*. Observe, that a fundamental sequence depends on the partitioning of an NTQ system $S = 1$ into blocks and on the choice of the solutions π_i.

2.4. NTQ systems with adjoined free variables

Sometimes we assume that there is a finite set of "dummy" variables T which actually do not occur in any of the equations S_i but are formally listed in them after the variables X_n. In this case the free group $F(T)$ occurs as a free factor in all the coordinate groups G_i, we distinguish variables from T as *free variables*. We may assume also that there is the empty equation $S_{n+1}(T)=1$ (thus adding an extra layer to the system) which extends the fundamental sequence related to the system allowing F-automorphisms of the last group $F*F(T)$. In particular, if $\eta: F*F(T)\to F$ is an arbitrary F-homomorphism (a specialization of variables from T in F) then it can be realized as the composition of the automorphism $t\to t\eta(t), t\in T$, and the $\pi_{n+1}: F*F(T)\to F$ which sends all $t\in T$ into 1. This remark allows one to treat uniformly occurrences of free variables in NTQ systems from [13,14], and in quadratic equations from [8]. Recall, that in the papers [13,14] instead of inserting dummy variables from T in the equations S_i it was simply assumed that there is an F-homomorphism $\pi_n: G_n\to F*F(T)$ and it was allowed to use arbitrary specializations $\eta: F*F(T)\to F$. We will use both these approaches in the sequent.

More generally, now we define an NTQ system Q with an adjoined partitioned set of free variables T. Suppose, the NTQ system $Q(X_1,\dots,X_n)=1$ is given in the form

$$\begin{aligned} Q_1(X_1,\dots,X_n)&=1,\\ &\vdots\\ Q_n(X_n)&=1,\end{aligned}$$

and T is a set of free variables partitioned into n disjoint subsets

$$T=\{t_1,\dots,t_{k_1}\}\cup\{t_{k_1+1},\dots,t_{k_2}\}\cup\dots\cup\{t_{k_{n-1}+1},\dots,t_{k_n}\}.$$

We assume that all variables T formally participate in every equation from Q as dummy variables.

Now we define the projections

$$\pi_i: F_{R(Q_i,\dots,Q_n)}*F(T)\to F_{R(Q_{i+1},\dots,Q_n)}*F(T)$$

as homomorphisms identical on $F_{R(Q_{i+1},\dots,Q_n)}*F(T)$ and such that

$$\pi_i: F_{R(Q_i,\dots,Q_n)}\to F_{R(Q_{i+1},\dots,Q_n)}*F(t_{k_{i-1}+1},\dots,t_{k_i})$$

(here we assume for uniformity that Q_{n+1} is the empty equation in T). To describe fundamental solutions of the system (Q,T) with respect to the homomorphisms π_i we consider automorphisms σ_i of $F_{R(Q_i,\dots,Q_n)}*F(T)$ which are identical on $F_{R(Q_{i+1},\dots,Q_n)}*F(T)$ and such that their restriction on $F_{R(Q_i,\dots,Q_n)}$ is an automorphism of $F_{R(Q_i,\dots,Q_n)}$. Now a fundamental solution of the system Q with adjoined free variables T and a fixed family of homomorphisms $\pi_1,\dots,\pi_n$ is a composition $\sigma_1\pi_1\cdots\sigma_n\pi_n\tau$ where σ_i are described above and $\tau: F*F(T)\to F$ is an arbitrary F-homomorphism. In Section 7 we put more restrictions on π_i and σ_i and consider fundamental sequences under this restrictions. In all such occasions we specify these restrictions precisely.

3. Splittings

3.1. Graphs

A *directed graph* X consists of a set of vertices $V(X)$ and a set of edges $E(X)$ together with two functions $\sigma: E(X)\to V(X)$, $\tau: E(X)\to V(X)$. For an edge $e\in E(X)$ the vertices $\sigma(e)$

and $\tau(e)$ are called the *origin* and the *terminus* of e. A *non-oriented graph* is a directed graph X with involution $- : E(X) \to E(X)$ which satisfies the following conditions:

$$\bar{\bar{e}} = e, \qquad e \neq \bar{e}, \qquad \sigma(\bar{e}) = \tau(e).$$

We refer to a pair $\{e, \bar{e}\}$ as a non-oriented edge.

A *path* p in a graph X is a sequence if edges $e_1, \ldots, e_n$ such that $\tau(e_i) = \sigma(e_{i+1})$, $i \in \{1, \ldots, n-1\}$. Put $\sigma(p) = \sigma(e_1)$, $\tau(p) = \tau(e_n)$. A path $p = e_1 \ldots e_n$ is *reduced* if $e_{i+1} \neq \bar{e}_i$ for each i. A path p is *closed* (or a *loop*) if $\sigma(p) = \tau(p)$.

3.2. Graphs of groups

A *graph of groups* $\Gamma = \mathcal{G}(X)$ is defined by the following data:

(1) a connected graph X;
(2) a function $\mathcal{G}$ which for every vertex $v \in V(X)$ assigns a group G_v, and for each edge $e \in E(X)$ assigns a group G_e such that $G_{\bar{e}} = G_e$;
(3) for each edge $e \in E(X)$ there are monomorphisms $\sigma : G_e \to G_{e\sigma}$ and $\tau : G_e \to G_{e\tau}$.

Let $\mathcal{G}(X)$ be a graph of groups and T a maximal subtree of X. The fundamental group $\pi(\mathcal{G}(X), T)$ of the graph of groups $\mathcal{G}(X)$ with respect to the tree T is defined by generators and relations as follows:

$$\pi\big(\mathcal{G}(X), T\big) = \big\langle (*_{v \in V(X)} G_v),\ t_e\ \big(e \in E(X)\big)\ \big|\ t_e = 1\ (e \in T),\ t_e^{-1} g t_e = g^{\tau}\ (g \in G_e),\ t_e t_{\bar{e}} = 1 \big\rangle.$$

It is known that $\pi(\mathcal{G}(X), T)$ is independent (up to isomorphism) of T. Therefore, we will omit sometimes the tree T from the notations and write simply $\pi(\mathcal{G}(X))$.

If some presentation is fixed, the non-trivial generators t_e will be called *stable letters*. The group $\pi(\mathcal{G}(X))$ can be obtained from the vertex groups by a tree product with amalgamation and then by HNN-extensions. The following lemma shows that subgroups of $\pi(\mathcal{G}(X))$ are again fundamental groups of some special graphs of groups related to $\mathcal{G}(X)$.

Lemma 2. [4] *Let $\mathcal{G}(X)$ be a graph of groups, and let $H \leqslant \pi(\mathcal{G}(X))$. Then $H = \pi(\mathcal{G}(Y))$ where the vertex groups of $\mathcal{G}(Y)$ are $H \cap gG_v g^{-1}$ for all vertices $v \in X$, and g runs over a suitable set of (H, G_v) double coset representatives, and the edge groups are $H \cap gG_e g^{-1}$ for all edges $e \in X$, where g runs over a suitable set of (H, G_e) double coset representatives.*

3.3. Definitions and elementary properties of splittings

Let $\pi(\mathcal{G}(X), T)$ be the fundamental group of graph of groups $\mathcal{G}(X)$ with respect to a maximal subtree T. Let $\phi : G \to \pi(\mathcal{G}(X), T)$ be an isomorphism of groups. In this event the triple $D = (G, (\mathcal{G}(X), T), \phi)$ is called a *splitting* of G. A splitting D is a $\mathbb{Z}$-splitting [abelian splitting] if every edge group is infinite cyclic [abelian]. Splittings of the type $G = A *_C B$ or $G = A*_C$, are called *elementary* $\mathbb{Z}$-splittings. We say that G *splits over* an element $c \in G$ if there exists an elementary $\mathbb{Z}$-splitting of G with the edge group generated by c.

An elementary abelian splitting D is called *essential* if the images of the edge group under the boundary monomorphisms do not have finite index in the corresponding vertex groups. A splitting is *reduced* if all vertex groups of valency one and two properly contain the images of groups of adjacent edges. An abelian splitting is called *non-degenerate* if its graph of groups is reduced and has an edge. A splitting is non-trivial if its graph of groups has an edge.

Recall, that a splitting of a group G is called 2-acylindrical if for every non-trivial element $g \in G$, the fixed set of g when acting on the Bass–Serre tree corresponding to the splitting has diameter at most 2. A splitting of a group is a *star of groups*, if its underlying graph is a tree T which has diameter 2.

Definition 7. Let G be a group and $\mathcal{K} = \{K_1, \dots, K_n\}$ be a set of subgroups of G. An abelian splitting D is called a *splitting modulo* $\mathcal{K}$ if every subgroup from $\mathcal{K}$ is conjugated into a vertex group in D.

3.4. Elementary transformations of graphs of groups and splittings

A *conjugation of a splitting* of G is a conjugation of $\pi(\mathcal{G}(X), T)$.

A *sliding* is a modification of a graph of groups according to the relation

$$(A_1 *_{C_1} A_2) *_{C_2} A_3 \cong (A_1 *_{C_1} A_3) *_{C_2} A_2$$

in the case when $C_1 \leqslant C_2$. More precisely, suppose a graph of groups $\Gamma = \mathcal{G}(X)$ contains vertices v_1, v_2, v_3 with vertex groups A_1, A_2, A_3, respectively, and edges $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$ with edge groups C_1 and C_2 and $\tau_{e_1}(C_1) \leqslant \sigma_{e_2}(C_2)$. We replace e_1 by the edge $\bar{e}_1 = (v_1, v_3)$ with edge group C_1 and $\sigma_{\bar{e}_1}(C_1) = \sigma_{e_1}(C_1)$, $\tau_{\bar{e}_1}(C_1) = \tau_{e_2}\sigma_{e_2}^{-1}\tau_{e_1}(C_1)$. Notice that two of the vertices may coincide.

If Γ is a graph of groups, and G_e is an edge group in Γ such that $\sigma(G_e) = G_{e\sigma}$, then *conjugation of the boundary monomorphism* σ is a replacement of the monomorphism σ by σ_h such that $\sigma_h(g) = h^{-1}\sigma(g)h$ for some $h \in G_{e\sigma}$ and any $g \in G_e$.

A splitting $G = A *_{C_1} B_1$ is obtained by *folding* from the splitting $G = A *_C B$ if C is a proper subgroup of C_1 and $B_1 = C_1 *_C B$. An *unfolding* is the inverse operation to folding. A splitting is *unfolded* if one cannot apply an unfolding to it.

Lemma 3. *Elementary transformations preserve the fundamental groups of graphs of groups up to isomorphism.*

Proof. The statement is obvious for a conjugation. Suppose the graph of groups $\Gamma_1 = \mathcal{G}_1(X_1)$ is obtained from $\Gamma = \mathcal{G}(X)$ by an elementary transformation. Different choice of T in the graph X defines an isomorphism of the fundamental group $\pi(\mathcal{G}(X), T)$. Therefore it is enough to prove the isomorphism of the fundamental groups for a suitable choice of T in X and T_1 in X_1. For sliding we consider only the case when vertices v_1, v_2, v_3 are different. In this case we choose the tree T such a way that edges e_1 and e_2 belong to T, and obtain T_1 by the sliding defined above from T. Then the isomorphism becomes obvious.

Suppose $\Gamma_1 = \mathcal{G}_1(X)$ is obtained from Γ by the conjugation of the boundary monomorphism σ. Consider first the case when it is possible to choose T such a way that $e \notin T$. In this case $G = \pi(\mathcal{G}(X), T)$ has a stable letter t corresponding to e and a relation $t^{-1}\sigma(g)t = \tau(g)$ for any $g \in G_e$. We replace $\sigma(g)$ by $\sigma_h(g)$ and the stable letter t in the presentation of $\pi(\mathcal{G}(X), T)$ by $t_h = h^{-1}t$. Then in these generators $\pi(\mathcal{G}_1(X), T)$ has the same presentation as $\pi(\mathcal{G}(X), T)$. Now consider the case when it is not possible to choose T such a way that $e \notin T$. Then removing e from X we obtain two connected components X_1 and X_2. Let $G_{e\sigma} \in X_1$, G_1 and G_2 are fundamental groups of the graphs of groups corresponding to X_1 and X_2. Then

$$\pi\big(\mathcal{G}(X), T\big) = G_1 *_{\sigma(G_e)=\tau(G_e)} G_2$$

and

$$\pi\big(\mathcal{G}_1(X), T\big) = G_1 *_{h^{-1}\sigma(G_e)h=h^{-1}\tau(G_e)h} \big(h^{-1}G_2h\big).$$

These groups are isomorphic.

Similarly, in the case of foldings we choose T so that the edge corresponding to C belongs to T. $\square$

3.5. Freely decomposable groups

Recall that a group G is *freely decomposable* if it is isomorphic to a non-trivial free product (in which there are at least two non-trivial factors). Otherwise, G is called *freely indecomposable*. A free decomposition

$$G = G_1 * \cdots * G_n * F(Y)$$

is called *Grushko's decomposition* if all the factors $G_1, \ldots, G_n$ are non-cyclic freely indecomposable groups, and $F(Y)$ is a free group with basis Y (perhaps empty). If there is another Grushko's decomposition

$$G = H_1 * \cdots * H_m * F(Z),$$

then $n = m$, corresponding factors G_i and H_i (after reordering) are conjugated, and $|Z| = |Y|$.

Definition 8. Let G be a group and H be a subgroup of G. We say that G has a *non-trivial free decomposition modulo* H, if

$$G \simeq G_1 * G_2,$$

$H \leqslant G_1$, and $G_2 \neq 1$.

Now we generalize the definition above.

Definition 9. Let G be a group and $K_1, \ldots, K_n$ be subgroups of G. We say that G has a non-trivial free decomposition modulo $K_1, \ldots, K_n$, if

$$G \simeq G_1 * G_2,$$

$K_1 \leqslant G_1$, and for each i there exists $g_i \in G$ such that $K_i^{g_i} \leqslant G_1$.

Lemma 4. *Let G be a group with a non-trivial splitting D of G. If D contains an edge with the trivial associated group then G is decomposable into a non-trivial free product.*

Proof. Let $D = \mathcal{G}(X)$ be a non-trivial splitting of G with an edge with the trivial associated subgroup. Denote by X^* the subgraph of X formed by all vertices in X and all edges in X whose associated groups are non-trivial. Assume first, that X^* is not connected and $Y_1, \ldots, Y_n$ $(n \geqslant 2)$ are the connected components of X^*. By collapsing in D every subgraph Y_i into a single vertex, say v_i, with the associated group $\pi_1(Y_i)$ one gets a graph of groups Z in which all the edge groups are trivial. Since all the vertex groups in D are non-trivial the groups $\pi_1(Y_i)$ are non-trivial. By Lemma 2 the group $\pi_1(Z)$ is the free product of its vertex groups and a free group, so it is freely decomposable. Observe, that $G \simeq \pi_1(Z)$ since collapses preserve the fundamental groups. Therefore G is freely decomposable.

Suppose now that X^* is connected. In this case a given maximal subtree T of X^* is also a maximal subtree of X. If e is an edge in X with the trivial associated subgroup G_e then $e \notin T$, so $t_e \neq 1$ in $G = \pi_1(X)$. Since $G_e = 1$ the infinite cyclic group $\langle t_e \rangle$ is a free factor of G. Hence G is freely decomposable. This completes the proof of the lemma. $\square$

Definition 10. Let G be a group and $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a set of subgroups of G. We say that a free decomposition of G

$$G \simeq G_1 * \cdots * G_k$$

is *compatible* with $\mathcal{K}$ if each subgroup in $\mathcal{K}$ is a conjugate of a subgroup of one of the factors G_j. Denote by $\mathcal{K}_j$ the set of all conjugates of subgroups in $\mathcal{K}$ which belong to G_j. This decomposition is called *reduced* if none of the G_j has a non-trivial compatible free decomposition modulo $\mathcal{K}_j$.

Proposition 1. *Let*

$$G = G_1 * \cdots * G_n * F(Y) = G = H_1 * \cdots * H_m * F(Z),$$

be two compatible with $\mathcal{K}$ reduced free decompositions, then $n = m$, corresponding factors G_i and H_i (after reordering) are conjugated, and $|Z| = |Y|$.

Proof. Consider a Bass–Serre tree T corresponding to the first decomposition. Subgroups from $\mathcal{K}$ fix some vertices of this tree. Each subgroup H_i acts on T, and, since it does not have a free decomposition compatible with $\mathcal{K}$, fixes a vertex of this tree. Therefore it is conjugated into some factor, say G_i, of the first decomposition. Conversely, each G_i is conjugated into some H_j. Each factor H_i, G_i is malnormal, therefore H_i and G_i are conjugated. The normal closure generated by $H_1 * \cdots * H_m$ and by $G_1 * \cdots * G_m$ is the same, therefore $|Y| = |Z|$. $\square$

Remark 1. In the notations above, if

$$G \simeq G_1 * \cdots * G_k$$

is a compatible with $\mathcal{K}$ decomposition of G, then we will always assume (taking a conjugation of G if necessary and renaming subgroups G_i) that $K_1 \leqslant G_1$.

3.6. Splittings of finitely generated fully residually free groups

Denote by $\mathcal{F}$ the class of all finitely generated fully residually free groups. The structure of groups from $\mathcal{F}$ was described in [13], where it was shown that these groups are subgroups of the Lyndon's group $F^{Z[x]}$. Here we state one corollary of the results mentioned above, which shows that non-abelian groups from $\mathcal{F}$ have non-trivial cyclic splitting.

Theorem 6. [13] *Every group $G \in \mathcal{F}$ can be obtained from free groups by finitely many operations of the following type*:

(1) *free products*;
(2) *free products with amalgamation along cyclic subgroups with at least one of them being maximal*;
(3) *separated HNN-extensions along cyclic subgroups with at least one of them being maximal*;
(4) *free extensions of centralizers* (*and for each element its centralizer extends at most once*).

Corollary 2. *Every freely indecomposable f.g. non-abelian fully residually free group has an essential $\mathbb{Z}$-splitting.*

3.7. Elliptic and hyperbolic subgroups

If H and K are subgroups of a group G, we say that H can be *conjugated into* K if it is a conjugate of a subgroup of K.

An element $g \in G$ [a subgroup $H \leqslant G$] is called *elliptic* in a given splitting of G if g [correspondingly, H] can be conjugated into a vertex group, and *hyperbolic* otherwise.

Lemma 5. [27] *Let G be a freely indecomposable group. If D_i is an elementary $\mathbb{Z}$-splitting of G with the edge group C_i $(i = 1, 2)$, then C_1 is hyperbolic* (*elliptic*) *in D_2 if and only if C_2 is hyperbolic* (*elliptic*) *in D_1.*

A pair of elementary $\mathbb{Z}$-splittings D_i $(i = 1, 2)$ is called *intersecting* if they form a hyperbolic-hyperbolic pair, namely C_1 is hyperbolic with respect to D_2 and C_2 is hyperbolic with respect to D_1.

Lemma 6. *Let $G \in \mathcal{F}$ and N a maximal abelian non-cyclic subgroup of G. Then the following holds*:

(1) *if $G = A *_C B$ is an abelian splitting of G then N is elliptic in this splitting*;
(2) *if $G = A*_C$ is an abelian splitting of G then one of the following holds*:
    (a) *N is elliptic in this splitting,*
    (b) *C is the centralizer $C_A(v)$ of some element $v \in A$, $C \leqslant N^g$ for some $g \in G$, and $G = A *_C N^g$ is an extension of the centralizer C.*

Proof. The first statement follows from the description of commuting elements in a free product with amalgamation (see, for example, [23]). The second one is a direct corollary of Lemma 2 and Theorem 5 from [7]. $\square$

Definition 11. An abelian splitting D of a group G is called *normal* if all maximal abelian non-cyclic subgroups of G are elliptic in D. By $\mathcal{D}(G)$ we denote the set of all normal splittings of G. Denote by $\mathcal{D}_F(G)$ the class of all normal splittings of G such that F belongs to a vertex group.

3.8. Quadratically hanging subgroups

Let $D = (G, (\mathcal{G}(X), T), \phi)$ be an abelian splitting of G. A vertex group $Q = G_v$, $v \in V$, is called *quadratically hanging* in D (shortly, QH-subgroup in D), if the following conditions hold:

(1) Q admits one of the following presentations:

$$\left\langle p_1, \dots, p_m, a_1, \dots, a_g, b_1, \dots, b_g \;\middle|\; \prod_{j=1}^{g} [a_i, b_i] \prod_{k=1}^{m} p_k \right\rangle, \quad g \geqslant 0,\ m \geqslant 1, \tag{6}$$

and if $g = 0$, then $m \geqslant 4$,

$$\left\langle p_1, \dots, p_m, a_1, \dots, a_g \;\middle|\; \prod_{j=1}^{g} a_i^2 \prod_{k=1}^{m} p_k \right\rangle, \quad g \geqslant 1,\ m \geqslant 1 \tag{7}$$

(in particular, Q is a free group);

(2) for every edge $e \in E$ outgoing from v, the edge group G_e is conjugate to one of the subgroups $\langle p_i \rangle$, $i = 1, \dots, m$;

(3) for each p_i there is an edge $e_i \in E$ outgoing from v such that G_{e_i} is a conjugate of $\langle p_i \rangle$.

Notice that for a freely indecomposable group G property (3) is automatically satisfied. Sometimes we refer to the elements p_i as to *boundary* elements.

A QH-subgroup Q of D is called a *maximal QH-subgroup* (shortly, MQH-subgroup) if for every elementary abelian splitting E of G with the edge group C, either Q is elliptic in E, or C is conjugate into Q, in which case E is induced from a $\mathbb{Z}$-splitting of Q.

Similarly, one can define a relativized version of a QH-subgroup. Let $\mathcal{K} = \{K_1, \dots, K_n\}$ be a set of subgroups of G and D be an abelian splitting of G modulo $\mathcal{K}$. A QH-subgroup Q of D is called a *maximal QH-subgroup* if the condition above, which defines maximal QH-subgroups, holds for every elementary abelian splitting E of G modulo $\mathcal{K}$.

Sometimes we say that a subgroup Q of G is a QH-subgroup of G (modulo a set of subgroups $\mathcal{K}$) if Q is a QH-subgroup in some splitting D of G (modulo $\mathcal{K}$).

Non-QH non-abelian vertex groups of D are called *rigid* in D.

3.9. QH-subgroups and quadratic equations

Let $D = (G, (\mathcal{G}(X), T), \phi)$ be a splitting of G. Assume that the maximal subtree T of X is chosen as follows. A maximal sub-forest in the subgraph of X spanned by all rigid vertices is extended to a maximal sub-forest in the subgraph of X spanned by all non-QH vertex groups, and then extended to T.

Let a maximal QH subgroup Q in D be given by a presentation

$$\prod_{i=1}^{n} [x_i, y_i] p_1 \cdots p_m = 1.$$

Using elementary transformations (see Section 3.4) one can change D in such a way that there are exactly m outgoing edges $e_1, \dots, e_m$ from the vertex corresponding to Q, and each e_i has the edge group equal to $\langle p_i \rangle$, i.e., $\sigma(G_{e_i}) = \langle p_i \rangle$.

If we consider a presentation of G by generators and relations corresponding to D, the relations corresponding to this vertex are

$$\prod_{i=1}^{n} [x_i, y_i] p_1 \cdots p_m = 1,$$

$$c_i^{z_i} = p_i, \quad i = 1, \dots, m, \qquad c_m = p_m,$$

where $\langle c_i \rangle$, $i = 1, \dots, m$, is the image of the edge group $\langle p_i \rangle$, $e_m \in T$ and $z_i = 1$ if $e_i \in T$, $i = 1, \dots, m-1$. Excluding p_i from the presentation of G we obtain the following relation:

$$S_Q = \prod_{i=1}^{n} [x_i, y_i] c_1^{z_1} \cdots c_{m-1}^{z_{m-1}} c_m = 1. \tag{8}$$

We term $S_Q = 1$ the *quadratic equation* corresponding to Q. Conversely, if G is a quotient of the free product

$$F(x_1, y_1, \dots, x_n, y_n, z_1, \dots, z_{m-1}) * H,$$

where $H \in \mathcal{F}$, over the normal subgroup generated by the left side of the equation $S_Q = 1$, then G can be represented as a fundamental group of the graph of groups Γ such that Γ contains a QH vertex with the vertex group Q given by the relation $\prod_{i=1}^{n} [x_i, y_i] p_1 \cdots p_m = 1$, with outgoing edges e_i, $i = 1, \dots, m$, and with the edge groups $\langle p_i \rangle$, $i = 1, \dots, m$.

Every homomorphism $\phi : G \to F$ gives rise to a solution of the equation

$$\prod_{i=1}^{n} [x_i, y_i] c_1^{\phi z_1} \cdots c_{m-1}^{\phi z_{m-1}} c_m^{\phi} = 1 \tag{9}$$

in F. Similarly, one can consider a non-orientable QH-subgroup.

3.10. Induced QH-subgroups

Let G be a f.g. group and $D = \mathcal{G}(X)$ an abelian splitting of G. For a QH-subgroup Q of D one can define a splitting D_Q of G as follows. Suppose the subgroup Q is associated in $\mathcal{G}(X)$ with a vertex $v \in X$ with outgoing edges $e_1, \dots, e_m$. Denote by $Y_1, \dots, Y_k$ the connected components of the graph $X - \{v, e_1, \dots, e_m\}$ and by $P_1, \dots, P_k$—the fundamental groups of the graphs of groups induced from $\mathcal{G}(X)$ on $Y_1, \dots, Y_k$. Collapsing every subgraph Y_i of X into a single vertex u_i results in an associated splitting of G which we denote by D_Q. By $\mathcal{K}_{D,Q}$, or simply by $\mathcal{K}_Q$, we denote the set of subgroups $\{P_1, \dots, P_m\}$ which we call the *Q-associated subgroups*.

Lemma 7. [19] *Let G be a finitely generated fully residually free group, $\Gamma = \Gamma(X)$ a cyclic* [*abelian*] *splitting of G, Q a QH-subgroup in Γ, and D_Q the associated splitting of G. If H is a finitely generated non-cyclic subgroup of G then one of the following conditions holds*:

(1) *H is a non-trivial free product*;
(2) *H is conjugated in G in one of the subgroups from $\mathcal{K}_Q$*;
(3) *H is freely indecomposable, and for some $g \in G$ $H \cap Q^g$ has finite index in Q^g. In this event $H \cap Q^g$ is a QH vertex group in H.*

Moreover, if $H_Q = H \cap Q$ is non-trivial and has infinite index in Q, then H_Q is a free product of cyclic groups generated by conjugates of powers of some boundary elements of Q and a free group F_1 which does not intersect any conjugate of a boundary element of Q.

3.11. Quadratic decomposition

By Theorem 5.6 from [27] for every f.g. freely indecomposable torsion free non-surface group G there exists a reduced (may be trivial) $\mathbb{Z}$-splitting D_{quadr} (a quadratic decomposition of G) with the following properties:

(a) every MQH-subgroup of G can be conjugated to a vertex group in D_{quadr}; every QH-subgroup of G can be conjugated into one of the MQH-subgroups of G; every vertex with a non-MQH vertex group is adjacent only to vertices with MQH vertex groups;
(b) if an elementary $\mathbb{Z}$-splitting $G = A *_C B$ or $G = A*_C$ is hyperbolic in another elementary $\mathbb{Z}$-splitting of G, then C can be conjugated into some MQH-subgroup;
(c) for every elementary $\mathbb{Z}$- [abelian] splitting $G = A *_C B$ or $G = A*_C$ from $\mathcal{D}(G)$ which is elliptic in each elementary $\mathbb{Z}$- [abelian] splitting from $\mathcal{D}(G)$, the edge group C can be conjugated into a non-MQH-subgroup of D_{quadr};
(d) if D'_{quadr} is another splitting that has properties (a)–(c), then it can be obtained from D_{quadr} by slidings, conjugations, and modifying boundary monomorphisms by conjugation.

Similarly, there exists a relativized version of a quadratic decomposition of G modulo a set of subgroups $\mathcal{K}$ of G (in this case all splittings that occur are splitting modulo $\mathcal{K}$).

3.12. JSJ decompositions

All elementary cyclic [abelian] splittings of a finitely presented (f.p.) torsion free freely indecomposable group are encoded in a splitting called a JSJ decomposition.

If H and K are subgroups of a group G, we say that H is *conjugated into K* if it is conjugated to a subgroup of K.

Theorem 7. [27, part of Theorem 7.1] *Let H be a f.p. torsion-free freely indecomposable group. There exists a reduced, unfolded $\mathbb{Z}$-splitting of H, called a JSJ decomposition of H, with the following properties:*

(1) *Every MQH-subgroup of H can be conjugated to a vertex group in the JSJ decomposition. Every QH-subgroup of H can be conjugated into one of the MQH-subgroups of H. Every non-MQH vertex group in the JSJ decomposition is elliptic in every $\mathbb{Z}$-splitting of H.*
(2) *If an elementary $\mathbb{Z}$-splitting $H = A *_C B$ or $H = A*_C$ is hyperbolic in another elementary $\mathbb{Z}$-splitting, then C can be conjugated into some MQH-subgroup.*

We call a splitting of a finitely presented group *almost reduced* if vertices of valency one and two properly contain the images of the edge groups except vertices between two MQH-subgroups that may coincide with one of the edge groups.

Theorem 8. [19] *Let H be a freely indecomposable f.g. fully residually free group. One can effectively find a normal almost reduced unfolded cyclic* [*abelian*] *splitting $D \in \mathcal{D}(H)$ of H, with the following properties*:

(1) *Every MQH-subgroup of H can be conjugated to a vertex group in D*; *every QH-subgroup of H can be conjugated into one of the MQH-subgroups of H*; *non-MQH-subgroups in D are of two types*: *maximal abelian and non-abelian, every non-MQH vertex group in D is elliptic in every cyclic* [*abelian*] *splitting in $\mathcal{D}(H)$.*
(2) *If an elementary cyclic* [*abelian*] *splitting $H = A *_C B$ or $H = A*_C$ is hyperbolic in another elementary cyclic* [*abelian*] *splitting, then C can be conjugated into some MQH-subgroup.*
(3) *Every elementary cyclic* [*abelian*] *splitting $H = A *_C B$ or $H = A*_C$ from $\mathcal{D}(H)$ which is elliptic with respect to any other elementary cyclic* [*abelian*] *splitting from $\mathcal{D}(H)$ can be obtained from D by a sequence of collapsings, foldings, conjugations and modifying boundary monomorphisms by conjugation.*
(4) *If $D_1 \in \mathcal{D}(H)$ is another splitting that has properties* (1), (2)*, then it can be obtained from D by slidings, conjugations, and modifying boundary monomorphisms by conjugation.*

We will call such a splitting a *cyclic* [*abelian*] *JSJ decomposition of H*. Similar result holds for the class of splittings $\mathcal{D}_F(H)$. Such a splitting will be called an *abelian* (*or cyclic*) *JSJ decomposition* of H modulo F.

Theorems 6 and 8 imply the following result.

Corollary 3. *Every freely indecomposable f.g. non-abelian fully residually free group which is not a surface group admits a non-degenerate cyclic* [*abelian*] *JSJ decomposition. Moreover, such a decomposition can be found effectively.*

3.13. JSJ decompositions modulo a system of subgroups

In this section we introduce JSJ decompositions of an arbitrary f.g. group G modulo a system of subgroups $\mathcal{K} = \{K_1, \dots, K_n\}$ of G. As always, if G is an F-group then we assume that $F \leqslant K_1$.

We start with the case when G has no non-trivial compatible free decomposition modulo $\mathcal{K}$.

Definition 12. Let G be a finitely generated group and $\mathcal{K} = \{K_1, \dots, K_n\}$ a set of subgroups of G. Suppose there is no non-trivial compatible free decomposition of G modulo $\mathcal{K}$. Then an abelian normal splitting $D \in \mathcal{D}(G)$ of G modulo $\mathcal{K}$ (possibly degenerate) is called a JSJ decomposition of G modulo $\mathcal{K}$ if the following conditions hold:

(1) Every MQH-subgroup of G modulo $\mathcal{K}$ can be conjugated to a vertex group in D; every QH-subgroup of G modulo $\mathcal{K}$ can be conjugated into one of the MQH-subgroups of G modulo $\mathcal{K}$; non-MQH-subgroups in D are of two types: maximal abelian and non-abelian, every non-MQH vertex group in D is elliptic in every abelian splitting in $\mathcal{D}(G)$ modulo $\mathcal{K}$.

(2) If an elementary abelian splitting $G = A *_C B$ or $G = A*_C$ modulo $\mathcal{K}$ is hyperbolic in another elementary abelian splitting modulo $\mathcal{K}$, then C can be conjugated into some MQH-subgroup.
(3) Every elementary abelian splitting modulo $\mathcal{K}$, $G = A *_C B$ or $G = A*_C$ from $\mathcal{D}(G)$ which is elliptic with respect to any other elementary abelian splitting modulo $\mathcal{K}$ from $\mathcal{D}(G)$ can be obtained from D by a sequence of collapsings, foldings, conjugations and modifying boundary monomorphisms by conjugation.
(4) If $D_1 \in \mathcal{D}(G)$ is another splitting modulo $\mathcal{K}$ that has properties (1), (2), then it can be obtained from D by slidings, conjugations, and modifying boundary monomorphisms by conjugation.

Theorem 9. [19] *Let G be a finitely generated fully residually free group and $\mathcal{K} = \{K_1, \ldots, K_n\}$ a set of subgroups of G. If there is no non-trivial compatible free decomposition of G modulo $\mathcal{K}$ then a JSJ decomposition of G modulo $\mathcal{K}$ exists and can be effectively found.*

Definition 13. Let G be a finitely generated group and $\mathcal{K} = \{K_1, \ldots, K_n\}$ a set of subgroups of G. Suppose

$$G = G_1 * \cdots * G_\ell \tag{10}$$

is a reduced compatible free decomposition of G modulo $\mathcal{K}$ and $\mathcal{K}_j = \{K_{j_1}, \ldots, K_{j_s}\}$ is the set of all subgroups in $\mathcal{K}$ conjugated into G_j, $j = 1, \ldots, \ell$. A JSJ decomposition of G is a splitting obtained from a splitting like 10 by replacing each G_j with a JSJ decomposition D_j of G_j modulo $\mathcal{K}_j$.

Theorem 10. [19] *Let G be a finitely generated fully residually free group and $\mathcal{K} = \{K_1, \ldots, K_n\}$ a set of subgroups of G. Then a JSJ decomposition of G modulo $\mathcal{K}$ exists and can be effectively found.*

3.14. Canonical automorphisms

Let $G = A *_C B$ be an elementary abelian splitting of G. For $c \in C$ we define an automorphism $\phi_c : G \to G$ such that $\phi_c(a) = a$ for $a \in A$ and $\phi_c(b) = b^c = c^{-1}bc$ for $b \in B$.

If $G = A*_C = \langle A, t \mid c^t = c',\ c \in C\rangle$ then for $c \in C$ define $\phi_c : G \to G$ such that $\phi_c(a) = a$ for $a \in A$ and $\phi_c(t) = ct$.

We call ϕ_c a *Dehn twist* obtained from the corresponding elementary abelian splitting of G. If G is an F-group, where F is a subgroup of one of the factors A or B, then Dehn twists that fix elements of the free group $F \leqslant A$ are called *canonical Dehn twists*. Similarly, one can define canonical Dehn twists with respect to an arbitrary fixed subgroup K of G.

Definition 14. Let $D \in \mathcal{D}(G)$ $[D \in \mathcal{D}_F(G)]$ be an abelian splitting of a group G and G_v be either a QH or an abelian vertex of D. Then an automorphism $\psi \in Aut(G)$ is called a canonical automorphism corresponding to the vertex G_v if ψ satisfies the following conditions:

(i) ψ fixes element-wise all other vertex groups in D up to conjugation (hence fixes up to conjugation all the edge groups);

(ii) if G_v is a QH vertex in D, then ψ is a Dehn twist [canonical Dehn twists] corresponding to some essential $\mathbb{Z}$-splitting of G along a cyclic subgroup of G_v;
(iii) if G_v is an abelian subgroup then ψ acts as an automorphism on G_v which fixes all the edge subgroups of G_v.

Definition 15. Let $D \in \mathcal{D}(G)$ $[D \in \mathcal{D}_F(G)]$ be an abelian splitting of a group G and e an edge in D. Then an automorphism $\psi \in Aut(G)$ is called a canonical automorphism corresponding to the edge e if ψ is a Dehn twist [canonical Dehn twist] of G with respect to the elementary splitting of G along the edge e which is induced from D.

Definition 16. Let $D \in \mathcal{D}(G)$ $[D \in \mathcal{D}_F(G)]$ be an abelian splitting of a group [F-group] G. Then the *canonical group* of automorphisms $A_D = A_D(G)$ of G with respect to D is the subgroup of $Aut(G)$ generated by all canonical automorphisms of G corresponding to all edges, all QH vertices, and all abelian vertices of D. In the case $D \in \mathcal{D}(G)$ we also include conjugation into the canonical group of automorphisms.

Lemma 8. [19] *Let G be a freely indecomposable F-group in $\mathcal{F}$, and D be an abelian JSJ decomposition of G. Then A_D is a direct product of abelian groups generated by canonical Dehn twists corresponding to edges of D between non-QH non-abelian subgroups, and groups of canonical automorphisms corresponding to MQH and abelian vertex groups.*

4. Lifting splittings

4.1. Lifting free decompositions into fundamental groups of compact surfaces

In this section we develop a technique to lift an elementary splitting from a factor-group into the ambient group. We use notation from the previous section.

Lemma 9. [35] *Let G be a fundamental group of a compact connected surface T, and either the boundary of T is empty or it consists of several components $E_1, \ldots, E_m$. Suppose that the simple closed curve E_i represents the conjugacy class ε_i, $i = 1, \ldots, m$. Let f be an epimorphism from G onto a free product $A * B$ such that for all i, $f(\varepsilon_i)$ contains an element either from A or from B. Then there is a simple closed curve d on T separating T into two pieces T_A and T_B such that $G = \pi_1(T_A) *_{\pi_1(d)} \pi_1(T_B)$ is an amalgamated product over a cyclic group $\pi_1(d)$ and $f(\pi_1(T_A)) = A$, $f(\pi_1(T_B)) = B$, $f(\pi_1(d)) = 1$.*

Corollary 4. *Let D be a splitting of a group G and Q a QH-subgroup in D. Suppose f is a homomorphism from G onto a non-trivial free product $A * B$ such that*:

(1) *the image $f(Q)$ is not elliptic in $A * B$*;
(2) *for every edge e in D adjacent to Q the image $f(G_e)$ of the edge group G_e is conjugated into A or B.*

Then f is a solution in $A * B$ of the quadratic equation $S_Q(X, P^f) = 1$, where $S_Q(X, P) = 1$ is the quadratic equation corresponding to Q (*see Section* 3.9). *Moreover, f is a solution of a system of two quadratic equations $S_1(X_1, P^f) = 1$ and $S_2(X_2, P^f) = 1$, where $X = X_1 \cup X_2$*

is a partition of X, P^f *are the same coefficients as in* $S_Q(X, P^f) = 1$, *and the size of each* $S_1(X_1, P^f) = 1$ *and* $S_2(X_2, P^f) = 1$ *is strictly less then that of* $S_Q(X, P^f) = 1$.

Proof. The conditions (1) and (2) above ensure (by the Kurosh theorem) that the image $f(Q)$ is a non-trivial free product which satisfies all the conditions of the Stallings' lemma. Therefore, Q splits into a non-trivial amalgamated product $Q = Q_1 *_c Q_2$ with $f(c) = 1$. This allows one to refine the splitting D to a new splitting D' by replacing Q with $Q_1 *_c Q_2$. Clearly Q_1 and Q_2 are QH-subgroups in the refined splitting D'. If T is the maximal subtree in D chosen according to Section 3.9, then we can chose a maximal subtree T' in D' such that $T \subseteq T'$. Let $S'_1(X_1, P) = 1$, $S'_2(X_2, P) = 1$ be the quadratic equations that correspond to QH-subgroups Q_1 and Q_2 in D' relative to the tree T'. Observe, that the edges adjacent to the QH-subgroups Q_1 and Q_2 are the same old edges as were adjacent to Q in D except for the new edge e' corresponding to the splitting $Q = Q_1 *_c Q_2$. Let p'_1 and p'_2 be the new boundaries in Q_1 and Q_2 that correspond to e'. Then the equations $S'_1 = 1$ and $S'_2 = 1$ can be written in the form

$$S_1(X_1, P_1) = c^{z'_1}, \qquad S_2(X_2, P_2) = c^{z'_2},$$

where $X_1 \cap X_2 = \emptyset$, $X = X_1 \cup X_2$ and $P = P_1 \cup P_2$. Application of f to the equations above immediately gives

$$S_1\big(X_1, P_1^f\big) = 1, \qquad S_2\big(X_2, P_2^f\big) = 1,$$

as required. □

4.2. *Lifting elementary splittings of QH-subgroups*

Let G be a f.g. fully residually free group and $\mathcal{K}$ a set of subgroups of G. Denote by G_1 a fully residually free quotient of G, $\kappa : G \to G_1$ the canonical epimorphism, and $\mathcal{K}_1 = \mathcal{K}^\kappa$.

Let E_1 be an elementary HNN splitting of G_1 modulo $\mathcal{K}_1$. We say that E_1 *lifts* into G relative to κ if there exists an abelian splitting E of G modulo $\mathcal{K}$ with only one vertex such that κ maps the stable letters of E into the stable letter of E_1, the edge groups of E—into the edge group of E_1, and the vertex group of E—into the vertex group of E_1. In this case we say that E is a κ*-lift* of E_1, E_1 is a κ*-image* of E, and write $E^\kappa = E_1$. In other words, E_1 is the κ-image of E if E is a one-vertex splitting of G such that κ is equal to the homomorphism induced from a morphism of graph of groups $E \to E_1$. Similarly, we define lifts of free products with cyclic amalgamation. Namely, let E_1 be an elementary splitting of G_1 modulo $\mathcal{K}_1$ as a free product with amalgamation. Then E_1 *lifts* into G relative to κ if there exists an abelian splitting E of G modulo $\mathcal{K}$ such that the underlying graph of E is a tree, κ maps vertex groups of E into vertex groups of E_1, and the edge groups of E—into the edge groups of E_1. Equivalently, E_1 is the κ-image of E if E is a cyclic splitting of K, which underlying graph is a tree, and a morphism of graphs of groups $E \to E_1$ induces κ. Again, in this case we say that E is a κ*-lift* of E_1, E_1 is the κ*-image* of E, and write $E^\kappa = E_1$. Sometimes, elementary splittings of G_1 have only "partial lifts," to describe this situation we introduce the following terminology. We say that an HNN (free product with amalgamation) splitting E_1 of G_1 modulo $\mathcal{K}_1$ *partially lifts* or, equivalently, the *edge of* E_1 *lifts* into G relative to κ if there exists an elementary HNN (free product with amalgamation) splitting E of G such that κ maps the stable letter of E into the stable letter of E_1 and the edge group of E—into the edge group of E_1. Clearly, if E_1 lifts into G then it partially lifts into G.

We say that a standard automorphism ϕ_1 of G_1 relative to an elementary splitting E_1 is the *κ-image* of a standard automorphism ϕ of G relative to a splitting E of G if $\phi\kappa = \kappa\phi_1$. In this event we also say that ϕ is a κ-lift of ϕ_1. It is not hard to see that, in the notation above, if an elementary $\mathbb{Z}$-splitting E_1 of G_1 lifts to a splitting E of K then for every edge e_i, $i = 1, \dots, m$, of E there exists a Dehn twist τ_i of G relative to e_i such that the κ-image of the product $\tau_1 \cdots \tau_m$ is a Dehn twist of G_1 relative to E_1. Indeed, suppose κ maps the generators c_i of the edge groups G_{e_i} of E into c^{n_i}, where c is a generator of the edge group of E_1. Denote by s_i an integer such that $n_i s_i = lcm(n_1, \dots, n_m)$ and by τ_i the Dehn twist of G relative to the edge e_i such that $\tau_i : t_i \to c_i^{s_i} t_i$. Then $(\tau_1 \cdots \tau_m)^\kappa$ is a Dehn twist of G_i relative to the edge of E_1, as required. It follows, that some subgroup of finite index of the group of Dehn twists of G_1 relative to E_1 lifts into the group of standard automorphisms of G relative to E.

Lemma 10. *Let G be a f.g. fully residually free group, $\mathcal{K}$ a set of subgroups of G such that G is freely indecomposable modulo $\mathcal{K}$, D an abelian splitting of G modulo $\mathcal{K}$, Q a QH-subgroup in D, and $\mathcal{K}_{D,Q}$ the set of Q-associated subgroups. Let $\kappa : G \to G_1$ be an epimorphism of G onto a fully residually free group G_1 such that Q^κ is not abelian and G does not have an elementary splitting over an element $q \in Q$ with $\kappa(q) = 1$. Then the following holds*:

(1) *if E_1 is a splitting of G_1 as an HNN-extension modulo $\mathcal{K}^\kappa_{D,Q}$ which is obtained from an HNN splitting of a QH-subgroup of G_1 over a non-boundary element, then E_1 lifts in G*;
(2) *if G_1 splits as two intersecting HNN-extensions E_1 and T_1 modulo $\mathcal{K}^\kappa_{D,Q}$ then E_1 and T_1 partially lift into corresponding intersecting HNN-extensions E and T of G modulo $\mathcal{K}_{D,Q}$.*

Proof. We divide the proof into a sequence of claims. The first claim shows how to embed the group G_1 into a new group H_1 with a suitable Lyndon length function.

Claim 1. *Let $G_1 = \langle H, t \mid c_1^t = c_2\rangle$ then there exists a f.g. fully residually free group $\bar{H}$ and an embedding $\theta_0 : H \to \bar{H}$ such that*:

(1) *there exists a free regular length function $\bar{L} : \bar{H} \to \mathbb{Z}^m$, where $\mathbb{Z}^m$ is lexicographically ordered, such that $\bar{L}(c_1) = \bar{L}(c_2)$*;
(2) *the centralizers of $\theta_0(c_1)$ and $\theta_0(c_2)$ in $\bar{H}$ are cyclic and conjugately separated*;
(3) *the monomorphism $\theta_0 : H \to \bar{H}$ extends naturally to a monomorphism*

$$\theta_1 : G_1 = \langle H, t \mid c_1^t = c_2\rangle \to H_1 = \langle \bar{H}, t \mid \theta_0(c_1)^t = \theta_0(c_2)\rangle,$$

where $\theta_1|_H = \theta_0$ and $\theta_1(t) = t$;
(4) *the group H_1 is fully residually free.*

Proof. Such a group H_1 together with the embedding $\theta_1 : G_1 \to H_1$ is constructed in [19, Section 12]. For more details see [18]. $\square$

Corollary 5. *In the notation above there exists a free regular length function $L_1 : H_1 \to \mathbb{Z}^{m+1}$ such that*:

(1) $L_{\max}(t) \neq 0$;
(2) $L_{\max}(h) = 0$ *for every $h \in \bar{H}$,*

where $L_{\max}: H_1 \to \mathbb{Z}^{m+1}/\mathbb{Z}^m \simeq \mathbb{Z}$ *is the length function induced from* $\bar{L}$ *on the leading coordinate of* $\mathbb{Z}^{m+1}$.

Denote by $E_1(H_1)$ the splitting $H_1 = \langle \bar{H}, t \mid \theta_0(c_1)^t = \theta_0(c_2)\rangle$ constructed in Claim 1 from the splitting E_1 of G_1.

Let $S_Q(X, P) = 1$ be the standard quadratic equation associated with the QH-subgroup Q in G. The homomorphism κ gives a solution X^κ of the equation $S_Q(X, P^\kappa) = 1$ in G_1, hence the homomorphism $\kappa\theta_1$ gives a solution $X^{\kappa\theta_1}$ of $S_Q(X, P^{\kappa\theta_1}) = 1$ in H_1. It follows that H_1 canonically embeds into the coordinate group $(H_1)_{R(S_Q)}$. Observe, that κ is a non-degenerate solution of $S_Q(X, P^\kappa) = 1$, i.e., the κ-image of any atom q from S_Q is non-trivial. Indeed, otherwise G would have an elementary splitting over such an atom $q \in Q$ with $\kappa(q) = 1$—contradicting the conditions of the lemma. This implies that $\bar{\kappa} = \kappa\theta_1$ is a non-degenerate solution of $S_Q(X, P^{\kappa\theta_1}) = 1$ in H_1. Now we consider two cases—whether the solution $\bar{\kappa}$ (or, equivalently, κ) is commutative or not.

Case 1. The solution $\bar{\kappa}$ of S_Q in H_1 is non-commutative, i.e., there are two consecutive atoms in S_Q whose $\bar{\kappa}$-images do not commute. In this case there exists an automorphism δ of the free group $F(X \cup P)$ that fixes the elements from P (so δ also can be viewed as an automorphism of $F(X \cup P^{\bar{\kappa}})$), fixes the element S_Q, and such that $X^{\delta\bar{\kappa}}$ is a solution in a general position of $S_Q(X, P^{\bar{\kappa}})$ in H_1 (see [12, Proposition 3]). Clearly, δ gives rise to an H_1-automorphism of the coordinate group $(H_1)_{R(S_Q)}$. Recall, that the quadratic equation $S_Q = 1$ is regular, so the radical $R(S_Q)$ is equal to the normal closure $ncl(S_Q)$ of the element S_Q in $H_1[X]$, hence $(H_1)_{R(S_Q)} = H_1[X]/ncl(S_Q)$. It follows now from Section 3.9 that Q embeds into $(H_1)_{R(S_Q)}$ as a QH-subgroup with the same defining relation (where the boundary elements are replaced by their $\bar{\kappa}$-images) as the QH-subgroup Q in G. We denote this embedding by θ'. Let κ' be the restriction of $\bar{\kappa}$ onto the subgroup generated by the canonical generators of all the stable letters and all the vertex groups in D, except for Q. Then θ' and κ' give rise to a unique homomorphism $\theta: G \to (H_1)_{R(S_Q)}$. Indeed, this follows immediately from the canonical presentations of the fundamental groups of the corresponding graphs of groups. If $\kappa_Q: (H_1)_{R(S_Q)} \to H_1$ is the homomorphism induced by the solution $\bar{\kappa}$ of S_Q in H_1 then $\bar{\kappa}$ is equal to the composition of θ and κ_Q. The claim below follows directly from the construction.

Claim 2. *If the splitting* $E_1(H_1)$ *is the* κ_Q*-image of some elementary splitting* $\bar{E}_1$ *of* $(H_1)_{R(S_Q)}$ *which is induced from a splitting* $\bar{E}_{Q^\theta}$ *of the QH-subgroup* Q^θ *then* E_1 *is* κ*-image of the splitting* E *of* G *which is induced from the isomorphic splitting* $\bar{E}_1^{(\theta')^{-1}}$ *of* Q *in* G.

To show that the splitting E_1 lifts into a splitting $\bar{E}_1$ of $(H_1)_{R(S_Q)}$ of the type described in Claim 2 we need to consider the generalized equations associated with S_Q over H_1. Since H_1 has a regular length function one can construct a finite system of generalized equations $\mathcal{E} = \{\Omega_1, \ldots, \Omega_l\}$ with constants from H_1 such that every solution of the equation $S_Q(X, P^{\bar{\kappa}}) = 1$ in H_1 comes from a corresponding solution of some equation from $\mathcal{E}$ (see [19]). It has been shown in [14, Proposition 8], that there exists an infinite family Φ of P-automorphisms of $F(X \cup P)$ (as well as H_1-automorphisms of $(H_1)_{R(S_Q)}$) such that for every infinite subset $\tilde{\Phi}$ of Φ the set of solutions $\tilde{\Phi}_\kappa = \{\phi\delta\kappa_Q \mid \phi \in \tilde{\Phi}\}$ discriminates $(H_1)_{R(S_Q)}$ into H_1 and it is a generic set of solutions. It follows then, that there is a generalized equation $\Omega \in \mathcal{E}$ such that some infinite subset $\tilde{\Phi}_\kappa$ of Φ_κ goes through Ω. As a consequence, we obtain that a discriminating set of solutions

$$\begin{tikzcd}
E & \bar{E} & E_\Omega \\
G \arrow[r, "\Theta"] \arrow[d, "\kappa"'] & H_{1R(S_Q)} \arrow[loop above, "\phi\in\Phi"] \arrow[r, "\Theta_\Omega"] \arrow[d, bend right, "\kappa_Q"'] \arrow[d, bend left, "\kappa_\phi"] & H_{1R(\Omega)} \arrow[dl, "\kappa_\Omega"] \\
G_1 = \langle H, t \mid c_1^t = c_2 \rangle \arrow[r, "\Theta_1"'] & H_1 = \langle \overline{H}, t \mid c_1^t = c_2 \rangle & \\
E_1 & E_1(H_1) &
\end{tikzcd}$$

Fig. 1.

$\tilde{\Phi}_\kappa$ of S_Q goes through Ω. This implies that the canonical homomorphism $\theta_\Omega : (H_1)_{R(S_Q)} \to (H_1)_{R(\Omega)}$ is an embedding. Fix an arbitrary solution $\kappa_\phi = \phi\delta\kappa_Q \in \tilde{\Phi}_\kappa$ (if $\kappa_Q \in \tilde{\Phi}_\kappa$ then we take $\kappa_\phi = \kappa_Q$) and denote by $\kappa_\Omega : (H_1)_{R(\Omega)} \to H_1$ the corresponding solution of Ω. By construction of Ω one has $\kappa_\phi = \theta_\Omega\kappa_\Omega$ (see Fig. 1).

Claim 3. *The splitting $E_1(H_1)$ lifts (relative to κ_Ω) into the splitting E_Ω of $(H_1)_{R(\Omega)}$ modulo $\mathcal{K}_{D,Q}^{\theta\theta_1}$ with edge groups conjugated into Q^θ in $(H_1)_{R(S_Q)}$ and this splitting induces a lift $\bar{E}$ of $E_1(H_1)$ in $(H_1)_{R(S_Q)}$ relative to κ_Q.*

Proof. The generalized equation Ω has an infinite set of solutions, say $\tilde{\Psi}$ corresponding to the set of solutions $\tilde{\Phi}_\kappa$ of the quadratic equation $S_Q = 1$. Observe, that for each solution $\psi \in \tilde{\Psi}$ there is a base μ in Ω such that μ^ψ is comparable with t in H_1 with respect to the length function L_1 on H_1 from Corollary 5 (otherwise the image of a finite product of bases cannot contain t). We call such a base a "big" base (other bases are called "small"). Now we apply the entire transformation to Ω. By construction the entire transformation on every step reduces the length of a given solution of Ω (with respect to the length function L_1). There are two possible cases for the carrier base:

(a) the carrier base is big;
(b) the carrier base is small.

If the carrier base is small, we cut the initial segment of the whole interval up to the beginning of the left-most big base, making it into a closed section and put this section at the end of the interval (see auxiliary transformation D2 in [14]). After applying these auxiliary transforms the resulting generalized equation is still quadratic, it has the same coordinate group, but has smaller complexity, and it is of the type (a).

In the case when the carrier base is big there are two possibilities. Either the carrier base overlaps, or it does not. If it does not overlap then in finitely many steps it becomes a transfer base and eventually is transferred. Notice that in this case it is transferred inside the active part of the equation (since the constant part is infinitely smaller then t). Moreover, during the transferring the big base the length of the interval becomes smaller by an element of the length comparable to t. Afterward, we are again either in the case (a) or in the case (b). The auxiliary transformations above allow one to return to the case (a). Now, the crucial observation is that one can transfer a big base only finitely many times (which is bounded by the number of occurrences of the letter t

in the normal form of the ψ-image of the whole interval). This shows that in finitely many steps one has to arrive to the case of an overlapping big base. If there are other big bases in this section then in transferring them the length of the big overlapping base decreases by a comparable length, so in finitely many steps it either becomes small or all other bases become small. In the case when the overlapping base gets small we put the section at the end of the generalized equation. Since the complexity decreases in finitely many steps we arrive to the case when the overlapping base is big and all other bases in the section are small.

Thus we may assume that the resulting equation begins with an overlapping pair $ux = xv$, where x is a big base and u is a product of small bases. Hence u^ψ does not contain t, so $u^\psi \in \bar{H}$. It is clear that v has the same length as u (since the equality $ux = xv$ comes from the generalized equation), so $v \in \bar{H}$, as well. Now, the equality $u^\psi x^\psi = x^\psi v^\psi$ holds in the HNN-extension H_1. Since the maximal abelian groups $\langle c_1\rangle$ and $\langle c_2\rangle$ are malnormal and conjugately separated in $\bar{H}$ one can deduce (using the normal form argument) that $x^\psi = ht$ and $u = hc_1^n h^{-1}$ for some natural n and $h \in \bar{H}$. Since there is no cancellation in generalized equations, the equality $ux = xv$ implies that $h \in \langle c_1\rangle$, so $x^\psi = c_1^m t$, $u^\psi = c_1^n$.

We cut the initial segment of the whole interval up to the end of the overlapping pair making it into a closed section and put this section at the end of the interval. If the new active part does not have big bases, then it is mapped into $\bar{H}$, otherwise we transform the generalized equation to obtain another overlapping pair with big carrier base. Since the complexity of the generalized equation decreases, this process will stop.

The quadratic equation corresponding to the whole quadratic part of the generalized equation is equivalent to a quadratic equation $S = 1$ in the standard form (see Section 7 of Chapter 1 in [22]). The group $(H_1)_{R(\Omega)}$ is isomorphic to a free product of a free group and $(H_1)_{R(S_{Q'})}$, where QH-subgroup Q' corresponds to the quadratic equation $S = 1$. Moreover, $(H_1)_{R(S_Q)}$ is embedded into $(H_1)_{R(S_{Q'})}$, therefore, Q^θ is a finite index subgroup in Q' and $(H_1)_{R(S_Q)}$ is a finite index subgroup in $(H_1)_{R(S_{Q'})}$. By the implicit function theorem [14], there exists an $(H_1)_{R(S_Q)}$-homomorphism from $(H_1)_{R(S_{Q'})}$ onto $(H_1)_{R(S_Q)}$. This homomorphism cannot have a non-trivial kernel, therefore $(H_1)_{R(S_{Q'})} = (H_1)_{R(S_Q)}$ and $Q' = Q^\theta$.

Since the set of solutions $\tilde{\Phi}_\kappa$ is generic, element u contains some base λ such that v either contains λ^{-1} or λ. Suppose that v contains λ^{-1}. Then for some v_1 and v_2 $x^{-1}uxv = x^{-1}uxv_1u^{-1}v_2$. Replacing variable x by $x_1 = xv_1$ we obtain $v_1^{-1}[x_1, u^{-1}]v_2 = 1$, where v_1 and v_2 do not contain u and x_1. Therefore, there is a splitting of Q^θ with a relation $u^{x_1} = v_1v_2^{-1}u$. The image of U is c^n, the image of x_1 is $tv_1^{\kappa\Omega}$. Let v_3 be a pre-image of $v_1^{\kappa\Omega}$ in G. Then we can take $x_1v_1v_3^{-1}$ as a stable letter for a splitting of G. It is mapped into $c^m t$. The case when v contains λ can be considered similarly. □

Case 2. The solution κ is commutative.

Suppose, first, that $S_Q = 1$ is orientable. If genus of S_Q is greater than one then there are two consecutive commutators, say $[x_1, y_1]$ and $[x_2, y_2]$ such that their κ-images commute. Since G_1 is fully residually free this implies that $[x_1, y_1]^\kappa = [x_2, y_2]^{\varepsilon\kappa}$ for some $\varepsilon \in \{-1, 1\}$. So (up to the obvious change of variables) $[x_1, y_1]^\kappa[x_2, y_2]^\kappa = 1$. Now, we may assume (again up to an isomorphism of Q that fixes the constants in S_Q) that the equation S_Q can be written in the form $S_Q = [x_1, y_1][x_2, y_2]S_1(X')$, where X' does not contain x_1, x_2, y_1, y_2. Therefore, Q splits over $q = [x_1, y_1][x_2, y_2]$ and $q^\kappa = 1$—contradicting the conditions of the lemma. Hence S_Q has genus at most 1. If genus of S_Q is zero, then Q is generated by the boundary elements p_i

all of whose κ-images commute—contradicting the condition that Q^κ is non-abelian. Suppose now that S_Q has genus one, i.e., S_Q has a commutator $[x, y]$ and c^z as neighbors. It is not hard to see that the map $\gamma : x \to x^{c^z x^{-1}}, y \to (c^z x^{-1})^{-1}, z \to z c^z x^{-1}$ defines an automorphism of $G[x, y, z]$ that fixes the word $c^z[x, y]$. If $\gamma\kappa$-images of the atoms c^z and $[x, y]$ do not commute then one can replace κ with $\gamma\kappa$ and proceed as in Case 1. If $\gamma\kappa$-images of the atoms c^z and $[x, y]$ commute then $\gamma\kappa$-images of $c^{zx^{-1}}$ and c^z also commute, hence (from commutation-transitivity of G_1) $[x^{\gamma\kappa}, (c^z)^{\gamma\kappa}] = 1$. Therefore, $\gamma\kappa$-images of x and $[x, y]$ commute in G_1. This implies that $\gamma\kappa$-images of x and y commute. Indeed, suppose not. Since G_1 is fully residually free one can discriminate $[x^{\gamma\kappa}, y^{\gamma\kappa}]$ into a free group, so the image of this commutator is non-trivial. It is easy to see that the equation $[x, [x, y]] = 1$ has only commutative solutions in free groups—contradiction. This shows that κ-images of x and y, together with the κ-images of all boundary elements p_i pair-wise commute—contradiction with the condition that Q^κ is non-abelian.

If S_Q is non-orientable of positive genus then all squares $(x_i^2)^\kappa$ commute with all the images of the boundary elements p_j^κ, so Q^κ is abelian—contradiction. This finishes Case 2.

We will now prove statement (2). Notice, that if $E_1 : G_1 = \langle H, t \mid c_1^t = c_2\rangle$ and $T_1 : G_1 = \langle H, t_1 \mid c_3^{t_1} = c_4\rangle$ are two intersecting splittings corresponding to splittings of Q_1, then c_1, c_2 written in normal form corresponding to T_1 must contain t_1. Therefore their pre-images under κ in G have to contain pre-images of t_1, and therefore there exist lifts E and T of E_1 and T_1 that intersect. □

4.3. Lifting QH-subgroups

In this section we show how to lift QH-subgroups from a factor-group into the ambient group.

Lemma 11. *Let G be a f.g. fully residually free group, $\mathcal{K}$ a set of subgroups of G, D an abelian splitting of G modulo $\mathcal{K}$, Q a QH-subgroup in D, and $\mathcal{K}_{D,Q}$ the set of Q-associated subgroups. Let $\kappa : G \to G_1$ be an epimorphism of G onto a fully residually free group G_1. Suppose D_1 is an abelian splitting of G_1 modulo $\mathcal{K}^\kappa_{D,Q}$ and Q_1 is a QH-subgroup of D_1. Then there exists a QH-subgroup $\bar{Q}$ of G modulo $\mathcal{K}_{D,Q}$ such that $Q_1 \geqslant \bar{Q}^\kappa$ and $[Q_1 : \bar{Q}^\kappa] < \infty$.*

Proof. *Case* 1. Suppose that G does not have an elementary splitting over an element q such that q belongs to Q and $\kappa(q) = 1$.

We will show now that the pre-image of Q_1 under κ must contain a QH-subgroup. By Lemma 10 we can lift every elementary HNN splitting along an element in Q_1 from G_1 to G. In particular, any two intersecting splittings E_1 and E_2 of Q_1 with edge groups $\langle c_1\rangle$ and $\langle c_2\rangle$ can be lifted to intersecting splittings with edge groups mapped by κ into $\langle c_1\rangle$ and $\langle c_2\rangle$.

Then Q contains a QH-subgroup $\bar{Q}$ that is mapped by κ into Q_1 and not into a normal subgroup generated by the boundary elements. Moreover, the boundaries of $\bar{Q}$ must be mapped to the (conjugates of the) boundaries of Q_1, therefore to the elliptic elements in D_1. By Lemma 7, $\bar{Q}^\kappa$ is a finite index subgroup of Q_1.

Therefore the decomposition D is naturally mapped by κ into the decomposition D_1.

Case 2. Suppose now that κ is transcendental as a homomorphism to G_1. In this case G splits over elements in QH-subgroups that are mapped by κ to the identity. We can first map these elements to the identity and consider the obtained group instead of G. □

4.4. Lifting elementary splittings

Lemma 12. *Let G be a f.g. fully residually free group, $\mathcal{K}$ a set of subgroups of G, D an abelian splitting of G modulo $\mathcal{K}$, Q a QH-subgroup in D, and $\mathcal{K}_{D,Q}$ the set of Q-associated subgroups. Let $\kappa : G \to G_1$ be an epimorphism of G onto a fully residually free group G_1. Then the following holds: if E_1 is an elementary splitting of G_1 modulo $\mathcal{K}^{\kappa}_{D,Q}$ then there exists a splitting E of G modulo $\mathcal{K}_{D,Q}$ such that $E^{\kappa} = E_1$.*

Proof. Let E_1 be an elementary $\mathbb{Z}$-splitting of G_1. Assume first, that E_1 is an HNN-extension $G_1 = \langle H, t \mid c_1^t = c_2\rangle$. In this case there exists a group H_1 and an embedding $\psi : G_1 \to H_1$ such that H_1 has a free regular length function $L_1 : H_1 \to \mathbb{Z}^m$ satisfying properties (1), (3), (4) of Claim 1. Moreover, H_1 is an HNN-extension of some group $\bar{H}$ with the same stable letter t. But in this case H_1 is also the extension of the centralizer of c_1 by an element comparable to t (denote it s).

We use the same scheme as in the proof of Lemma 10 and obtain that $(H)_{1R(\Omega)}$ either has a splitting as an HNN-extension or as an extension of a centralizer of c_1. If $(H_1)_{R(\Omega)}$ splits as an HNN-extension (or multiple HNN-extension) modulo $\mathcal{K}_{D,Q}$ that induces the splitting of $(H_1)^{\kappa}_{R(\Omega)}$ as an HNN-extension with stable letter t, then G splits as an HNN-extension (multiple HNN-extension) with κ-image E_1.

Similarly we can prove the lemma for amalgamated products. □

5. Solutions and quotients

5.1. Minimal solutions and maximal standard quotients

Let G and K be H-groups and $A \leqslant Aut_H(G)$ a group of H-automorphisms of G. Two H-homomorphisms ϕ and ψ from G into K are called *A-equivalent* (symbolically, $\phi \sim_A \psi$) if there exists $\sigma \in A$ such that $\phi = \sigma\psi$ (i.e., $g^{\phi} = g^{\sigma\psi}$ for $g \in G$). Obviously, $\sim_A$ is an equivalence relation on $\mathrm{Hom}_H(G, K)$. By $\sim_{A_D}$ we denote the equivalence relation with respect to $A_D = A_D(G)$. If the splitting D is fixed or it is uniquely determined from the context then we write simply $\sim$ instead of $\sim_{A_D}$.

Let G be an F-group generated by a finite set X (over F) and D a fixed abelian splitting of G. Let $\bar{F} = F(A \cup Y)$ a free group with basis $A \cup Y$ and $\phi_1, \phi_2 \in \mathrm{Hom}_F(G, \bar{F})$. We write $\phi_1 < \phi_2$ if there exists an automorphism $\sigma \in A_D$ and an endomorphism $\pi \in \mathrm{Hom}_F(\bar{F}, \bar{F})$ such that $\phi_2 = \sigma^{-1}\phi_1\pi$ and

$$\sum_{x\in X} |x^{\phi_1}| < \sum_{x\in X} |x^{\phi_2}|.$$

Notice that if $Y = \emptyset$ then $\phi_1 < \phi_2$ if and only if $\phi_1 \sim \phi_2$ and $\sum_{x\in X} |x^{\phi_1}| < \sum_{x\in X} |x^{\phi_2}|$. This provides a natural way to chose representatives in $\sim$-equivalence classes of automorphisms. An F-homomorphism $\phi : G \to F$ is called *minimal* if ϕ is $<$-minimal in its $\sim$-equivalence class. In particular, if $S(X, A) = 1$ is a system of equations over $F = F(A)$ and $G = F_{R(S)}$ then $X \cup A$ is a generating set for G over F. In this event, one can consider *minimal solutions* of $S = 1$ in $\bar{F}$ with respect to X and D (see Fig. 2).

Definition 17. Let D be an abelian splitting of G. Denote by R_D the intersection of the kernels of all minimal (with respect to D) F-homomorphisms from $\mathrm{Hom}_F(G, F)$. Then G/R_D is called

$$\begin{tikzcd} F_{R(S)} \arrow[d, "\phi_2"'] & F_{R(S)} \arrow[l, "\sigma"'] \arrow[d, "\phi_1"] \\ \bar{F} & \bar{F} \arrow[l, "\pi"] \end{tikzcd}$$

Fig. 2. $\phi_1 <_D \phi_2$.

the *maximal standard quotient* of G relative to D and the canonical epimorphism $\eta : G \to G/R_G$ is the *canonical projection*.

Lemma 13. *Let D be a fixed abelian splitting of G. Then every F-homomorphism from G onto F can be presented as composition of a canonical* (*relative to D*) *automorphism of G, the canonical epimorphism $G \to G/R_D$, and an F-homomorphism from G/R_D onto F.*

Lemma 14. *Let G be a finitely generated fully residually free group, D an abelian splitting of G, and $\eta : G \to G/R_D$ the canonical projection of G on its maximal standard quotient. Then the following holds*:

(1) *the restriction of η onto a rigid subgroup of D is a monomorphism*;
(2) *for every abelian vertex group A of D the restriction of η onto the subgroup B of A generated by the images of all the edge groups adjacent to A is a monomorphism.*

Proof. Let H be a rigid subgroup of D. The group G is fully residually F, hence if h is a non-trivial element in H then there exists a homomorphism $\phi \in \mathrm{Hom}(G, F)$ (ϕ is an F-homomorphism, if G is an F-group) such that $h^\phi \neq 1$. Now, there exists a standard (with respect to D) automorphism $\sigma \in A_D$ and a minimal automorphism $\phi_{\min}$ such that $\phi = \sigma\phi_{\min}$. It follows from the definition of the standard automorphisms that σ acts on H by conjugations, say $h^\sigma = g^{-1}hg$ for some $g \in G$. Therefore,

$$h^\phi = h^{\sigma\phi_{\min}} = \left(g^{-1}hg\right)^{\phi_{\min}} = \left(g^{\phi_{\min}}\right)^{-1} h^{\phi_{\min}} g^{\phi_{\min}} \neq 1.$$

Hence $h^{\phi_{\min}} \neq 1$, so $h \notin R_D$, as required in (1).

A similar argument proves (2). □

5.2. Maximal standard quotients relative to a set of homomorphisms

In this section we introduce a relative version of the notion of maximal standard quotients. Let G be an F-group, D a fixed abelian splitting of G, and Φ a subset of $\mathrm{Hom}_F(G, F)$. Similar to Section 5.1 one can introduce minimal homomorphisms in Φ with respect to the splitting D, and the subgroup $R_{D,\Phi}$ of G as the intersection of kernels of the minimal homomorphisms from Φ. The quotient group $G/R_{D,\Phi}$ is called the *maximal standard quotient* of G relative to D and Φ. Notice that the analogs of Lemma 17 still holds in this case, and if Φ is a discriminating set of homomorphisms then Lemma 13 also holds.

5.3. Extended automorphisms and equivalence relations $\sim_{AEQ}$ and $\sim_{MAX}$

We will see later that if D is a cyclic JSJ decomposition of an F-group $G \in \mathcal{F}$, then the maximal standard quotient G/R_D is a proper quotient of G. However, if D is an arbitrary abelian JSJ decomposition of G modulo $\mathcal{K} = \{K_1, \dots, K_n\}$ then G/R_D may be equal to G. In this case it is convenient sometimes to consider suitable extensions of the relation $\sim$ (introduced in Section 5.1).

Let S be an elementary abelian splitting of G modulo $\mathcal{K}$, i.e., $G = A *_C B$ or $G = A*_C = \langle A, t \mid c^t = c',\ c \in C\rangle$ modulo $\mathcal{K}$. Suppose, for certainty, that $F \leqslant A$. Let $\psi : G \to F$ be an F-homomorphism from G into F and $C^\psi \leqslant \langle c_0\rangle$, where $\langle c_0\rangle$ is a maximal abelian subgroup of F. For an arbitrary $d \in \langle c_0\rangle$ we define a homomorphism $\psi_d : G \to F$ as follows. If $G = A *_C B$ then

$$\psi_d(a) = \psi(a) \quad \text{for } a \in A, \qquad \psi_d(b) = \psi(b)^d \quad \text{for } b \in B.$$

If $G = \langle A, t \mid c^t = c',\ c \in C\rangle$ then

$$\psi_d(a) = \psi(a) \quad \text{for } a \in A, \qquad \psi_d(t) = d\psi(t).$$

By $\sim_S$ we denote the following binary relation on $\mathrm{Hom}_F(G, F)$ (in the notation above)

$$\sim_S = \bigl\{(\psi, \psi_d) \bigm| \psi \in \mathrm{Hom}_F(G, F),\ d \in \langle c_0\rangle\bigr\}.$$

Now let D be an abelian JSJ decomposition of G modulo $\mathcal{K}$. Suppose M is an abelian vertex group in D. Then M is a direct product $M = M_1 \times M_2$, where M_1 is the minimal direct summand of M containing all the edge groups of M in D (so the subgroup generated by the edge groups of M has a finite index in M_1). Denote by G' the subgroup of G which is the fundamental group of the splitting D' obtained from D by removing the direct summand M_2 from the vertex M. Clearly, G splits as an extension of centralizer $C_{G'}(M_1)$ of the group G' by M_2. We fix a basis $g_1, \dots, g_s$ of the free abelian group M_2 (if $M_2 \neq 1$).

Now let $\theta : G \to F$ be an F-homomorphism and $M^\theta \leqslant \langle c_0\rangle$, where $\langle c_0\rangle$ is a maximal abelian subgroup of F. Then for every tuple $d = (d_1, \dots, d_s) \in \langle c_0\rangle^s$ the map

$$\theta_d : g_i \to d_i g_i^\theta, \quad i = 1, \dots, s,$$

extends to a homomorphism $\theta_d : M_2 \to F$. Now the restriction of the homomorphism θ on G' and the homomorphism $\theta_d : M_2 \to F$ give rise to a homomorphism $G \to F$ which we define by the same symbol θ_d. We refer, sometimes, to the homomorphism ψ_d and θ_d as obtained from ψ and θ by *extended automorphisms*.

By $\sim_M$ we denote the following binary relation on $\mathrm{Hom}_F(G, F)$ (in the notation above)

$$\sim_M = \bigl\{(\theta, \theta_d) \bigm| \theta \in \mathrm{Hom}_F(G, F),\ d \in \langle c_0\rangle^s\bigr\}.$$

We extend the relation $\sim$ to the equivalence relation $\sim_{AE}$ generated by $\sim$, all the binary relations $\sim_M$ where M runs over all abelian vertex groups in D, and all the binary relations $\sim_S$ where S runs over all elementary splittings of G corresponding to the edges of D.

Now we extend the relation $\sim$ to the equivalence relation $\sim_{AEQ}$ generated by $\sim$, all the binary relations $\sim_M$ where M runs over all abelian vertex groups in D, and all the binary relations $\sim_S$, where S runs over all elementary splittings of G modulo $\mathcal{K}$. It follows from the properties of JSJ decompositions that $\sim_{AEQ}$ is the equivalence relation generated by $\sim$, all the relations $\sim_M$, and all the binary relations $\sim_S$ where S runs over all elementary splittings of G corresponding to the edges of D and also over all elementary splittings of G conjugated into QH-subgroups of D.

We say that two F-homomorphisms $\phi, \psi \in \mathrm{Hom}_F(G, F)$ are *MAX*-equivalent (and write $\phi \sim_{MAX} \psi$) if there exists $\theta \in \mathrm{Hom}_F(G, F)$ such that $\phi \sim_{AEQ} \theta$ and θ coincides up to conjugation with ψ on the fundamental group of every connected component of the graph of groups obtained from D by removing from D all QH-subgroups and the edges adjacent to them.

5.4. *Sufficient splittings*

Let $H \leqslant G$. A family of H-homomorphisms

$$\Psi = \{\psi : G \to H\}$$

is called *separating* if for any non-trivial $g \in G$ there exists $\psi \in \Psi$ such that $\psi(g) \neq 1$ in H.

Definition 18. Let G be a group, $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a set of subgroups of G and F is a fixed subgroup of K_1. We say that there is a *sufficient splitting* of G modulo $\mathcal{K}$ if one of the following holds:

(i) G is freely decomposable modulo $\mathcal{K}$,
(ii) there exists a reduced $\mathcal{K}$-compatible free decomposition of G in which at least one factor G_j has an abelian splitting D_j such that all the subgroups from $\mathcal{K}_j$ (subgroups of G_j that are conjugates of subgroups from $\mathcal{K}$) are elliptic in D_j and such that G_j/R_{D_j} is a proper quotient of G_j (i.e., $R_{D_j} \neq 1$).

If G is freely indecomposable then an abelian splitting D of G is sufficient if and only if the standard maximal quotient G/R_D is a proper quotient of G, i.e., the set of minimal (relative to D) homomorphisms is not separating. Observe also, that in this case G has a sufficient abelian splitting [modulo $\mathcal{K}$] if and only if there exists a sufficient abelian JSJ decomposition of G [modulo $\mathcal{K}$] (in this event every abelian JSJ decomposition of G [modulo $\mathcal{K}$] is sufficient).

5.5. *Maximal standard fully residually free quotients*

Let G be an F-group, generated (as an F-group) by a finite set X, D an abelian splitting of G, and G/R_D the maximal standard quotient of G relative to D. Denote by $\langle X \mid S\rangle$ an F-presentation of G/R_D in generators $X \cup A$ (where $F = F(A)$). Then the coordinate group $F_{R(S)}$ is a quotient of G/R_D. If $S_1 = 1, \ldots, S_k = 1$ are finite systems that determine the irreducible components of the algebraic set $V_F(S)$ then (see [1])

$$R(S) = R(S_1) \cap \cdots \cap R(S_k)$$

and the radicals $R(S_i)$ are uniquely defined (up to reordering). Notice that the coordinate groups $F_{R(S_i)}$ are fully residually free and they are quotients of G/R_D hence of G.

Definition 19. Let G be a finitely generated F-group and D an abelian splitting of G. Then the groups $F_{R(S_i)}$, defined above, are called the *standard maximal fully residually free quotients* of G relative to D. The canonical epimorphism $\eta_i : G \to F_{R(S_i)}$ are called the *canonical projections*.

The following lemma immediately follows from Lemma 13 and the definition of irreducible components of algebraic sets.

Lemma 15. *Let D be a fixed abelian splitting of G. Then every F-homomorphism from G onto F can be presented as composition of a canonical* (*relative to D*) *automorphism of G, a canonical epimorphism $\eta_i : G \to F_{R(S_i)}$ and an F-homomorphism from $F_{R(S_i)}$ onto F.*

Lemma 16. *Let G be a finitely generated fully residually free group and D an abelian splitting of G. Then there exists a standard maximal fully residually free quotient L of G relative to D such that the restriction of the canonical projection $\eta : G \to L$ onto a subgroup H of G is a monomorphism for the following subgroups H*:

(1) *H is a rigid subgroup in D*;
(2) *H is an edge subgroup in D*;
(3) *H is the subgroup of an abelian vertex groups A in D generated by the canonical images in A of the edge groups of the edges of D adjacent to A.*

Proof. Let $\eta_i : G \to L_i$, $i = 1, \ldots, k$, be the canonical projections of G onto the maximal standard fully residually free quotients of G. Suppose that for each i there exists a non-trivial element u_i from a rigid subgroup H_i of D such that $\eta_i(u_i) = 1$. Every F-group discriminated by F is an F-domain (see [1, Sections 1.3, 1.4]), so G as a fully residually free group is an F-domain, hence there exist elements $f_1, \ldots, f_k \in F$ such that $w = [u_1^{f_1}, \ldots, u_k^{f_k}] \neq 1$. Therefore there exists an F-homomorphism $\phi : G \to F$ such that $w^\phi \neq 1$. Now, there is a minimal solution $\phi_0 : G \to F$ and a standard automorphism $\sigma \in A_D$ such that $\phi = \sigma\phi_0$. Observe, that σ acts on elements u_i by conjugation, so

$$w^\phi = \left[\left(u_1^{f_1}\right)^\sigma, \ldots, \left(u_k^{f_k}\right)^\sigma\right] = \left[u_1^{g_1}, \ldots, u_k^{g_k}\right]^{\phi_0} = \left[\left(u_1^{\phi_0}\right)^{g_1^{\phi_0}}, \ldots, \left(u_k^{\phi_0}\right)^{g_k^{\phi_0}}\right]$$

for some $g_1, \ldots, g_k \in G$. Since $w^\phi \neq 1$ this implies that $u_i^{\phi_0} \neq 1$ for each $i = 1, \ldots, k$, and the statement (1) follows.

A similar argument proves (2) and (3). □

5.6. Transcendental solutions

Let $S = 1$ be a consistent irreducible system of equations over F and $K = F_{R(S)}$ the coordinate group of $S = 1$. A solution $\psi : K \to F$ of the system $S = 1$ is called *transcendental* if ψ factors through an F-group L which is a non-trivial free product $L_1 * L_2$ with $F \leqslant L_1$, i.e., there exist F-epimorphisms $\lambda : K \to L_1 * L_2$ and $\gamma : L_1 * L_2 \to F$ with $\psi = \lambda\gamma$.

In this case we sometimes say that ψ *splits* over $L_1 * L_2$. It is easy to see that the transcendental solution ψ of $S = 1$ also splits over $F * F_2$ for some free group F_2. Indeed, if $L_2^\gamma \leqslant F$ is a non-trivial subgroup of F and F_2 an isomorphic copy of L_2^γ with an isomorphism $\mu : L_2^\gamma \to F_2$ then one can define a homomorphism $L_1 * L_2$ onto $F * F_2$ via the restriction $\gamma|_{L_1}$ of γ on L_1

and the composition $\gamma|_{L_2}\mu : L_2 \to F_2$, and then define an F-map from $F * F_2$ onto F using $\mu^{-1} : F_2 \to F$. If $L_2^\gamma = 1$ then one can take an epimorphism $\mu : L_2 \to F_2$ from L_2 onto a non-trivial free group F_2 (since L_2 is residually free and non-trivial), then map $L_1 * L_2$ onto $F * F_2$ using the homomorphisms $\gamma|_{L_1}$ and μ, and afterward send F_2 to 1.

Now we generalize this definition to solutions into arbitrary groups and also consider a relativized version of it.

Let K, K_1 be finitely generated fully residually free groups such that $F \leqslant K_1$ and $F \leqslant H \leqslant K$. An F-homomorphism $\psi : K \to K_1$ is called *transcendental* relative to H if ψ factors through a non-trivial free product $L_1 * L_2$ such that L_1 is mapped to K_1 and (in the notation above) $H^\lambda \leqslant L_1$. Again, the argument similar to the one given above proves the following lemma.

Lemma 17. *Let K, K_1 be finitely generated fully residually free groups such that $F \leqslant H \leqslant K$ and $F \leqslant K_1$. Then an F-homomorphism $\psi : K \to K_1$ is transcendental if and only if ψ splits over $K_1 * K_2$ where K_2 is an isomorphic copy of a non-trivial subgroup of K_1.*

5.7. Systems of reducing quotients

Let H, K be finitely generated fully residually free groups such that $F \leqslant H \leqslant K$ and K does not have a sufficient splitting modulo H. Below for any subset P of K by $R(P)$ we denote the intersection of the kernels of all F-homomorphisms $K \to F$ which send every element of P to 1, i.e., $R(P)$ is the minimal normal subgroup of K containing r and such that $K/R(P)$ is discriminated by F. We refer to $R(P)$ as to the *radical* subgroup generated by P.

Let $K_1 = K/R(P)$ be a quotient of K, $\kappa : K \to K_1$ the canonical epimorphism, and $H_1 = H^\kappa$ the canonical image of H in K_1.

An elementary abelian splitting of K_1 modulo H_1 which does not lift into K is called a *new* splitting.

Definition 20. In the notation above the quotient K_1 is called *reducing* if one of the following holds:

(1) K_1 has a non-trivial free decomposition modulo H_1;
(2) K_1 has a new elementary abelian splitting modulo H_1;
(3) $K_1 = H_1$.

Let D be a JSJ decomposition of a group $K \in \mathcal{F}$. We say that a homomorphism $\phi : K \to K_1$ is *special* if ϕ either maps an edge group of D to the identity or maps a non-abelian vertex group of D to an abelian subgroup.

It is convenient to introduce the following notation. Let G be a f.g. group and $D = \mathcal{G}(X)$ an abelian splitting of G. Denote by $Y_1, \ldots, Y_k$ the connected components of the graph obtained from X by removing all the QH-vertices and all the edges adjacent to them. Let $P_1, \ldots, P_k$ be the fundamental groups of the graphs of groups induced from $\mathcal{G}(X)$ on $Y_1, \ldots, Y_k$. Collapsing every connected component Y_i into a single vertex u_i results in an associated splitting of G which we denote by D_{RA} (RA here is for "rigid" and "abelian"). By $\mathcal{K}_{D,RA}$, or simply by $\mathcal{K}_{RA}$, we denote the set of subgroups $\{P_1, \ldots, P_m\}$ associated with D.

Lemma 18. *Let K be a finitely generated fully residually free group and D an abelian splitting of K. Let $\kappa : K \to K_1$ be a non-special epimorphism of K onto a fully residually free group K_1.*

Suppose D_1 is a JSJ decomposition of K_1 modulo $\mathcal{K}^{\kappa}_{D,RA}$ and Q_1 is a QH-subgroup of D_1. Then there is a QH-subgroup Q of K modulo $\mathcal{K}_{D,RA}$ such that Q^{κ} is a finite index subgroup of Q_1.

Proof. Suppose first that K does not have an elementary splitting over an element q such that q belongs to a QH-subgroup of K modulo $\mathcal{K}_{D,RA}$ and $\kappa(q) = 1$. Notice, that all boundary elements of all QH-subgroups of D belong to subgroups from $\mathcal{K}_{D,RA}$, hence their κ-images are either trivial or elliptic in D_1. By Lemma 7 there is a QH-subgroup Q' in K such that $Q_2 = Q'^{\kappa} \cap Q_1$ is a finite index subgroup of Q_1.

Now we consider the case when κ is transcendental as a homomorphism to K_1. In this case K splits over elements in QH-subgroups that are mapped by κ to the identity. We can first map these elements to the identity and consider the resulting group instead of K. □

Lemma 19. *Let $F \leqslant H \leqslant K$ be finitely generated fully residually free groups such that K does not have a sufficient splitting modulo H. Let $K/R(P)$ be a reducing quotient of K and Φ a set of $\sim$-minimal homomorphisms from K to F that discriminates $K/R(P)$. Then there is a discriminating sub-family Φ_0 of Φ such that for each $\phi \in \Phi_0$ there are infinitely many pair-wise non-$\sim$-equivalent homomorphisms from $K/R(P)$ to F extending $\phi|_H$ and all these extensions together discriminate $K/R(P)$.*

Proof. Let $K_1 = K/R(P)$ and D be a JSJ decomposition of K modulo H, $\kappa : K \to K_1$ the canonical epimorphism, and D_1 a JSJ decomposition of K_1 modulo H^{κ} such that edge groups connecting rigid subgroups are maximal abelian and each edge groups connecting an abelian vertex group A to a rigid vertex group coincides with the isolator in A of the subgroup generated by the edge groups.

The conclusion of the lemma obviously holds if K_1 has type (1). Therefore, if there is a discriminating sub-family Φ_1 of Φ of transcendental solutions, then the conclusion of the lemma holds for Φ_1. Indeed, if there exists a quotient of K_1 that has type (1), then the lemma holds for K_1.

If there is no such a discriminating subfamily, then there is a discriminating subfamily of non-transcendental solutions. Therefore we can suppose that K_1 is of type (2), and no one of the homomorphisms from Φ is transcendental. If the image of some rigid subgroup S of D does not belong to a conjugate of a rigid subgroup in D_1 or to a conjugate of the isolator of a subgroup generated by the edge groups of an abelian subgroup (or the image of an edge group does not belong to a conjugate of an edge group or a rigid subgroup), then the statement of the lemma follows. Using D_1 we can change images of the elements of S infinitely many times such a way that this change is not a simultaneous conjugation. Abelian subgroups of D are mapped either into rigid subgroups or into edge groups. Similarly, if the images of stable letters in D are not stable letters in D_1 and not elliptic elements, the statement follows.

For the rest we suppose that images of rigid subgroups of D belongs to conjugates of rigid subgroups in D_1 or to conjugates of the isolators of subgroups generated by edge groups of abelian subgroups and images of edge groups belong to conjugates of edge groups or rigid subgroups. In particular, the lemma is true for the case when D does not have QH-subgroups. So we suppose now that D has QH-subgroups. In this case all the splittings of QH-subgroups of D_1 are induced from splittings of QH-subgroups of K by Lemma 18. All the other splitting are also induced from splittings of K. The lemma is proved. □

A system of reducing quotients $\mathcal{R} = \{K/R(r_1), \ldots, K/R(r_s)\}$ of K is called a *complete reducing* system if each quotient is discriminated by $\sim$-minimal homomorphisms and each homomorphism from K onto F that factors through a reducing quotient is $\sim_{MAX}$-equivalent to a homomorphism that factors through one of the quotients in $\mathcal{R}$. Given a finite complete reducing system $\mathcal{R}$ one can delete from $\mathcal{R}$ all the groups which are proper quotients of other groups from $\mathcal{R}$. For simplicity we will always assume that a complete reducing system $\mathcal{R}$ is minimal, i.e., no factor from $\mathcal{R}$ is a proper quotient of another factor.

Lemma 20. *Let $F \leqslant H \leqslant K$ be finitely generated fully residually free groups such that K does not have a sufficient splitting modulo H. Then there exists a finite complete reducing system for K. Moreover, such a system can be found effectively.*

Proof. Let $Sol_{\min}$ be the set of all minimal solutions from $\psi : K \to F$ that factors through at least one reducing factor of K.

For each reducing quotient $K_1 = F_{R(S_1)}$ of K there is a discriminating family of homomorphisms Φ such that for each homomorphism $\phi \in \Phi$ the restriction $\phi : H \to F$ by Lemma 19 can be extended by infinitely many ways to such solutions, moreover, for almost all solutions from Φ (except for solutions satisfying a fixed proper equation) the restriction $\phi|_H$ can be extended by infinitely many ways to $\sim$-minimal solutions. Considering the elimination process for the set $Sol_{\min}$ modulo H, we obtain a finite number of proper quotients of K that have new splitting modulo H: $\mathcal{R} = \{K/R(r_1), \ldots, K/R(r_s)\}$, a finite number of proper quotients of K that do not have new splittings modulo H and finite number of terminal vertices when the generalized equations are transferred to the parametric part corresponding to H. There is a generic family of solutions for each K_1 that corresponds to branches that end with quotients from $\mathcal{R}$. One can choose generic families such a way that all solutions in the families (except for a finite number) are reducing. Therefore by the parametrization theorem each solution of S_1 is reducing. Each quotient from $\mathcal{R}$ either has a non-trivial free decomposition modulo H or has an elementary abelian splitting modulo H which is not inherited from a splitting of K. Therefore, we can consider the system consisting of the family $\mathcal{R}$ and quotients $K/R(r_i)$ of the second type as the canonical reducing system. $\square$

5.8. *Algebraic and reducing solutions*

Let $F \leqslant H \leqslant K$ be finitely generated fully residually free groups such that $K = F_{R(S)}$ does not have a sufficient splitting modulo H and $\mathcal{R} = \{K/R(r_1), \ldots, K/R(r_s)\}$ a complete reducing system for K.

Now we define algebraic and reducing solutions of $S = 1$ in F with respect to $\mathcal{R}$. Let $\phi : H \to F$ be a fixed F-homomorphism and Sol_ϕ the set of all homomorphisms from K onto F which extend ϕ. A solution $\psi \in Sol_\phi$ is called *reducing* if there exists a solution $\psi' \in Sol_\phi$ in the $\sim_{MAX}$-equivalence class of ψ which satisfies one of the equations $r_1 = 1, \ldots, r_k = 1$. All non-reducing non-special solutions from Sol_ϕ are called *K-algebraic* (modulo H and ϕ). We denote the set of all K-algebraic solutions from Sol_ϕ by $\mathcal{A}lg_\phi$. A reducing solution that is $\sim$-equivalent to a solution that satisfies one of the equations $r_1 = 1, \ldots, r_k = 1$ is called *strongly reducing*.

Note, that in Section 9 and further we will call a solution algebraic if it is not *MAX*-equivalent to a solution that factor through a corrective extension of a canonical NTQ group for a reducing quotient from $\mathcal{R}$.

6. Finiteness theorem for groups with no sufficient splitting

Finiteness results are very important because we are going to deal with an iterative process of verification of sentences. We will work with equations with different families of variables bounded by different quantifiers. Some variables will be also considered as parameters.

6.1. Formulation of the results and proof of the existence of a bound

Here we consider a simple situation of the type described above. Suppose we have two irreducible systems of equations $T(\hat{X}) = 1$ and $S(\hat{X}, \hat{Y}) = 1$ with the coordinate groups H and K such that the identical map $\hat{X} \to \hat{X}$ extends to an F-monomorphism $H \to K$. In this section we show that there is a constant N such that for every solution of $T(\hat{X}) = 1$ there are at most N equivalence classes (with respect to the relation $\sim_{AEQ}$, and, therefore, with respect to $\sim_{MAX}$) of algebraic solutions of $S(\hat{X}, \hat{Y}) = 1$.

Theorem 11. *Let H, K be finitely generated fully residually free groups such that $F \leqslant H \leqslant K$ and K does not have a sufficient splitting modulo H. Let D be an abelian JSJ decomposition of K modulo H (which may be trivial). There exists a constant $N = N(K, H)$ such that for each F-homomorphism $\phi : H \to F$ there are at most N algebraic pair-wise non-equivalent with respect to $\sim_{AEQ}$ and, therefore, with respect to $\sim_{MAX}$, homomorphisms from K to F that extend ϕ.*

Moreover, the constant N for the number of $\sim_{MAX}$-non-equivalent homomorphisms can be found effectively.

Proof. In the proof we extensively use the technique of generalized equations described in [14, Sections 4.3 and 5]. The reader has to be familiar with these sections of [14].

Let a group K with a subgroup $H \leqslant K$ be a counterexample to the statement of the theorem about *MAX*-equivalence classes. Denote by X_H a finite generating set of H and X_K a finite set of elements of K such that $X_H \cup X_K$ generates K. Let $S(X_H \cup X_K)$ be finite set of relators of K with respect to the set of generators $X_H \cup X_K$. Then $K = F_R(S)$. Then for every $m \in \mathbb{N}$ there exists an F-homomorphism $\phi_m : H \to F$ and m minimal non-*MAX*-equivalent algebraic solutions $\psi_m^{(1)}, \ldots, \psi_m^{(m)} : K \to F$ of $S = 1$ extending ϕ_m. The sequence $\gamma = \{\phi_m\}_{m \in \mathbb{N}}$ satisfies the following property:

(∗) For every $N \in \mathbb{N}$ there exists a homomorphism $\phi \in \gamma$ and minimal algebraic non-*MAX*-equivalent solutions $\psi_1, \ldots, \psi_N : K \to F$ of $S = 1$ extending ϕ.

Sequences of solutions that satisfy condition (∗) are called *growing sequences*. Observe from the construction that any infinite subsequence of the sequence γ is a growing sequence.

Let $\mathcal{G}E(S)$ be the finite set of generalized equations corresponding to the equation $S = 1$. Each generalized equation $\Omega \in \mathcal{G}E(S)$ can be viewed as a parametric generalized equation with H-bases as parameters. It follows that there exists an equation $\Omega \in \mathcal{G}E(S)$ and a finite subsequence γ_Ω of the sequence γ such that every solution from γ^Ω factors through the group $F_{R(\Omega)}$ (with respect to the canonical F-homomorphism $\kappa : K \to F_{R(\Omega)}$). Moreover, since the group K is fully residually free, as well as the groups $F_{R(\Omega)}$, $\Omega \in \mathcal{G}E(S)$, one may assume that for the chosen equation Ω the homomorphism κ is a monomorphism. After renaming homomorphism in γ_Ω and, perhaps taking a subsequence, we may assume for simplicity that $\gamma_\Omega = \{\phi_m\}_{m \in \mathbb{N}}$ and for every m there exist minimal non-*MAX*-equivalent algebraic solutions $\psi_m^{(1)}, \ldots, \psi_m^{(m)} : K \to F$

of $S = 1$ extending ϕ_m. For each minimal solution $\psi_m^{(i)}$ we associate a fixed solution $\theta_m^{(i)}$ of Ω which is minimal possible for $\psi_m^{(i)}$. We consider only those generalized equations Ω for which $K \leqslant F_{R(\Omega)}$ and $K \leqslant \bar{F}_{R(\Omega)}$, where $\bar{F}_{R(\Omega)}$ is a quotient of $F_{R(\Omega)}$ discriminated by the family $\Theta = \{\theta_m^{(i)},\ i = 1, \ldots, m,\ m \in \mathbb{N}\}$. Generalized equations, for which K is not embedded into $\bar{F}_{R(\Omega)}$ result some of the equations from $\mathcal{R}$. We construct the tree $T(\Omega, \Theta)$ modulo H, namely, with H-bases as parameters as described in [14, Section 5.3].

We will show now that $T(\Omega, \Theta)$ is a finite tree. Solutions of Ω that belong to Θ are obtained from minimal solutions of Ω with respect to the group of automorphisms corresponding to the quadratic part by a bounded number of these automorphisms (the boundary depends only on Ω). Indeed, the group of automorphisms of a closed connected quadratic section of Ω is a group of automorphisms of a free product of a free group and a group of quadratic equation with coefficients being products of constant bases (for the quadratic part). The group of automorphisms corresponding to a QH-subgroup of K modulo H has finite index in the group of automorphisms of this quadratic equation. If a splitting in the coordinate group of Ω that is hyperbolic with respect to some other splitting does not intersect non-trivially any of the QH-subgroups of K modulo H, then we can always take a solution of Ω minimal with respect to the group of automorphisms corresponding to the QH-subgroup of $F_{R(\Omega)}$ modulo H containing the edge group of this splitting. Therefore Case 12 in the construction of $T(\Omega, \Theta)$ (see [14, Section 5.3]) can appear only a bounded number of times. We can also suppose that all periodic structures in vertices of type (2) are non-singular (see [14, Section 5.4]) and suppose that Θ contains only solutions of Ω corresponding to periodic structures with the set $C^{(2)}$ empty and $BT = BT_0$ (see [14, Lemma 27]). Therefore Case 15 [14, Section 5.3] can only appear a bounded number of times. If, reducing (Ω, Θ) to the terminal equations we obtain some free variables because the boundary between h_i and h_{i+1} does not touch any base, we can consider $\ker(\Omega)$ instead of Ω. Indeed, in this case $\bar{F}_{R(\Omega)}$ is the quotient of $F_{R(\ker(\Omega))}$. Therefore Cases 7–10 [14, Section 5.3] can only appear a bounded number of times in the subtree of $T(\Omega, \Theta)$ corresponding to the actual paths for a minimal solution. Hence the tree $T(\Omega, \Theta)$ is finite. Free variables in the generalized equations corresponding to vertices of this tree cannot appear because K is freely indecomposable modulo H.

In the leaf vertices v of $T(\Omega, \Theta)$ we obtain generalized equations of three types:

(1) generalized equations Ω_v with intervals labeled by generators h_t of H;
(2) generalized equations Ω_w such that $F_{R(\Omega_w)}$ is a proper quotient of $F_{R(\Omega)}$, and the image of K in $\bar{F}_{R(\Omega_w)}$ is a proper quotient of K;
(3) generalized equations Ω_w such that $F_{R(\Omega_w)}$ is a proper quotient of $F_{R(\Omega)}$, and the image of K in $\bar{F}_{R(\Omega_w)}$ is isomorphic to K.

If we have case (3) we apply the leaf-extension transformation [14, Section 5.6] at this leaf vertex, and again only consider minimal solutions of Ω_w with respect to the groups of automorphisms. The number of times when we have case (3) is finite. Therefore after a finite number of steps we end up with case (1) or (2). In case (2) we obtain an equation from the family $\mathcal{R}$.

Consider now case (1). Take one terminal generalized equation such that its coordinate group contains K as a subgroup and the intervals labeled by generators h_t of H. Denote this generalized equation by Ω_v. We will show that the number of distinct solutions of $S = 1$ is bounded for each value of h_t by some number depending only on Ω_v. Let $\bar{F}_{R(\Omega)}$ be the quotient discriminated by solutions of Ω_v that belong to Θ. Construct a cut equation Π from Ω_v (see [14, Section 5.7]). The intervals of Π are labeled by values of h_t. This cut equations has the set of intervals $\mathcal{E}$, the set of variables M, the set of parameters $\bar{H} = \{h_t\}$ that correspond to generators of H. For each solution $\beta = \phi : H \to F$ there is an F-homomorphism $\alpha : F[M] \to F$ induced by a solution of Ω_v and every α induces a solution of Ω_v which induces a solution of Ω and of $S(\hat{X}, \hat{Y}) = 1$. Different solutions of $S = 1$ correspond to different solutions of Π. Therefore there is a sequence of homomorphisms $\{\phi_m\}$, where $\phi_m : H \to F$, such that for each ϕ_m there are al least m distinct solutions of the cut equation $\alpha_m^{(i)}$, $i = 1, \dots, m$.

Let $\stackrel{\circ}{=}$ denote graphical equality. We take a copy of the set M and denote it by $\hat{M}$. For each interval $\sigma \in \mathcal{E}$ such that $f_X(\sigma) = h_t$ and $f_M(\sigma) = h_t(y_1, \dots, y_n)$, $y_i \in M$, we consider a graphic equation in the form

$$h_t(x_1, \dots, x_n) \stackrel{\circ}{=} h_t(y_1, \dots, y_n), \tag{11}$$

where $x_1, \dots, x_n$ are copies of $y_1, \dots, y_n$ in $\hat{M}$, and consider the system W of all such graphic equations for all $\sigma \in \mathcal{E}$. Each pair of n-tuples $x_i = y_i^{\alpha_m^{(j)}} = y_{im}^{(j)}$, $y_i = y_i^{\alpha_m^{(k)}} = y_{im}^{(k)}$, $j, k = 1, \dots, m$, $i = 1, \dots, n$, is a solution of W. Let $\bar{F}_{R(W)}$ be the subgroup discriminated by all such solutions of $W = 1$ corresponding to all solutions of $S = 1$ from the set $\{\psi_m^{(j)}, j = 1, \dots, m, m \in \mathbb{N}\}$ (we call this set a *growing sequence*).

For each $h_{tm} = h_t^{\phi_m}$ there are at least m distinct solutions of $S = 1$. They induce at least m distinct solutions of Π: $(y_{1m}^{(j)}, \dots, y_{nm}^{(j)})$, $j = 1, \dots, m$. Let $z_\delta^{(jk)}$ be a word joining a boundary δ which is a boundary of some $y_{tm}^{(j)}$ with the corresponding boundary of $y_{tm}^{(k)}$. Consider the family of all such words, and let $Z = \{z_\delta\}$ be the corresponding family of variables. Then we can rewrite the system W of graphic equations in variables $x_1, \dots, x_n$ and Z. There are equations that say that different expressions of y_i in terms of x_i and variables from Z are the same. Denote an irreducible system of equations in variables $X = \{x_1, \dots, x_n\}$ and Z discriminated by solutions $\{y_{tm}^{(j)}, z_\delta^{(jk)}\}$ of the system W by $\bar{W} = 1$. Then $F_{R(\bar{W})}$ is isomorphic to $\bar{F}_{R(W)}$. Let M_m be the sum of length of all h_{tm}. One can choose a growing subsequence $\{h_{t\ell}, y_{i\ell}^{(j)},\ j = 1, \dots, \ell,\ i = 1, \dots, n,\ \ell \in \mathbb{N}\}$ such that each ratio $|y_{i\ell}^{j}|/M_\ell$ approaches a fixed number between 0 and 1. Indeed, in each bounded sequence $\{|y_{im}^{(j_m)}|/M_m\}$ there is a convergent subsequence $\{|y_{i\ell}^{(j_\ell)}|/M_\ell\}$ and the set of limit points of the infinitely many such subsequences is a bounded set. Therefore infinitely many subsequences converge to the same limit point. Consider such subsequence instead of the original sequence of solutions. We call a variable from $X \cup Z$ *short* if it becomes infinitely small comparatively to M_m as m approaches infinity. Then $X = X_1 \cup X_2$, where variables from X_2 are short for this subsequence, and for each variable $x \in X_1$ the value $\frac{|x|}{M_m} > \lambda$ for some fixed number λ. We will call variables from X_1 *long*. All variables from Z are short. Below we will always consider a suitable subsequence instead of the initial sequence.

Notice, that each y_i must appear at least twice in Eq. (11), otherwise we can just remove the equation containing y_i from the system W. Suppose y_i occurs at least twice. Let δ_1, δ_2 be the initial and terminal boundaries for the first occurrence, δ_3, δ_4 be the initial and terminal boundaries for the second occurrence, $z_i = z_{\delta_i}$, $i = 1, 2, 3, 4$. We have equations $y_i = z_1 x_i z_2$, $y_i = z_3 x_i z_4$. These equations imply

$$x_i^{-1} z_3^{-1} z_1 x_i = z_4 z_2^{-1}. \tag{12}$$

Suppose x_i is long, we know that z_1, z_2, z_3, z_4 are short. For each i we have $x_{i,m} = A(i,m)^{r_m} A_1(i,m)$, where $A(i,m) = A_1(i,m) A_2(i,m)$ and $z_3^{-1} z_1 = A(i,m)^t$ for some word

$A(i,m)$. Each x_i can occur more than two times. If it occurs p times, then there is a tuple of such numbers $t = (t_1, \dots, t_{p-1})$.

There are two cases.

(a) If t is bounded, then we can partition a sequence into a finite number of subsequences such that for each of them $z_3 = z_1$, $z_4 = z_2$, and $y_i = z_1 x_i z_2$ for any occurrence of y_i. The maximal difference between beginnings (and ends) of $y_{i,m}^{(j)}$ and $y_{i,m}^{(k)}$ for $x_i \in X_1$ is short. Therefore, for all m, and for each occurrence of x_i in h_{tm}, the interval labeled by h_{tm} contains a (maximal) subinterval covered by all $y_i^{(j)}, y_i^{(k)}$ corresponding to this occurrence, for all j. Remove all such subintervals from all h_{tm} for all m. Removing one such subinterval we obtain two new intervals. Denote by h'_{sm} the labels of new intervals.

(b) If t is unbounded for some long variable x_i, we introduce a new variable d_i (called a *period*) and the sequence of solutions discriminates a group generated by X, Z, d_i which is an extension of a centralizer of an element d_i with the subgroup generated by X and d_i being elliptic.

Suppose we have case (a). The maximal difference between beginnings (and ends) of $y_{i,m}^{(j)}$ and $y_{i,m}^{(k)}$ for $x_i \in X_1$ is short. Therefore, for all m, and for each occurrence of x_i in h_{tm}, the interval h_{tm} contains a maximal subinterval covered by all $y_{i,m}^{(j)}, y_{i,m}^{(k)}$ corresponding to this occurrence, for all j, k. Remove all such subintervals from all h_{tm} for all m. Removing one such subinterval we obtain two new intervals. Denote by h'_{sm} the new intervals. Define the same way the notion of second order long and short variables. In addition to variables, the new intervals contain also the pieces of the long variables from the previous step: at most two pieces for each long variable. We can write a system of equations for the new intervals in terms of short variables and variables from Z. In the same way we can show that for a suitable subsequence the position of the second order long variables is fixed on the intervals h'_{sm} modulo the second order short differences. Eventually we will have a situation such that there are no kth order long variables. Taking a suitable subsequence we can suppose that one of the short variables x_r of kth order is maximal. Then there exists a variable $z \in Z$ such that for each m and j the inequality $m|y_{r,m}^{(j)}| < |z_m|$ holds for the values on the mth step of the sequence. One can partition the set $X = \{x_1, \dots, x_n\} = X_{\text{short}} \cup X_{\text{long}}$, where X_{short} are variables from X short on step k, and X_{long} are all the other variables. One can always find a subsequence of solutions such that values of variables from Z are infinitely small in comparison with variables from X_{long} and grow as $m \times |x|$, where x is the maximal variable from X_{short}. As above we denote by $\bar{W} = 1$ an irreducible system of equations in variables X and Z discriminated by these solutions. If on some step we had case (b), we still can construct this systems $\bar{W} = 1$ but it will be a system in variables X, Z and periods d_i's.

The system $\bar{W} = 1$ written in variables $X \cup Z$ and periods consists of two parts: the system in variables X and periods and the system in variables $Z \cup X_{\text{short}}$ and periods. Let $Y = \{y_1, \dots, y_n\}$ and $\hat{X}_Y, \hat{Y}_Y$ be the natural image of the generators $\hat{X}, \hat{Y}$ of the group K in the subgroup $\langle Y \rangle$ of $F_{R(\bar{W})}$. If we consider the process of transformation of the generalized equation Ω to Ω_v in the opposite direction (we suppose that this process is the same for all solutions of the sequence that we are considering, otherwise we can go to a subsequence), we can see that specializations of elements $\hat{X}_Y, \hat{Y}_Y$ can be expressed without cancellation in terms of specializations of elements X and Z. Indeed, the specializations of $\hat{X}, \hat{Y}$ were cut into maximal cancellable pieces (bases) according to the cancellation tree, and then these pieces were further cut in the following very special way. On each step there were two bases λ, μ such that $\mu = \lambda\mu_1$ and the transformation of this step replaced μ and $\Delta(\mu)$ by μ_1 and $\Delta(\mu_1)$ and did not change any other bases. Suppose $v \in \hat{X}_Y \cup \hat{Y}_Y$, then v has the form

$$v = z_{i_1}x_{i_1}z_{i_2}z_{j_2}x_{i_2}\cdots x_{i_{m-1}}z_{i_{m-1}}z_{j_{m-1}}x_{i_m}z_{i_m},$$

where $x_{i_1},\ldots,x_{i_m} \in X^{\pm 1}$ and $z_{i_1},\ldots,z_{i_m},z_{j_2},\ldots,z_{j_{m-1}} \in Z^{\pm 1}$. Denote by $\hat{Z}$ the set of these products $z_{i_1}, z_{i_2}z_{j_2},\ldots,z_{i_{m-1}}z_{j_{m-1}}, z_m$ which participate in representation of elements $\hat{X}, \hat{Y}$. Either $z_{i_k}z_{j_k} = 1$ or there is no cancellation in the specialization $z_{i_k}z_{j_k}$ because such specialization corresponds to a label of a path in the cancellation tree.

Denote the subgroup of $F_{R(\bar{W})}$ generated by X, $\hat{Z}$ and periods by $F_{R(U)}$ and the corresponding system of equations by $U = 1$. Denote the subgroup generated by X_{short}, $\hat{Z}$ and periods by $F_{R(P)}$ and the corresponding system of equations by P.

If $F_{R(P)}$ is a non-trivial free product modulo the subgroup generated by X_{short} and periods, then homomorphisms from the set $\{\psi_m^{(j)},\ j = 1,\ldots,m,\ m \in \mathbb{N}\}$ can be represented as compositions of the form $\psi = \lambda\gamma$, where λ is an epimorphism from K onto $F * F_1$, where F_1 is a free group, sending H onto F, and γ is an F-homomorphism from $F * F_1$ onto F (specialization). All such homomorphisms discriminate a proper quotient of K, relations corresponding to maximal fully residually free quotients corresponding to this proper quotient we include into the set $\mathcal{R}$.

Therefore, we suppose that $F_{R(P)}$ is freely indecomposable modulo the subgroup generated by X_{short} and periods. The group $F_{R(P)}$ has a splitting modulo the subgroup generated by X_{short} and periods because otherwise the length of variables $\hat{Z}$ would be bounded by a function of $|x|$, $x \in X_{\text{short}}$. This means that there is a splitting of $F_{R(U)}$ modulo the subgroup generated by $x_1,\ldots,x_n$ and periods and the length of variables $\hat{Z}$ grows because they obtained by an application of Dehn twists along this splitting.

Let $\bar{K}$ be the image of K in $F_{R(U)}$ and $\bar{H}$ be the image of H. Let D be a JSJ decomposition of $F_{R(U)}$ modulo $\langle X, d_i, i \in I\rangle$. Notice that in the case when $F_{R(U)}$ has a splitting modulo $\langle X, d_i, i \in I\rangle$ that induces a splitting of the image of one of the rigid subgroups of K (the image of one of the rigid subgroups does not belong to a rigid subgroup of D), then the initial sequence of solutions must be reducing. Therefore, the images of the rigid subgroups of K belong (up to conjugation) to rigid subgroups in D or to isolators of the subgroups generated by the edge groups in abelian vertex groups. Similarly, the images of the edge groups have to belong to edge subgroups or to rigid subgroups.

Suppose now that the JSJ decomposition of K modulo H has QH-subgroups. Suppose also that K is a counterexample of minimal regular size to the statement of the theorem for *MAX*-equivalence classes. Denote the image of K in $F_{R(U)}$ by $\bar{K}$. If a sequence of homomorphisms from $F_{R(U)}$ to F constructed from Θ (denote it Θ_1) contains a growing subsequence of transcendental homomorphisms, then Θ contains a growing subsequence of transcendental homomorphisms. We suppose now that all homomorphisms from Θ_1 are not transcendental. Let D be a JSJ decomposition of $F_{R(U)}$ modulo $\langle X\rangle$. It must be non-trivial. Then $\bar{K}$ cannot be conjugated into the fundamental group of a proper subgraph of D, because K is generated by $\hat{Z}$ and $h_t \in X$. Since the family of homomorphisms Θ is algebraic, rigid subgroups and edge groups of K must be mapped into either rigid subgroups of D or edge groups or isolators of subgroups generated by the edge groups in abelian vertex groups. Therefore each QH-subgroup of D contains conjugates of subgroups of QH vertex groups of K as subgroups of finite index. All the splittings of $\bar{K}$ that are induced from D are, therefore, induced from splittings of K. Therefore, $F_{R(U)}$ does not have a sufficient splitting modulo $\langle X\rangle$.

If the sequence Θ_1 contains a growing subsequence of $F_{R(U)}$-algebraic solutions, then we suppose that Θ_1 coincides with this subsequence. Otherwise we take instead of Θ_1 a growing subsequence of reducing solutions corresponding to a fixed reducing quotient of $F_{R(U)}$ (denote it by T). This is indeed a growing sequence, because it consists of solutions from different *MAX*-classes for T and corresponding to solutions in Θ from different *MAX*-classes for K. Now we apply the same procedure to T until we obtain a quotient B of $F_{R(U)}$ that has a splitting but not a sufficient splitting modulo $\langle X\rangle$ and such that there is a growing sequence of algebraic not-*MAX*-equivalent homomorphisms from B into a free group. The size of the quadratic system corresponding to B cannot be less than the size of the one corresponding to K because K is a counterexample of minimal size. Therefore K has the same QH-subgroups as the JSJ decomposition D_B for B. If D_B has less number of edges between rigid subgroups, then we can apply induction on the number of these edges. If D_B has the same number of such edges as K, then D has the same number of such edges and the same QH-subgroups as K. This contradicts the fact that the length of variables $\hat{Z}$ grows because they obtained by an application of Dehn twists along some splitting of $F_{R(U)}$ modulo $\langle X\rangle$. This proves the existence of the number N for *MAX*-classes in Theorem 11.

We will now prove the existence of the boundary for the number of *AEQ*-classes. Let a group K with a subgroup $H \leqslant K$ be a counterexample of minimal size to the statement of the theorem about *AEQ*-equivalence classes.

It is enough to prove that the number of *AEQ*-classes is bounded for the case when H is the fundamental group of the graph of groups obtained from D by deleting QH vertices and edges adjacent to them. The reduction to this case is as follows. By Lemma 8 Dehn twists along the edges of D commute with canonical automorphisms corresponding to QH-subgroups. Similarly, if $\psi : K \to F$ is a homomorphism, then the homomorphisms ψ_{ab} and ψ_{ba} obtained by extended automorphisms are the same for a belonging to an edge group and b belonging to a QH-subgroup. Let $Y_1, \ldots, Y_k$ be connected components the graph of groups obtained from D by deleting QH vertices and edges adjacent to them and $H_1, \ldots, H_k$ be the fundamental groups of these connected components, where $F \leqslant H_1$. Let $\bar{H} = \langle H_1, \ldots, H_k\rangle$ and $\bar{K}$ be the fundamental group of the graph of groups $\bar{D}$ with one vertex with vertex group $\bar{H}$ and other vertices and edges being QH vertices of D and edges adjacent to them ($\bar{K} = K * F(t_2, \ldots, t_k)$, where $t_2, \ldots, t_k$ correspond to new stable letters). Clearly, $\bar{K}$ does not have a sufficient splitting modulo $\bar{H}$ and has the same size as K. Therefore, below we suppose that H is the fundamental group of the graph of groups obtained from D by deleting QH vertices and edges adjacent to them.

Denote by X_H a finite generating set of H and X_K a finite set of elements of K such that $X_H \cup X_K$ generates K. Let $S(X_H \cup X_K)$ be finite set of relators of K with respect to the set of generators $X_H \cup X_K$. Then for every $m \in \mathbb{N}$ there exists an F-homomorphism $\phi_m : H \to F$, and homomorphisms $\phi_{i,m} : H_i \to F$ $(i = 2, \ldots, k)$ and m minimal non-*AEQ*-equivalent algebraic modulo H solutions $\psi_m^{(1)}, \ldots, \psi_m^{(m)} : K \to F$ of $S = 1$ extending ϕ_m.

The same way as we did in the proof of the statement for *MAX*-equivalence classes we can construct the group $F_{R(U)}$ that has a splitting modulo the subgroup generated by $x_1, \ldots, x_n$ and periods. Let T be the terminal group T of the fundamental sequence for $F_{R(U)}$ modulo the subgroup generated by $x_1, \ldots, x_n$ and periods such that a growing subsequence of homomorphisms from $F_{R(U)}$ to F factors through this fundamental sequence. Either there is a splitting of T as a free product modulo the subgroup generated by $x_1, \ldots, x_n$ and periods or there exists a splitting of the image of a rigid subgroup of $F_{R(U)}$ in T. All solutions of $U = 1$ from the growing sequence belong to a bounded number of *MAX*-classes. Indeed, all splittings of all the reducing quotients of T that are freely indecomposable modulo the image of the subgroup generated by $x_1, \ldots, x_n$ and periods correspond to splittings of QH-subgroups of D (by Lemmas 11 and 10), and solutions of $U = 1$ are constructed from *AEQ*-minimal solutions of $S = 1$. Therefore there is a growing sequence of solutions of $U = 1$ modulo the subgroup of T generated by all non-QH-subgroups and denoted by $\hat{H}$. Since T modulo this subgroup does not have a reducing quotient that is freely indecomposable modulo $\hat{H}$, T modulo $\hat{H}$ has a growing sequence of *AEQ*-non-equivalent algebraic homomorphisms into F. Since K was a counterexample of minimal size, the size of T and, therefore, the size of $F_{R(U)}$ has to be the same as the size of K. Therefore $F_{R(U)}$ and T have the same QH-subgroups as K. Since K cannot be conjugated into a fundamental group of a proper subgraph of groups of the graph of groups corresponding to T, this implies that the length of variables from $\hat{Z}$ grows because the solutions are obtained by the application of the increasing number of Dehn twists along the splitting of $F_{R(U)}$ corresponding to QH-subgroups, contradiction. □

6.2. *Effectiveness*

Now we will show how to find the boundary for the number of *MAX*-equivalence classes effectively. One can bound effectively the tuple t using the first part of Lemma 27 from [14]. After t is bounded, the rest of the proof of Theorem 11 is effective.

6.3. *Examples and counterexamples*

In this section we illustrate Theorem 11 on some simple examples. We show also that the theorem fails if $\sim_{AEQ}$-equivalence is replaced by $\sim$-equivalence, or even by $\sim_{AE}$-equivalence.

Example 1 *(One AEQ-class, no uniform bounds on the number of $\sim$-classes).* Let $K = F[x, y]$ (so $S(x, y) = 1$ is the trivial equation $1 = 1$) and $H = \langle x, x^y\rangle * F$. Then $N = 1$ and there is no uniform bound on the number of $\sim$-equivalence classes.

Proof. Indeed, $K \simeq \langle H, y \mid y^{-1}xy = x^y\rangle$ is an HNN-extension of H which is a JSJ of K modulo H. Let $\phi : x \to u, \phi : x^y \to u^v$. Then a solution $\psi : K \to F$ extends ϕ if and only if $\psi : x \to u, \psi : y \to u_0^m v$, where u_0 is the root of u in F, and $m \in \mathbb{Z}$. It follows $\phi_1 : x \to u, \psi : y \to v$ is the only $\sim_{AE}$-minimal solution (as well as the only $\sim_{AEQ}$-minimal solution) that extends ϕ. Therefore, $N = 1$. Notice that if $u = a^n$ then all the solutions of the type $\psi_m : x \to a^n$, $\psi_m : y \to a^m v$, where $-\frac{n}{2} \leqslant m < \frac{n}{2}$, are $\sim$-minimal. Hence there is no a uniform bound on the number of $\sim$-classes. □

Example 2 *($N = 0$, a single maximal reducing quotient).* Let $K = A * F$, where A is a free abelian group with basis x, y (so $S(x, y) = [x, y] = 1$), and $H = \langle y, F\rangle = \langle y\rangle * F$. Then $N = 0$ and there is a system consisting of a single reducing quotient $R/R(r_1)$, where $r_1 = x$.

Proof. Indeed, the JSJ decomposition of K modulo H is an amalgamated product of A and H with amalgamation over $\langle y\rangle$. For each homomorphism $\phi : H \to F$ there is only one $\sim_{AEQ}$-equivalence class of solutions extending ϕ, since every such class contains the solution $\psi : x \to 1$, $\psi : y \to y^\phi$, which is the unique minimal solution in this class. Hence, $N = 0$, $r = x$. □

Example 3 *(No uniform bounds on $\sim_{AE}$-classes, finitely many $\sim_{AEQ}$-classes, a single $\sim_{MAX}$-class).* Let

$$K = \langle x, y\rangle *_{[x,y]=p} \left(F * \langle p\rangle\right)$$

and $H = F * \langle p\rangle$. Then there are no uniform bounds on the number of $\sim_{AE}$-classes, there are only finitely many $\sim_{AEQ}$-classes, and there is only one $\sim_{MAX}$-class.

Proof. K is a free product of $\langle x, y\rangle$ and H with amalgamation $[x, y] = p$, which is, in fact, a JSJ decomposition of K modulo H. It is not hard to see that the set of all $\sim$-minimal solutions discriminates K, so K does not have a sufficient splitting modulo $\langle p\rangle$. Let $\phi^{(n)} : p \to [a^n, b]$ be an F-homomorphism from H into $F = F(a, b)$. Solutions $\psi^{(n,m)} : x \to a^n$, $\psi^{(n,m)} : y \to a^m b$, extend $\phi^{(n)}$. Observe, that if $-\frac{n}{2} \leqslant m < \frac{n}{2}$ then these solutions are pair-wise non-$\sim$-equivalent and $\sim$-minimal (see Example 1). It follows that there is no any uniform upper bound on the number of $\sim$-classes. Notice, that for the group K $\sim$-equivalence and $\sim_{AE}$-equivalence are the same. Indeed, all the extended automorphisms that are allowed to use in $\sim_{AE}$ correspond to the edge p, but $\phi(p)$ is a commutator in F for any solution ϕ, hence it is not a proper power, so the extended automorphism can be realized by the Dehn twists with respect to the edge p.

Theorem 11 says that there are only finitely many *AEQ*-classes. There is no any easy explanation of this result even for our particular equation $[x, y] = p$ (one has to follow the proof of the theorem), but, at least, we can show that for a given n all the solutions $\psi^{(n,m)}$ are *AEQ*-equivalent. Indeed, one can rewrite the equation $[x, y] = p$ in the HNN form $y^{-1}xy = xp$, which gives an elementary splitting of K. Clearly, $\phi^{(n)}(x) = a^n$, so there is an extended homomorphism $x \to \psi^{(n,m)}(x)$, $y \to a^m\psi^{(n,m)}$. This shows that $\psi^{(n,m)}$ is *AEQ*-equivalent to $x \to a^n$, $y \to b$.

Observe, that an arbitrary F-homomorphism from K to F which extends a given F-homomorphism $\phi : H \to F$ fixes all the non-QH-subgroups of the JSJ of K modulo H, so any two such homomorphisms are *MAX*-equivalent. Hence, there is only one equivalence class with respect to $\sim_{MAX}$. $\square$

Example 4. Let

$$K = \langle x_1, y_1, x_2, y_2\rangle *_{[x_1,y_1][x_2,y_2]=p} \big(\langle p\rangle * F\big).$$

Then:

(1) there exists a number N such that for any F-homomorphism $\phi : \langle p\rangle * F \to F$ such that $\phi(p)$ has genus 2 the set of all solutions from K into F that extend ϕ partitions into at most N *AEQ*-classes, i.e., if $\phi(p)$ has genus 2 then solutions extending ϕ are K-algebraic;
(2) one of the reducing equations $r = 1$ is $x_1 = 1$.

Example 5. If D does not have abelian and QH vertex groups, and the graph is a tree, the number of equivalence classes with respect to $\sim$ is finite.

Add to K elements d_j, $j \in J$, that commute with the edge groups of the JSJ decomposition D_K of K modulo H, and consider groups discriminated by minimal (with respect to $\sim$) algebraic solutions and generators of the images in F of the edge groups in place of variables d_j, $j \in J$. Denote any of the finite family of such fully residually free groups by K_E. Denote the edge groups of K by G_{e_j}, $j \in J$. If the images of all the subgroups $\langle G_{e_j}, d_j\rangle$ were cyclic in K_E, the number of equivalence classes with respect to $\sim$ would be finite. One can choose a subtree T including all the vertices with non-abelian groups in the decomposition D such that for any edge $e_j \in T$ the subgroup $\langle G_{e_j}, d_j\rangle$ is cyclic. Indeed, let G_{v_0} be the vertex group containing H. Join to it as many non-abelian vertex groups as possible by edges with cyclic groups $\langle G_{e_j}, d_j\rangle$ and collapse these edges. We will have a group $\bar{G}_{v_0}$. If there is no non-abelian vertex groups left, T is the maximal subtree and the graph itself. Suppose some non-abelian subgroups are not in $\bar{G}_{v_0}$. Let e_i, $i \in I$, be the set of edges joining $\bar{G}_{v_0}$ to all the other non-abelian subgroups. Let $\bar{G}$ be a group generated by all the vertex groups that are not in $\bar{G}_{v_0}$ and elements d_i, $i \in I$. The group $\bar{G}$ cannot have a splitting not induced from D, because each splitting of $\bar{G}$ can be extended to a splitting of K modulo H. Therefore this splitting is a JSJ decomposition of $\bar{G}$. This is impossible because solutions were minimal with respect to this splitting. Therefore $\bar{G}_{v_0}$ contains all the non-QH vertex groups.

7. Canonical embeddings of finitely generated fully residually free groups in NTQ groups

In this section we construct the standard quotients of freely indecomposable finitely generated fully residually free groups and the canonical embeddings of such groups into NTQ groups.

7.1. JSJ decompositions are sufficient splittings

Here we prove that a non-degenerate JSJ $\mathbb{Z}$-decomposition D of a finitely generated freely indecomposable fully residually free group G is a sufficient splitting of G, i.e., the standard maximal quotient G/R_D is a proper quotient of G. Below we use results and constructions from [14, Sections 4.3 and 5] and [19, Section 7] assuming that the reader is familiar with these results.

Theorem 12. *Let G be a finitely generated freely indecomposable* [*freely indecomposable modulo F*] *fully residually free group G. Then a non-degenerate JSJ $\mathbb{Z}$-decomposition* [*modulo F*] *of G is a sufficient splitting of G.*

Proof. We consider only the case of JSJ decompositions modulo F the other case (not modulo F) is similar.

Let G be a finitely generated freely indecomposable modulo F fully residually free group G and D a non-degenerate JSJ $\mathbb{Z}$-decomposition of G modulo F. By $\mathcal{G}E(U)$ we denote the finite set of the initial generalized equations associated with $U = 1$ (see [19, Section 4]). Recall that every equation $\Omega \in \mathcal{G}E(U)$ comes equipped with the canonical homomorphism $\eta : G \to F_{R(\Omega)}$. Fix an equation $\Omega \in \mathcal{G}E(U)$ with the homomorphism η. We construct a *maximal quotients* tree $T_{mq}(\Omega)$ as follows. If η is not a monomorphism then $T_{mq}(\Omega)$ consists of one vertex with the group $F_{R(\Omega)}$ assigned to it. Suppose η is a monomorphism. Let $T_{\mathrm{dec}}(\Omega)$ be the decomposition tree which has been constructed in [19, Section 7]. Suppose v is a leaf vertex of $T_{\mathrm{dec}}(\Omega)$ and Ω_v is the generalized equation associated with v in $T_{\mathrm{dec}}(\Omega)$. By the construction the group $F_{R(\Omega_v)}$ is a proper quotient of $F_{R(\Omega)}$. If the image of G in $F_{R(\Omega_v)}$ under the quotient epimorphism $F_{R(\Omega)} \to F_{R(\Omega_v)}$ is not a proper quotient of G, then we extend the tree $T_{\mathrm{dec}}(\Omega)$ by gluing up the tree $T_{\mathrm{dec}}(\Omega_v)$ to the vertex v (by identifying the vertex v in $T_{\mathrm{dec}}(\Omega)$ with the root vertex in $T_{\mathrm{dec}}(\Omega_v)$). Denote the resulting tree by $T_{\mathrm{dec}}(\Omega)'$. Again, if there is a leaf vertex $v' \in T_{\mathrm{dec}}(\Omega)'$ such that the canonical image of G in the group associated with v' is not a proper quotient of G then we extend $T_{\mathrm{dec}}(\Omega)'$ by gluing up the tree $T_{\mathrm{dec}}(\Omega'_v)$ to the vertex v' (we refer to this operation as to *leaf extension*). More formally, let $T_0 = T_{\mathrm{dec}}(\Omega)$. Suppose a tree T_i with associated groups and homomorphisms is already constructed. If there is a vertex $w \in T_i$ such that the image of G in the group associated with w is not a proper quotient of G then we glue up the tree $T_{\mathrm{dec}}(\Omega_w)$ to the vertex w (by identifying w with the root vertex in $T_{\mathrm{dec}}(\Omega_w)$) and denote the resulting tree by T_{i+1}.

Claim 1. *The sequence*

$$T_0 \subset T_1 \subset T_2 \subset \cdots \tag{13}$$

is finite.

Indeed, suppose the sequence (13) is infinite. Then the graph $T = \bigcup_{i=1}^{\infty} T_i$ is infinite. Then by König's lemma there is an infinite branch in T, say,

$$v_0 \to v_1 \to v_2 \to \cdots. \tag{14}$$

Therefore there exists an infinite subsequence, say,

$$v_{i_0} \to v_{i_1} \to v_{i_2} \to \cdots$$

such that for every vertex v_{i_j} in this sequence at some step the tree $T_{\mathrm{dec}}(\Omega_{v_{i_j}})$ was glued to the vertex v_{i_j}. It follows that the homomorphism corresponding to the edge $v_{i_j-1} \to v_{i_j}$ is a proper epimorphism. Hence there are infinitely many proper epimorphisms associated with the edges of the branch (14). This implies that there exists an infinite sequence of epimorphisms of finitely generated residually free groups which contains infinitely many proper epimorphisms—contradiction with the equationally Noetherian property of free groups. This proves the claim.

Denote the union tree T of the finite sequence (13) by $T_{mq}(\Omega)$. Now we recall the description of homomorphisms $G \to F$ in terms of the trees $T_{mq}(\Omega)$, $\Omega \in \mathcal{GE}(U)$, from the paper [19].

For a given Ω fix some terminal vertex w of $T_{mq}(\Omega)$. Let v_0 be the initial vertex of $T_{\mathrm{dec}}(\Omega)$, v_1 be the terminal vertex of $T_{\mathrm{dec}}(\Omega)$, v_2 be the terminal vertex of $T_{\mathrm{dec}}(\Omega_{v_1})$, and so on, such that $v_n = w$ for some n. The sequence of vertices $v_1, \ldots, v_n$ defines a branch $b: v_0 \to v_1 \to \cdots \to v_n$ in $T_{mq}(\Omega)$ from v_0 to w. Let G_b be the image of G under composition of the homomorphisms $\pi(v_i, v_{i+1})$ along b (see the construction of the tree $T_0(\Omega)$ from [19]), so $G_b \leqslant F_{R(\Omega_{v_n})}$. Given a branch b in $T_{mq}(\Omega)$ one can associate with b a set of homomorphisms from $F_{R(\Omega)}$ into $F_{R(\Omega_w)}$ (Razborov's fundamental sequences) that "go along b" (see [19, Theorem 8.1, Section 8]). The main property of the tree $T_{mq}(\Omega)$ is that every solution of Ω in F goes along one of the branches of $T_{mq}(\Omega)$. The restrictions of such homomorphism onto the image G^η of G in $F_{R(\Omega)}$ give a set of induced homomorphisms from $G^\eta \to G_b$. Composing these homomorphisms with the canonical homomorphism η one gets a set of induced homomorphisms $G \to G_b$. The next result (description of homomorphisms from G into F) says that every homomorphism $\phi: G \to F$ can be described as the composition $\phi = \eta\psi_b\xi$ for suitable Ω, b, and ξ, where ψ_b is a homomorphism along the branch b and ξ is a homomorphism $G_b \to F$.

Now, each homomorphism ψ_b that goes along the branch b can be transformed by some consecutive applications of canonical automorphisms corresponding to vertices of the branch b (which are the canonical automorphisms of the decomposition D_i of $F_{R(\Omega_{v_i})}$, associated with the branch $v_i \to v_{i+1}$ of $T_{\mathrm{dec}}(\Omega_{v_i})$, $i = 1, \ldots, n$ (see [19, Theorem 7.7]) into the solution $G \to G_b$ along b satisfying a proper equation, say E_b, which is equivalent to the conjunction of the defining relations of G_b (recall that G_b is a proper quotient of G, so E_b does not belong to the radical of U). Notice, that there are finitely many branches b in $T_{mq}(\Omega)$, and a solution along a branch b satisfies the equation E_b. It follows that every minimal solution satisfies the disjunction $\bigvee_b E_b$ of the equations E_b where b runs over all branches. It is known that any finite disjunction over F is equivalent to a single equation (see [14]). In particular, $\bigvee_b E_b$ is equivalent to some equation E. Observe, that $E \notin R(U)$. Indeed, if $E \in R(U)$ then

$$V(U) = V(E) \cap V(U) = \bigcup_b \bigl(V(E_b) \cap V(U)\bigr)$$

and $V(E_b) \cap V(U) \neq V(U)$. This implies that $V(U)$ is a reducible algebraic set in F—contradiction with the condition that $G = F_{R(U)}$ is a fully residually free group (see [1]). The theorem is almost proven. However, the standard automorphisms that were used to transform ψ_b are not precisely the canonical automorphisms of G relative to D (they are not even automorphisms of G). To finish the proof it suffices to show that the same thing happens to the minimal solutions of U relative to the canonical automorphisms of G with respect to the JSJ decomposition D. The idea is to show that the minimal solutions of U (with respect to D) are some short solutions with respect to the branches of $T_{mq}(\Omega)$.

By Lemma 7 for each QH-subgroup Q in $F_{R(\Omega_{v_i})}$ one of the following two cases holds. Either the intersection $Q_1 = G \cap Q^g$ of G with some conjugate Q^g of Q is of finite index in Q^g and in this case Q_1 is a QH-subgroup of G, or G is conjugated into the fundamental group P of a connected component of the graph of groups obtained from D by removing the vertex Q and the edges adjacent to Q (i.e., $P \in \mathcal{K}_Q$, see Section 3.10). Notice, that case (1) in Lemma 7 cannot take place because G is freely indecomposable.

If G is conjugated into P then each solution of $U = 1$ can be obtained from a solution of Ω_{v_i} which is minimal with respect to the canonical group of automorphisms corresponding to Q.

If G has trivial intersection with all conjugates of some edge group of D_i, then again each solution of $U = 1$ can be obtained from a solution of Ω_{v_i} minimal with respect to the group generated by the canonical Dehn twists along this edge. If the intersection $G \cap Q^g$ has a finite index, then it is a QH-subgroup Q_1 of G and the group of automorphisms of Q_1 induced by canonical automorphisms of Q^g has finite index in the group of canonical automorphisms of Q^g. Let $\bar{F}_{R(\Omega_{v_i})}$ be the factor that contains G in some Grushko decomposition of $F_{R(\Omega_{v_i})}$. Let $A(\bar{F}_{R(\Omega_{v_i})})$ be the group of outer canonical automorphisms of $\bar{F}_{R(\Omega_{v_i})}$ induced from canonical automorphisms of $F_{R(\Omega_{v_i})}$. By Lemma 7, the group of canonical automorphisms of $\bar{F}_{R(\Omega_{v_i})}$ corresponding to Q^g (where Q is a QH-subgroup of $\bar{F}_{R(\Omega_{v_i})}$) that induce canonical automorphisms of G has finite index in the subgroup of $A(\bar{F}_{R(\Omega_{v_i})})$ corresponding to Q. Denote by $A_G(\bar{F}_{R(\Omega_{v_i})})$ the subgroup of $A(\bar{F}_{R(\Omega_{v_i})})$ generated by canonical Dehn twists corresponding to the splittings of $\bar{F}_{R(\Omega_{v_i})}$ that induce non-trivial splittings of G. The subgroup A_{ind} of $A_G(\bar{F}_{R(\Omega_{v_i})})$ consisting of automorphisms that induce automorphisms of G, has finite index in $A_G(\bar{F}_{R(\Omega_{v_i})})$. Similarly, by [19, Section 13.3], each abelian vertex group of G^η (denote it by A) is conjugated into one of the abelian vertex groups of $F_{R(\Omega_{v_0})}$ (denote it by B). Then $B = B_1 \oplus B_2$, where A has finite index in B_1. By [19, Lemma 6.7], if one can shorten a homomorphism from $F_{R(\Omega_{v_0})}$ to F by pre-composing it with a canonical automorphism corresponding to B, then one can shorten it by pre-composing first with a canonical automorphism that is identical on B_1 (therefore, does not change homomorphisms from G^η to G_b) and then with a canonical automorphism of B_1 that fixes the edge groups.

This implies that using canonical automorphisms corresponding to the JSJ decomposition of G, we can transform each solution ϕ of $U = 1$ into a solution that is not longer than ϕ and satisfies one of the finite number of proper equations (which can be effectively found) or, equivalently, satisfies a proper equation which represents this disjunction of equations. Since a minimal solution in the equivalence class of ϕ factors through some branch of $T_{mq}(\Omega)$ for one of the finite number of generalized equations Ω_w, it must satisfy this proper equation. □

7.2. Standard maximal fully residually free quotients of G relative to JSJ decompositions

Let G be a finitely generated freely indecomposable (modulo F) fully residually free group, and D a non-degenerate abelian JSJ decomposition of G. Let $\mathcal{L}$ be a (finite) collection of all standard maximal fully residually free quotients of G relative to D. By Lemma 16 there exists at least one standard maximal fully residually free quotient $L \in \mathcal{L}$ such that the restrictions of the canonical epimorphism $\eta : G \to L$ onto the rigid subgroups of D, onto the edge subgroups in D, and onto the subgroups of the abelian vertex groups A generated by the images of all the edge groups of edges adjacent to A, are monomorphisms. We refer to such quotients $L \in \mathcal{L}$ as to *principal* standard maximal fully residually free quotients (or principal quotients).

Fix a principal quotient $L \in \mathcal{L}$ together with the canonical epimorphisms $\eta : G \to L$. Denote by $\mathcal{H} = \{H_1, \ldots, H_q\}$ the set of all rigid and edge subgroups of G relative to D. Consider the Grushko decomposition of L compatible with $\mathcal{H}^\eta$:

$$L = L_1 * \cdots * L_\alpha * F(t_1, \ldots, t_\beta), \tag{15}$$

where $F(t_1, \ldots, t_{\beta_i})$ is the maximal free factor that does not contain conjugates of any of the subgroups from $\mathcal{H}^\eta$. We call elements $t_1, \ldots, t_{\beta_i}$ *free parameters* (or *free variables*, when we view homomorphisms as solutions of equations) and refer to the group $\bar{G} = L_1 * \cdots * L_\alpha$ as to the *base* of the quotient L.

Lemma 21. *Let L be a principal quotient of G with respect to D. Then in the decomposition* (15) *every factor L_i contains a conjugate of a subgroup from $\mathcal{H}^\eta$.*

Proof. Indeed, clearly, the images of abelian vertex subgroups of D are elliptic in (15), so up to conjugation they are contained in some factors containing the images of corresponding edge groups. Hence it suffices to consider the images of the QH-subgroups from D. Let Q be a QH-subgroup from D. If $\eta_i(Q)$ is elliptic in (15) then again up to conjugation it lies in a factor containing conjugates of an edge group. If $\eta_i(Q)$ is not elliptic in (15) then it admits a free decomposition. Now the result follows from Lemma 9. □

7.3. Canonical extensions of the base groups of principal quotients

Let L be a principal quotient of G with the base $\bar{G}$. In this section we construct a canonical extension G^* of the base group $\bar{G}$ which is the fundamental group of a graph of groups Γ obtained from a single vertex v with the associated vertex group $G_v = \bar{G}$ by adding finitely many edges corresponding to extensions of centralizers (viewed as amalgamated products) and finitely many QH vertices connected only to v as depicted in Fig. 3.

Combining foldings and slidings, we can transform D into an abelian decomposition in which each vertex with non-cyclic abelian subgroup that is connected to some rigid vertex, is connected to only one vertex which is rigid. We suppose from the beginning that D has this property.

We start with the description of the extensions G_i of $\bar{G}_i$. Fix an integer i, $i = 1, \ldots, k$, and, to simplify notation, denote $\bar{G}_i$ by $\bar{G}$, η_i by π, and G_i by G^*. Let $\bar{G} = P_1 * \cdots * P_\alpha$ be the Grushko decomposition of $\bar{G}$. Then by construction of $\bar{G}$, each factor in this decomposition contains a

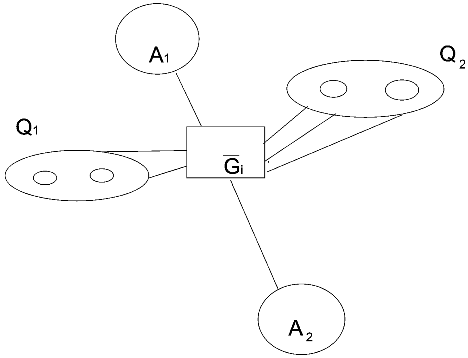

Fig. 3. The graph of groups Γ_i of the coordinate group of the quasi-quadratic system over $\bar{G}_i$.

conjugate of the image of some rigid subgroup or an edge group in D. Let $g_1, \ldots, g_l$ be a fixed finite generating set of $\bar{G}$. For an edge $e \in D$ we fix a generator d_e of the cyclic edge group G_e (or generators of an abelian edge group connecting a non-cyclic abelian vertex group to a rigid subgroup). The required extension G^* of $\bar{G}$ is constructed in three steps. On each step we extend the centralizers $C_{\bar{G}}(\pi(d_e))$ of some edges e in D or add a QH-subgroup. Simultaneously, for every edge $e \in D$ we associate an element $s_e \in C_{G^*}(\pi(d_e))$.

Step 1. Let E_{rig} be the set of all edges between rigid subgroups in D. For each edge $e \in E_{\mathrm{rig}}$ we do the following. If

$$G = A *_{\langle d_e\rangle} B \quad \text{and} \quad \bar{G} = \pi(A) *_{\langle \pi(d_e)\rangle} \pi(B) \quad \text{or} \quad \bar{G} = C *_{\langle \pi(d_e)\rangle}$$

or

$$G = A *_{\langle d_e\rangle} \quad \text{and} \quad \bar{G} = \pi(A) *_{\langle \pi(d_e)\rangle} \quad \text{or} \quad \bar{G} = C *_{\langle \pi(d_e)\rangle} D$$

then we delete the edge e from E_{rig}. In this case we associate the trivial element $s_e = 1$ with e. Denote the resulting set of edges by E'. One can define an equivalence relation $\sim$ on E' assuming for $e, f \in E'$ that

$$e \sim f \quad \Leftrightarrow \quad \exists g_{ef} \in \bar{G} \ \left(g_{ef}^{-1} C_{\bar{G}}\big(\pi(e)\big) g_{ef} = C_{\bar{G}}\big(\pi(f)\big)\right).$$

Let E be a set of representatives of equivalence classes of E' modulo $\sim$.

Now we construct a group $G^{(1)}$ by extending every centralizer $C_{\bar{G}}(\pi(d_e))$ of $\bar{G}$, $e \in E$ as follows. Let

$$[e] = \{e = e_1, \ldots, e_{q_e}\}$$

and $y_e^{(1)}, \ldots, y_e^{(q_e)}$ be new letters corresponding to the elements in $[e]$. Then put

$$G^{(1)} = \left\langle \bar{G}, y_e^{(1)}, \ldots, y_e^{(q_e)} \ (e \in E) \ \middle| \ \left[C\big(\pi(d_e)\big), y_e^{(j)}\right] = 1, \ \left[y_e^{(i)}, y_e^{(j)}\right] = 1 \ (i, j = 1, \ldots, q_e)\right\rangle.$$

One can associate with $G^{(1)}$ the following system of equations over $\bar{G}$:

$$\left[\bar{g}_{es}, y_e^{(j)}\right] = 1, \qquad \left[y_e^{(i)}, y_e^{(j)}\right] = 1, \quad i, j = 1, \ldots, q_e, \ s = 1, \ldots, p_e, \ e \in E, \tag{16}$$

where $y_e^{(j)}$ are new variables and the elements $\bar{g}_{e1}, \dots, \bar{g}_{ep_e}$ are constants from $\bar{G}$ which generate the centralizer $C(\pi(d_e))$. We assume that the constants $\bar{g}_{ej}$ are given as words in the generators $g_1, \dots, g_l$ of $\bar{G}$. We associate the element $s_{e_i} = y_e^{(i)}$ with the edge $e = e_i$.

Step 2. Let A be a non-cyclic abelian vertex group in D and A_e the subgroup of A generated by the images in A of the edge groups of edges adjacent to A. Then $A = Is(A_e) \times A_0$ where $Is(A_e)$ is the isolator of A_e in A (the minimal direct factor containing A_e) and A_0 a direct complement of $Is(A_e)$ in A. Notice, that the restriction of π on $Is(A_e)$ is a monomorphism (since π is injective on A_e and A_e is of finite index in $Is(A_e)$). For each non-cyclic abelian vertex group A in D we extend the centralizer of $\pi(Is(A_e))$ in $G^{(1)}$ by the abelian group A_0 and denote the resulting group by $G^{(2)}$. Observe, that since $\pi(Is(A_e)) \leqslant \bar{G}$ the group $G^{(2)}$ is obtained from $\bar{G}$ by extending finitely many centralizers of elements from $\bar{G}$.

If the abelian group A_0 has rank r then the system of equations associated with the abelian vertex group A has the following form:

$$[y_p, y_q] = 1, \qquad [y_p, \bar{d}_{ej}] = 1, \quad p, q = 1, \dots, r,\ j = 1, \dots, p_e, \tag{17}$$

where y_p, y_q are new variables and the elements $\bar{d}_{e1}, \dots, \bar{d}_{ep_e}$ are constants from $\bar{G}$ which generate the subgroup $\pi(Is(A_e))$. We assume that the constants $\bar{d}_{ej}$ are given as words in the generators $g_1, \dots, g_l$ of $\bar{G}$.

Denote by

$$\bar{S}_i(y_1, \dots, y_t, g_1, \dots, g_l) = 1$$

the union of the system (16) and all the systems (17) for every abelian non-cyclic vertex group A in D. Here $y_1, \dots, y_t$ are all the new variables that occur in Eqs. (16), (17).

Step 3. A QH-subgroup Q such that $\pi(Q)$ is a QH-subgroup of $\bar{G}$ of the same size (see Section 1.6) is called a *stable QH-subgroup*. Let Q be a non-stable QH-subgroup in D. Suppose Q is given by a presentation

$$\prod_{i=1}^{n} [x_i, y_i] p_1 \cdots p_m = 1,$$

where there are exactly m outgoing edges $e_1, \dots, e_m$ from Q and $\sigma(G_{e_i}) = \langle p_i \rangle$, $\tau(G_{e_i}) = \langle c_i \rangle$ for each edge e_i. We add a QH vertex Q to $G^{(2)}$ by introducing new generators and the following quadratic relation:

$$\prod_{i=1}^{n} [x_i, y_i] \bigl(c_1^{\pi_i}\bigr)^{z_1} \cdots \bigl(c_{m-1}^{\pi_i}\bigr)^{z_{m-1}} c_m^{\pi_i} = 1 \tag{18}$$

to the presentation of $G^{(2)}$. Observe, that in the relations (18) the coefficients in the original quadratic relations for Q in D are replaced by their images in $\bar{G}$.

Similarly, one introduces QH vertices for non-orientable QH-subgroups in D. We refer to Section 3.9 for details on quadratic relations associated with non-orientable QH-subgroups.

The resulting group is denoted by $G^* = G^{(3)}$.

7.4. Embedding G into the canonical extensions G^*

In this section for every principal quotient L of G we construct a canonical embedding $G \to G^*$.

Let

$$S = S(z_1, \dots, z_m, g_1, \dots, g_l) = 1$$

be union of the quadratic systems of the type (18) for orientable and non-orientable non-stable QH-subgroups in D. Here the set of variables $\{z_1, \dots, z_m\}$ is union of all variables in quadratic equations of the type (18). In our notation D^* is a splitting of G^* with the following vertex groups: factors $P_1, \dots, P_\alpha$ in the compatible reduced free decomposition of $\bar{G}$, abelian vertex groups corresponding to relations (16), (17), and QH-subgroups added to $\bar{G}$. Let T^* be a maximal subtree in the underlying graph Γ^*. So far we defined elements s_e only for edges between two rigid subgroups of D.

Now we define an F-homomorphism $\psi : G \to G^*$ and show that ψ is injective. In the process we will associate elements s_e to the remaining edges of D. We need the following lemma.

Lemma 22.

(1) *Let $H = A *_{\langle d \rangle} B$ and $\pi : H \to \bar{H}$ be a homomorphism such that the restrictions of π on A and B are injective. Put*

$$H^* = \langle \bar{H}, y \mid [C_{\bar{H}}(\pi(d)), y] = 1 \rangle.$$

Then for every $u \in C_{H^*}(\pi(d))$, $u \notin C_{\bar{H}}(\pi(d))$, a map

$$\psi(x) = \begin{cases} \pi(x), & x \in A, \\ \pi(x)^u, & x \in B. \end{cases}$$

gives rise to a monomorphism $\psi : H \to H^*$.

(2) *Let $H = \langle A, t \mid d^t = c \rangle$ and $\pi : H \to \bar{H}$ be a homomorphism such that the restriction of π on A is injective. Put*

$$H^* = \langle \bar{H}, y \mid [C_{\bar{H}}(\pi(d)), y] = 1 \rangle.$$

Then for every $u \in C_{H^*}(\pi(d))$, $u \notin C_{\bar{H}}(\pi(d))$, a map

$$\psi(x) = \begin{cases} \pi(x), & x \in A, \\ u\pi(x), & x = t, \end{cases}$$

gives rise to a monomorphism $\psi : H \to H^*$.

Proof. We prove (1), a similar argument proves (2). Clearly, the map ψ agrees on the intersection $A \cap B$, so it defines a homomorphism $\psi : H_{\mathrm{rig}} \to G$. Straightforward verification shows that ψ maps reduced forms of elements in H_{rig} into reduced forms of elements in G. This proves that ψ is injective. □

We define the homomorphism $\psi : G \to G^*$ with respect to the splitting D of G. Let T be the maximal subtree of D. First, we define ψ on the fundamental group of the graph of groups induced from D on T. Notice that the subgroup F is elliptic in D, so there is a rigid vertex $v_0 \in T$ such that $F \leqslant G_{v_0}$. Mapping π embeds G_{v_0} into $\bar{G}$, hence into G^*.

Let P be a path $v_0 \to v_1 \to \cdots \to v_n$ in T that starts at v_0. With each edge $e_i = (v_{i-1} \to v_i)$ between two rigid vertex groups we have already associated the element s_{e_i}. Let us associate elements to other edges of P:

(a) if v_{i-1} is a rigid vertex, and v_i is either abelian or QH, then $s_{e_i} = 1$;
(b) if v_{i-1} is a QH vertex, v_i is rigid or abelian, and the image of e_i in the decomposition D^* of G^* does not belong to T^*, then s_{e_i} is the stable letter corresponding to the image of e_i;
(c) if v_{i-1} is a QH vertex and v_i is rigid or abelian, and the image of e_i in the decomposition of G^* belongs to T^*, then $s_{e_i} = 1$;
(d) if v_{i-1} is an abelian vertex with $G_{v_{i-1}} = A$ and v_i is a QH vertex, then s_{e_i} is an element from A that belongs to A_0.

Since two abelian vertices cannot be connected by an edge in Γ, and we can suppose that two QH vertices are not connected by an edge, these are all possible cases.

We now define the embedding ψ on the fundamental group corresponding to the path P as follows:

$$\psi(x) = \pi(x)^{s_{e_i} \cdots s_{e_1}} \quad \text{for } x \in G_{v_i}.$$

This map is a monomorphism by Lemma 22. Similarly we define ψ on the fundamental group of the graph of groups induced from D on T. We extend it to G using the second statement of Lemma 22.

7.5. Canonical systems of equations associated with JSJ decompositions

Theorem 13. *Let $U(X) = 1$, $X = \{x_1, \ldots, x_n\}$, be a finite irreducible system of equations over F such that $G = F_{R(U)}$ is freely indecomposable* (*modulo F if $U(X) = 1$ has non-trivial coefficients*). *Let D be a JSJ $\mathbb{Z}$-decomposition of G. Then one can effectively construct finitely many equations*

$$V_1(X) = 1, \quad \ldots, \quad V_s(X) = 1$$

with coefficients in F and finitely many systems

$$U_{D,1} = 1, \quad \ldots, \quad U_{D,k} = 1$$

over F with embeddings

$$\phi_1 : F_{R(U)} \to F_{R(U_{D,1})}, \quad \ldots, \quad \phi_k : F_{R(U)} \to F_{R(U_{D,k})}$$

such that:

(1) *$V_i \notin R(U)$, $i = 1, \ldots, s$.*

(2) $U_{D,i}(z_1, \ldots, z_m, y_1, \ldots, y_t, g_1, \ldots, g_l) = 1$ *is a system of the following type*:

$$S(z_1, \ldots, z_m, g_1, \ldots, g_l) = 1,$$
$$\bar{S}_i(y_1, \ldots, y_t, g_1, \ldots, g_l) = 1,$$
$$\bar{U}_i(g_1, \ldots, g_l) = 1,$$

where $\bar{U}_i = 1$ is irreducible, $S = 1$ is a non-degenerate quadratic system in variables $z_1, \ldots, z_m$ over $\bar{G}_i = F_{R(\bar{U}_i)}$, variables $z_1, \ldots, z_m$ are standard generators of QH-subgroups in D and the stable letters corresponding to non-tree edges adjacent to QH-subgroups (*with respect to a fixed maximal subtree of the underlying graph of D*), *$\bar{S}_i = 1$ corresponds to extensions of centralizers of elements from $F_{R(\bar{U}_i)}$. There is a splitting D_i^* of the coordinate group of the system $U_{D,i}$ modulo free factors of $\bar{G}_i$; MQH vertex groups in D_i^* correspond to equations from the system $S = 1$, abelian vertex groups correspond to equations from the system $\bar{S}_i = 1$.*

(3) *$F_{R(\bar{U}_i)} * F(t_1, \ldots, t_{\beta_i})$ is a proper quotient of $F_{R(U)}$ with the canonical epimorphism $\eta_i : F_{R(U)} \to F_{R(\bar{U}_i)} * F(t_1, \ldots, t_{\beta_i})$, $i = 1, \ldots, k$, which can be found effectively.*

(4) *$V(U) = p_1(V(U_{D,1})) \cup \cdots \cup p_k(V(U_{D,k})) \cup V(U \wedge V_1) \cup \cdots \cup V(U \wedge V_s)$, where the word mappings $p_i = (p_{i,1}, \ldots, p_{i,n})$, $i = 1, \ldots, k$, correspond to the embeddings $\phi_1, \ldots, \phi_k$, i.e., for every i and every $x_j \in X$*

$$\phi_i(x_j) = p_{i,j}(z_1, \ldots, z_m, y_1, \ldots, y_t, g_1, \ldots, g_l).$$

(5) *One can extend a canonical epimorphism η_i to an epimorphism $\eta_i^* : F_{R(U_{D,i})} \to F_{R(\bar{U}_i)} * F(t_1, \ldots, t_{\beta_i})$ such that $\phi_i \eta_i^* = \eta_i$ and η_i^* is identical on $F_{R(\bar{U}_i)}$ so that the following holds. Let $\Phi_i = \{\phi_i \sigma \eta_i^* \psi \mid \sigma \in A_{D_i^*},\ \psi \in V(\bar{U}_i)\}$. Then $\Phi_i \subseteq p_i(V(U_{D,i}))$ and*

$$V(U) = \Phi_1 \cup \cdots \cup \Phi_k \cup V(U \wedge V_1) \cup \cdots \cup V(U \wedge V_s).$$

Proof. If $G = F_{R(U)}$ is a surface group, we take the system $S = 1$ the same as $U = 1$, $\bar{S}_i = 1$ and $\bar{U}_i = 1$ trivial. If G is a free abelian group, we take $\bar{S} = 1$ the same as $U = 1$, and $S = 1$, $\bar{U} = 1$ trivial. If G is cyclic, we do not construct $U_i = 1$.

We constructed a finite number of proper fully residually free quotients of G: $\bar{G}_1, \ldots, \bar{G}_k$ where the systems $S = 1 \wedge \bar{S}_i = 1$ have solutions. Let $\bar{G}_i = F_{R(\bar{U}_i)}$. Now $U_{D,i}(z_1, \ldots, z_m, y_1, \ldots, y_t, g_1, \ldots, g_l) = 1$ is the system

$$S(z_1, \ldots, z_m, g_1, \ldots, g_l) = 1,$$
$$\bar{S}_i(y_1, \ldots, y_t, g_1, \ldots, g_l) = 1,$$
$$\bar{U}_i(g_1, \ldots, g_l) = 1.$$

We have already constructed the systems

$$U_{D,1} = 1, \quad \ldots, \quad U_{D,k} = 1$$

and embeddings

$$\phi_1 : F_{R(U)} \to F_{R(U_{D,1})}, \quad \ldots, \quad \phi_k : F_{R(U)} \to F_{R(U_{D,k})}.$$

Recall that $L_1, \ldots, L_{k+s}$ is a family of all standard maximal fully residually free quotients for G such that for each $i = 1, \ldots, k$ all the restrictions of π_i onto rigid subgroups of D, onto edge subgroups in D, and onto the subgroups of the abelian vertex groups A generated by the images of all the edge groups of edges adjacent to A, are monomorphisms. Consider now fully residually free quotients $L_{k+1}, \ldots, L_{k+s}$. For each quotient L_{k+i} there exists an element V_i from a rigid subgroup or from A_e for some abelian vertex group A such that $\eta_i(V_i) = 1$. Since all automorphisms from A_D only conjugate rigid subgroups, all solutions of $U = 1$ that can be represented as a composition $\sigma\pi_i$, where $\sigma \in A_D$, satisfy the equation $V_i = 1$.

In order to prove statements (4) and (5) we just notice that the set $\phi_i\sigma$, where $\sigma \in A_{D^*}$ is the set of canonical automorphisms from A_D except the automorphisms corresponding to stable QH-subgroups and edges between rigid subgroups for which we did not extend their centralizers. Using these automorphisms one can transform an arbitrary solution into a solution in one of the maximal fully residually free quotients. One can extend a canonical epimorphism η_i to an epimorphism $\eta_i^* : F_{R(U_{D,i})} \to F_{R(\bar{U}_i)} * F(t_1, \ldots, t_{\beta_i})$ such that $\phi_i\eta_i^* = \eta_i$, η_i^* maps letters extending the centralizers of edge groups in D into the identity and η_i^* is identical on $F_{R(\bar{U}_i)}$. $\square$

The statement that the construction described in the theorem is effective follows from Theorem 8 and the results of the next section.

7.6. *The canonical embedding tree $T_{CE}(G)$*

Now we are going to describe a *canonical embedding tree* for the coordinate group of an irreducible system.

Let $G = F_{R(U)}$ be the coordinate group of the irreducible system $U = 1$ over F. We construct a canonical embedding tree $T_{CE}(G)$ by induction. We suppose that G is freely indecomposable. Observe from the beginning that $T_{CE}(G)$ is a rooted tree oriented from the root v_0. We associate the group G with v_0. Let $\bar{U}_i = 1$, $i = 1, \ldots, k$, be the systems corresponding to $U = 1$ from Theorem 13. Notice, that $F_{R(\bar{U}_i)}$ is a subgroup of a standard fully residually free quotient of G, so $F_{R(\bar{U}_i)}$ is fully residually free, hence the systems $\bar{U}_i = 1$, $i = 1, \ldots, k$, are irreducible. We add new vertices $v_{11}, \ldots, v_{1k}$ and add an edge e_i from v_0 to v_{1i}. We associate the groups $F_{R(\bar{U}_i)}$ and $F_{R(\bar{U}_i)} * F(t_1, \ldots, t_{k_i})$ with the vertex v_{1i}, and the canonical epimorphism $\pi_{1i} : F_{R(U)} \to F_{R(\bar{U}_i)} * F(t_1, \ldots, t_{k_i})$ (where $F_{R(\bar{U}_i)}$ is a group from Theorem 13) with the edge e_{1i}, $i = 1, \ldots, t$. Take the Grushko's decomposition (modulo F)

$$F_{R(\bar{U}_i)} = *_j F_{R(\bar{U}_{ij})} * F(Z_i)$$

of $F_{R(\bar{U}_i)}$. Every system $\bar{U}_{ij} = 1$ again satisfies Theorem 13 ($F_{R(\bar{U}_{ij})}$ is a freely indecomposable fully residually free group) so we can repeat the argument above for $\bar{U}_{ij} = 1$ thus continuing the construction of the tree. By induction we construct a rooted tree $T_{CE}(G)$ in which every vertex has only finitely many outgoing edges. We claim that this tree is finite. Indeed, every branch of the tree gives a sequence of epimorphisms of fully residually free groups. Since F is equationally Noetherian every chain of proper quotients terminates in finitely many steps, so every branch of the tree is finite. Now the result follows from the König's lemma. The constructed tree is called the *canonical embedding tree*. Section 7.7 explains the name.

We will construct also the *augmented canonical embedding tree*, $T_{ACE}(G)$. To do this we add to T_{CE} vertices corresponding to the groups $F_{R(U \wedge V_i)}$ $(i = 1, \ldots, s)$ from Theorem 13 on each level, add edges from v_0 to these vertices, and glue corresponding trees for $F_{R(U \wedge V_i)}$.

7.7. Canonical NTQ system

In this section for each branch b of the tree T_{CE} we effectively construct a canonical NTQ system S_b together with a canonical embedding $G \to F_{R(S_b)}$.

For each branch of T_{CE} and each edge from vertex v in this branch with the associated group $F_{R(U_v)}$, consider the corresponding systems of quadratic equations and equations corresponding to extensions of centralizers obtained when Theorem 13 is applied to the system $U_v = 1$. Denote the system from Theorem 13 by associating the system $S_v = S_{q,v} \cup S_{ab,v} = 1$ with the corresponding edge from v.

Let $b = (v_0, v_1, \ldots, v_n)$ be a branch of the tree $T_{CE}(G)$ which begins at the root v_0 and ends at a leaf vertex v_n. We define the NTQ group G_b that corresponds to branch b of the tree as $G_b = F_{R(S_1,\ldots,S_n)}$, where S_i corresponds to quadratic equations and commutativity equations of the decomposition of the group on the $(i-1)$st level of the canonical embedding tree.

Proposition 2. *The group G is canonically embedded into the NTQ group G_b.*

Denote by ϕ_i the canonical embeddings from the proposition. Let $S(b): S_1(X_1, \ldots, X_n) = 1, \ldots, S_n(X_n) = 1$ be the above NTQ system. To each group G_b we assign the set of all homomorphisms from $F_{R(S(b))}$ in F of the form

$$\sigma_1 \pi_1 \cdots \sigma_k \pi_k \tau,$$

where σ_i is a canonical automorphism of $F_{R(S_i,\ldots,S_n)}$ identical on the variables from $X_{i+1}, \ldots, X_k$, and on free variables above this level (if $i = 1$ there are no such variables), π_k is a homomorphism from $F_{R(S_i,\ldots,S_n)} * F(t_1, \ldots, t_{k_{i-1}})$ onto $F_{R(S_{i+1},\ldots,S_n)} * F(t_1, \ldots, t_{k_i})$ that coincides with η_k^* on $F_{R(S_i,\ldots,S_n)}$ and is identical on the variables from $X_{i+1}, \ldots, X_n$ and on all free variables $t_1, \ldots, t_{k_{i-1}}$ from the higher levels, τ is an epimorphism from $F * F(t_1, \ldots, t_{k_n})$ to F identical on F. It is not hard to see that this set of homomorphisms is a fundamental set (fundamental sequence) for G_b. We can always suppose (adding, if necessary, a finite number of branches to T_{ACE}) that all homomorphisms that we consider satisfy the condition given in Sections 7.8 and 7.9 below.

Denote by $V_{\mathrm{fund}}(S(b))$ a fundamental sequence for $S(b) = 1$.

Proposition 3. *Let $G = F_{R(U)}$, where $U = 1$ is a finite irreducible system of equations. Let $\Psi_b = \{\phi\psi_b\}$, where $\psi_b \in V_{\mathrm{fund}}(S(b))$ and ϕ_i is the embedding from Proposition* 2 *corresponding to the branch b of T_{CE}. Then $V(U)$ is the union of solution sets Ψ_b for all branches of T_{CE} and solutions of $U \cap V_i$, $i = 1, \ldots, r$, for additional equations $V_1 = 1, \ldots, V_r = 1$ constructed on each level of the embedding tree according to Theorem* 13.

Definition 21. If $V_{\mathrm{fund}}(S(b))$ is a fundamental sequence for $S(b) = 1$, we define the *dimension* of $V_{\mathrm{fund}}(S)$ $(\dim(V_{\mathrm{fund}}(S)))$ as the maximal number of free variables in the solutions in this sequence.

Notice that $\dim(V_{\mathrm{fund}}(S))$ is the sum $k_1 + \cdots + k_n$.

7.8. First restriction on fundamental sequences

In this subsection the system $\bar{U}_i = 1$ will be denoted just $\bar{U} = 1$. Let

$$F_{R(\bar{U})} * F(t_1, \ldots, t_k) = P_1 * \cdots * P_q * \langle t_1 \rangle * \cdots * \langle t_k \rangle \tag{19}$$

be a reduced free decomposition of $F_{R(\bar{U})} * F(t_1, \ldots, t_k)$ modulo F and $\pi = \eta_i : F_{R(U)} \to F_{R(\bar{U}_i)} * F(t_1, \ldots, t_{\beta_i})$ the canonical epimorphism from Theorem 13. Let P be the subgroup generated by variables X and standard coefficients C of a regular quadratic equation $Q_i = 1$ corresponding to some fixed MQH-subgroup in the JSJ decomposition of $F_{R(U)}$. Consider a free decomposition $\pi(P) = K_1 * \cdots * K_p * \langle t_{k_{j_1}} \rangle * \cdots * \langle t_{k_{j_2}} \rangle$ inherited from the free decomposition (19) such that each standard coefficient is conjugated into some K_j, and each K_j has a conjugate of some coefficient. Then there is a canonical automorphism that transforms X into variables X_1 with the following properties:

(1) the family X_1 can be represented as a disjoint union of sets of variables $X_{11}, \ldots, X_{1t}$;
(2) every solution of $U = 1$ can be transformed by a canonical automorphism corresponding to $Q_i = 1$ into a solution of the system obtained from $U = 1$ by replacing $Q_i = 1$ by a system of several quadratic equations $Q_{i1}(X_{11}, C) = 1, \ldots, Q_{it}(X_{1t}, C) = 1$ with standard coefficients from C;
(3) each quadratic equation $Q_{ij} = 1$ either is coefficient-free or has coefficients from C which are conjugated into some K_r;
(4) X_1^{π} is a solution of the system $Q_{i1}(X_{11}, C^{\pi_1}) = 1, \ldots, Q_{it}(X_{1t}, C^{\pi_1}) = 1$;
(5) if $Q_{ij} = 1$ is coefficient-free, then X_{ij}^{π} is a solution of maximal possible dimension;
(6) if $Q_{ij} = 1$ is not coefficient-free then $Q_{ij} = 1$ is not equivalent to a system of two disjoint quadratic equations $Q_{ij1}(X_{ij1}) = 1$ and $Q_{ij2}(X_{ij2}) = 1$ such that $Q_{ij1}(X_{ij1}) = 1$ is coefficient-free and $Q_{ij2}(X_{ij2}) = 1$ has non-trivial coefficients from C which are conjugated into some K_r and such that X_{ij}^{π} is a solution of the system $Q_{ij2}(X_{ij1}, C^{\pi}) = 1$.

Suppose $Q_{ip} = 1$ is some equation in variables X_{1p} in this family which has coefficients from C. Each homomorphism in a fundamental sequence of homomorphisms from $F_{R(U)}$ to F is a composition $\sigma_1 \pi \phi$, where σ_1 is a canonical automorphism of $F_{R(U)}$, and ϕ is a homomorphism from $F_{R(\bar{U})} * F(t_1, \ldots, t_k)$ to F. We will include into the fundamental sequence only the compositions $\sigma_1 \pi \phi$ for which $\{\phi\}$ is non-special and satisfies the following property: for each j, $Q_{ij} = 1$ is not split into a system of two quadratic equations $Q_{ij1}(X_{ij1}) = 1$ and $Q_{ij2}(X_{ij2}) = 1$ with disjoint sets of variables such that $Q_{ij1} = 1$ is coefficient-free and $Q_{ij2} = 1$ has coefficients from C which are conjugated into some K_r and such that $X_{ij1}^{\pi_1 \phi}$ is a solution of $Q_{ij1} = 1$ and $Q_{ij2}(X_{ij2})^{\pi \phi} = 1$.

7.9. Second restriction on fundamental sequences

Suppose the family of homomorphisms $\sigma_1 \pi_1 \ldots \sigma_n \pi_n \tau$ is a fundamental sequence, corresponding to the NTQ system $Q(X_1, \ldots, X_n) = 1$:

$$\begin{aligned} Q_1(X_1, \ldots, X_n) &= 1, \\ &\vdots \\ Q_n(X_n) &= 1 \end{aligned}$$

adjoint with free variables T. Here the restriction of σ_i on $F_{R(Q_i,\ldots,Q_n)}$ is the canonical automorphism on $F_{R(Q_i,\ldots,Q_n)}$, identical on variables from $X_{i+1},\ldots,X_n$ and on all free variables from the higher levels, $\pi_i : F_{R(Q_i,\ldots,Q_n)} * F(t_1,\ldots,t_{k_{i-1}}) \to F_{R(Q_{i+1},\ldots,Q_n)} * F(t_1,\ldots,t_{k_i})$.

We include in the fundamental sequence only such homomorphisms that give non-abelian images of the regular subsystems of $Q_i = 1$ on all levels (the rest can be included into a finite number of fundamental sequences), and do not factor through a fundamental sequence $V_{\text{fund}}(Q(b_1))$ such that the tuple of dimensions $(k_1,\ldots,k_k)$ for b_1 is greater than the corresponding tuple for b in the lexicographic order.

We can suppose that all fundamental sequences that we consider satisfy the following properties. Let $F_{R(Q_i,\ldots,Q_n)}$ be a free product of some factors. Then

(1) the images of abelian factors under π_i are different factors of $F(t_{k_{i-1}+1},\ldots,t_{k_i})$;
(2) the images under π_i of factors which are surface groups are different factors of $F(t_{k_{i-1}+1},\ldots,t_{k_i})$;
(3) if some quadratic equation in $Q_i = 1$ has free variables in this fundamental sequence, then these variables correspond to some variables among $t_{k_{i-1}+1},\ldots,t_{k_i}$, the images under π_i of coefficients of quadratic equations cannot be conjugated into $F(t_{k_{i-1}+1},\ldots,t_{k_i})$;
(4) different factors in the free decomposition of $F_{R(Q_i,\ldots,Q_n)}$ are sent into different factors in the free decomposition of $F_{R(Q_{i+1},\ldots,Q_n)} * F(t_{k_{i-1}+1},\ldots,t_{k_i})$;
(5) the images in F of the edge groups of the JSJ decompositions on all levels are non-trivial.

Proposition 4. *One can effectively construct a finite set of equations $W_1 = 1,\ldots,W_l = 1$ not in the radical of $U = 1$ such that all fundamental solutions corresponding to a branch of T_{CE} for $U = 1$ that do not satisfy to the first or second restriction above satisfy one of the equations $W_1 = 1, \ldots, W_s = 1$.*

7.10. The canonical embedding theorem

In this section we summarize the results of the previous sections in one theorem. This theorem plays a crucial part throughout the rest of the paper.

Theorem 14. *Let $U = 1$ be a finite irreducible system of equations with coefficients in F, $G = F_{R(U)}$ the coordinate group of $U = 1$, and $T_{CE}(G)$ the canonical embedding tree for G. For a branch b of $T_{CE}(G)$ put*

$$\Psi_b = \bigl\{\phi\psi_b \bigm| \psi_b \in V_{\text{fund}}\bigl(S(b)\bigr)\bigr\},$$

where S_b is the canonical NTQ system along the branch b and $\phi_b : G \to G_b$ is the canonical embedding of G along the branch b from Proposition 2 into the coordinate group G_b of S_b. Then every solution of $U = 1$ in F either belongs to Ψ_b for some branch b in T_{CE}, or it satisfies one of the finitely many equations $V_1 = 1, \ldots, V_r = 1$ effectively constructed in Proposition 3, or it satisfies one of the finitely many equations $W_1 = 1, \ldots, W_s = 1$ effectively constructed in Proposition 4.

7.11. Canonical embeddings relative to subgroups and collection of homomorphisms

In this section we show how one can generalize Theorems 13 and 14 into several directions. First of all, the results hold if in the formulations of the theorems the condition that D is a JSJ

$\mathbb{Z}$-decomposition is replaced by the condition that D is an abelian JSJ decomposition which is a sufficient splitting of G. It is not hard to check that a similar argument works in this case also. Secondly, the results still hold if the condition that D is a JSJ decomposition modulo F is replaced by the condition that D is a JSJ decomposition modulo a finite family of subgroups of G. Again, the same argument works in this case also. Thirdly, given a set of F-homomorphisms $\Phi \subseteq \mathrm{Hom}(G, F)$ one may consider the standard maximal quotient $G/R_{D,\Phi}$ instead of G/R_D (see Section 5.2 for the corresponding definitions) and repeat the construction of the canonical embedding using $G/R_{D,\Phi}$ in the place of G/R_D—the result still holds and the proof comes from a similar argument. In all these generalizations the same argument works.

7.12. Induced NTQ systems

In proving Theorem 13 we had a situation where we had to extract a NTQ system for the group G from the NTQ system for a larger group, the group of a generalized equation $F_{R(\Omega)}$. This extraction can be done in a similar way for a subgroup of a NTQ group.

Let $S = 1$ be an NTQ system over a group G

$$\begin{aligned} &S_1(X_1, X_2, \dots, X_n, A) = 1,\\ &S_2(X_2, \dots, X_n, A) = 1,\\ &\quad\vdots\\ &S_n(X_n, A) = 1 \end{aligned}$$

and $\pi_i : G_i \to G_{i+1}$ a fixed G_{i+1}-homomorphism (a solution of $S_i(X_1, \dots, X_n) = 1$ in $G_{i+1} = F_{R(S_{i+1},\dots,S_n)}$, $G_{n+1} = G$, see Section 2.3). Let K be a finitely generated F-subgroup of $F_{R(S)}$. Then there exists a system $W(Y) = 1$ such that $K = F_{R(W)}$. It may happen that the group $F_{R(S)}$ is too large for G. We will describe here how to embed G more economically into a NTQ group $F_{R(Q)}$ such that $F_{R(Q)} \leqslant F_{R(S)}$ and assign to $Q = 1$ a fundamental sequence that includes all the solutions of $W = 1$ relative to $S = 1$.

Canonical automorphisms on different levels for $Q = 1$ will be induced by canonical automorphisms for $S = 1$, mappings between different levels for $Q = 1$ will be induced by mappings for $S = 1$. Without loss of generality we can suppose that $F_{R(S)}$ is freely indecomposable modulo F. The top quadratic system of equations $S_1(X_1, \dots, X_n) = 1$ corresponds to a splitting D of $F_{R(S)}$. The non-QH non-abelian subgroups of D are factors in a free decomposition of $\langle X_2, \dots, X_n\rangle$. Consider the induced splitting of G denoted by D_G. This splitting may give a free factorization $G = G_1 * \cdots * G_k$, where $F \leqslant G_1$. Consider each factor separately. Increasing G_1 by a finite number of suitable elements from abelian vertex groups of $F_{R(S)}$ we join together non-QH non-abelian subgroups of D_G which are conjugated into the same non-QH non-abelian subgroup of D by elements from abelian vertex groups in $F_{R(S)}$. In this way we obtain a group $\bar{G}_1$ such that $D_{\bar{G}_1}$ does not have edges between non-QH non-abelian subgroups, and generators of edge groups connecting non-QH non-abelian subgroups to abelian subgroups not having roots in $\langle X_2, \dots, X_N\rangle$. Therefore, the relations for $\bar{G}_1$ give the radical of a quadratic system solvable in $\langle X_2, \dots, X_n\rangle$. Moreover, $\pi_1(G_1) = \pi_1(\bar{G}_1)$ because we added elements from abelian subgroups that are sent by π_1 into the images under π_1 of the edge groups. Now consider separately each factor of $\pi_1(G_1) \cap \langle X_2, \dots, X_n\rangle$ and enlarge it the same way. Working similarly with each G_i we consider all the levels of $S = 1$ from the top to the bottom and obtain a group that we denote by H_1. Then we repeat the whole construction for H_1 in place of G, obtain H_2 and repeat the construction again. We will eventually stop, namely obtain that $H_i = H_{i+1}$, because every time when we repeat the construction we decrease one of the following characteristics:

(1) the number of free factors in the free decomposition on some level of H_i;
(2) if the number of free factors does not decrease, then the number of edges and vertices of the induced decomposition on some level of H_i decreases;
(3) if the number of free factors in the free decomposition on some level of H_i, and the number of edges and vertices does not decrease, then the size of H_i is decreased.

We end up with an NTQ system $Q = 1$ such that $G \leqslant F_{R(Q)} \leqslant F_{R(S)}$ and the image of the top i levels of $F_{R(Q)}$ on the level $j + 1$ is the same as the image of G on this level. Each QH-subgroup of the induced system is a finite index subgroup in some QH-subgroup of $S = 1$. The size of the induced system is the size of the tuple of these larger QH-subgroups.

Similarly if we have a fundamental sequence (and NTQ system) modulo a subgroup we can define induced fundamental sequence (and NTQ system) modulo a subgroup.

7.13. Relative dimension

Let $W = 1$ be a system of equations and $S = 1$ an NTQ system. Assume also that an embedding $\alpha : F_{R(W)} \to F_{R(S)}$ is given. Denote by $V_{\mathrm{fund}}(S)$ the fundamental sequence of solutions of $S = 1$. Consider homomorphisms $G \to F * F_1$ of the type $\alpha\phi$, where $\phi \in V_{\mathrm{fund}}(S)$, which map G onto a free group $F * F_1$. The *dimension of the system* $W = 1$ *relative* to $V_{\mathrm{fund}}(S)$ is the maximal rank of F_1. We denote it $\dim_S(W)$.

7.14. Generic families of solutions for NTQ systems

In Sections 7.7 and 7.9 we constructed fundamental sequences of solutions of the system $U = 1$ corresponding to each branch of the tree T_{ACE}. Similarly we can construct fundamental sequences of solutions of the canonical NTQ systems for the system $U = 1$ modulo a subgroup $H = \langle \bar{h} \rangle$, where $\bar{h} = \{h_1, \dots, h_t\}$ of the group $F_{R(U)}$. Each such fundamental sequence terminates with a group that does not have a sufficient splitting modulo H.

We will give now a definition of a generic family of solutions of a NTQ system modulo a subgroup H. First we recall the definition of a generic family of solutions of a NTQ system.

Definition 22. A family of solutions Ψ of a regular NTQ system $W(X, A) = 1$ is called *generic* if for any equation $V(X, Y, A) = 1$ the following is true: if for any solution from Ψ there exists a solution of $V(X^{\psi}, Y, A) = 1$, then $V = 1$ admits a complete W-lift.

A family of solutions Θ of a regular quadratic equation $S(X) = 1$ over a group G is called *generic* if for any equation $V(X, Y, A) = 1$ with coefficients in G the following is true: if for any solution $\theta \in \Theta$ there exists a solution of $V(X^{\theta}, Y, A) = 1$ in G, then $V = 1$ admits a complete S-lift. If the equation $S(X) = 1$ is empty ($G_{R(S)} = G * F(X)$) we always take as a generic family a sequence of growing different Merzljakov's words (defined in [14, Section 4.4]).

Let $W(X, A) = 1$ be a NTQ system that consists of equations $S_1(X_1, \dots, X_n) = 1, \dots, S_n(X_n) = 1$. A family of solutions Ψ of $W(X, A) = 1$ is called *generic* if $\Psi = \Psi_1 \cdots \Psi_n$, where Ψ_i is a generic family of solutions of $S_i = 1$ over G_{i+1} if $S_i = 1$ is a regular quadratic system, and Ψ_i is a discriminating family for $S_i = 1$ if it is a system of the type U_{com}.

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}^k$ the set of all k-tuples of elements from $\mathbb{N}$. For $s \in \mathbb{N}$ and $p \in \mathbb{N}^k$ we say that the tuple p is *s-large* if every coordinate of p is greater then s. Similarly, a subset $P \subset \mathbb{N}^k$ is *s-large* if every tuple in P is s-large. We say that the set P is *unbounded* if for any $s \in \mathbb{N}$ there exists an s-large tuple in P.

A family of automorphisms

$$\Gamma_P = \{\phi_{j,p},\ j \in N,\ p \in P\}$$

constructed in [14, Sections 7.1, 7.3] (alternatively, see [20, Section 5]) is called *positive s-large* if j and P are s-large. It is called *positive unbounded* if P and j are unbounded. In [14] this family was constructed for quadratic equations. We also constructed there a positive unbounded family of solutions $\{\psi_{j,p} = \phi_{j,p}\beta\}$ of a regular quadratic equation. Such a family was proved to be generic. The same family of automorphisms can be considered for a QH-subgroup of a fully residually free group. A solution of a system of equations $[y_i, y_j] = [y_i, u] = 1$ corresponding to an abelian vertex group on some level of a NTQ group is called s-large if $y_1 = u^{s_1}, \ldots, y_k = u^{s_1 \cdots s_k}$, $s_i > s$.

Definition 23. Let $W(X, A) = 1$ be a NTQ system modulo H that consists of equations $S_1(X_1, \ldots, X_n) = 1, \ldots, S_n(X_n) = 1$. Let $G_{n+1} = F_{R(T(Z))}$ be the terminal group of this system, G_{n+1} does not have a sufficient splitting modulo H. A family of solutions Ψ of $W(X, A) = 1$ is called *generic* if it can be represented as $\Psi = \Psi_1 \cdots \Psi_n \Phi$, satisfying the following conditions.

(1) Ψ_i is a generic family of solutions of $S_i = 1$ over G_{i+1} if $S_i = 1$ is a regular quadratic system, and Ψ_i is a discriminating family for $S_i = 1$ if it is a system of the type U_{com}.
(2) Φ is a family of algebraic solutions $\{Z_n, \bar{h}_n\}$ of $T = 1$ such that for each value $\bar{h}_n$ a corresponding NTQ system $N(Y, Z, \bar{h}_n) = 1$ for $T(Z, \bar{h}_n) = 1$ satisfies conditions of Section 7.9 and solution Y_n, Z_n is positive s-large for $s = n \max |h_{n,i}|$ for each MQH and abelian subgroup on each level of the system $N(Y, Z, \bar{h}_n) = 1$ and a family of large different Merzljakov words (depending on n) for empty equations (words (17) from [14]).

8. Algorithms over fully residually free groups

8.1. Algorithms for equations and coordinate groups

In this section we collect some results on algorithmic problems concerning equations over free groups and their coordinate groups. We assume below that a coordinate group G is given by a finite system of equations $S(X) = 1$ over F in such a way that $G = F_{R(S)}$.

Theorem 15. *The following statements are true*:

(1) *there is an algorithm which for a given finite system of equations $S(X) = 1$ over F and a given group word $w(X)$ in $X \cup A$ determines whether $w(X)$ is equal to* 1 *in $F_{R(S)}$ or not*;
(2) *there is an algorithm which for a given finite systems of equations $S(X) = 1$, $T(X) = 1$ over F decides whether or not $R(S) = R(T)$.*

Proof. (1) Let $S(X) = 1$ be a finite system of equations over F and w a group word in the alphabet $X \cup A$. Then the universal sentence

$$\Phi_{S,w} = \forall X \left(S(X) = 1 \to w(X) = 1\right)$$

is true in the free group F if and only if $w \in R(S)$. Since the universal theory $Th_{\forall}(F)$ is decidable [24] one can effectively check whether or not $\Phi_{S,w} \in Th_{\forall}(F)$. This proves (1). Now (2) follows immediately from (1). □

Theorem 16. ([13], see also Theorem 37 below.) *There is an algorithm which for a given system of equations $S(X) = 1$ over F finds finitely many NTQ systems $Q_1 = 1, \ldots, Q_n = 1$ over F and F-homomorphisms $\phi_i : F_{R(S)} \to F_{R(Q_i)}$ such that any F-homomorphism $\lambda : F_{R(S)} \to F$ factors through one of the homomorphisms ϕ_i, $i = 1, \ldots, n$.*

Corollary 6. *There is an algorithm which for a given finite system of equations $S(X) = 1$ over F finds finitely many groups $G_1, \ldots, G_n \in \mathcal{F}$ (given by finite presentations in generators $X \cup A$) and epimorphisms $\phi_i : F_{R(S)} \to G_i$ such that any homomorphism $\phi : F_{R(S)} \to F$ factors through one of the epimorphisms $\phi_1, \ldots, \phi_n$.*

Proof. The result follows from Theorems 16 and 29. □

Theorem 17. [12] *Given an NTQ system Q over F one can effectively find an embedding $\phi : F_{R(Q)} \to F^{\mathbb{Z}[t]}$.*

Theorem 18. *Given a finite system of equations $S = 1$ over F and an F-homomorphism $\phi : F_{R(S)} \to H$ of $F_{R(S)}$ into an NTQ F-group H one can effectively decide whether ϕ is a monomorphism or not.*

Proof. Let $G = F_{R(S)}$ for a finite system $S = 1$ over F and H an NTQ F-group. Suppose $\phi : F_{R(S)} \to H$ is a homomorphism given by the set of images $X^{\phi} \subseteq H$. By Theorem 29 one can effectively find a finite set, say T, of defining relations of the subgroup $G^{\phi} = \langle X^{\phi} \rangle \leqslant H$ with respect to the generating set X^{ϕ}. It follows that $R(S) \leqslant R(T)$. Clearly, ϕ is a monomorphism if and only if $R(S) = R(T)$. The latter can be checked effectively by Theorem 15. □

Theorem 19. *Given a finite irreducible system of equations $S = 1$ over F one can effectively find an NTQ system Q over F and an embedding $\phi : F_{R(S)} \to F_{R(Q)}$.*

Proof. By Theorem 16 one can effectively find finitely many NTQ systems $Q_1, \ldots, Q_n$ over F and F-homomorphisms $\phi_i : F_{R(S)} \to F_{R(Q_i)}$ such that any F-homomorphism $\lambda : F_{R(S)} \to F$ factors through one of the homomorphisms ϕ_i, $i = 1, \ldots, n$. It is known (see [1,12,13]) that in this case at least one of the homomorphisms ϕ_i is monic. Now by Theorem 18 one can effectively check which homomorphisms among $\phi_1, \ldots, \phi_n$ are monic. This proves the theorem. □

Theorem 20. *There is an algorithm which for a given finite irreducible system of equations $S(X) = 1$ over F finds a finite representation of the coordinate group $F_{R(S)}$ with respect to the generating set $X \cup A$.*

Proof. By Theorem 19 one can effectively find an NTQ system Q over F and an embedding $\phi : F_{R(S)} \to F_{R(Q)}$. It follows that one can effectively find the finite generating set $X^\phi \cup A$ of the subgroup $F^\phi_{R(S)}$ of the group $F_{R(Q)}$. Now, since the group $F_{R(Q)}$ belongs to $\mathcal{F}$ the result follows immediately from Theorem 29. □

Corollary 7. *For every finite irreducible system of equations $S = 1$ one can effectively find the radical $R(S)$ by specifying a finite set of generators of $R(S)$ as a normal subgroup.*

Theorem 21. [12] *There is an algorithm which for a given finite system of equations $S(X) = 1$ over F finds its irreducible components.*

Proof. We give here another, more direct, proof of this result. By Corollary 6 one can effectively find finitely many groups $G_1, \ldots, G_n \in \mathcal{F}$, given by finite presentations $\langle X \cup A \mid T_i \rangle$, and epimorphisms $\phi_i : F_{R(S)} \to G_i$ such that any homomorphism $\phi : GF_{R(S)} \to F$ factors through one of the epimorphisms $\phi_1, \ldots, \phi_n$. It is not hard to see that the systems $T_i = 1$ are irreducible over F and

$$V_F(S) = V_F(T_1) \cup \cdots \cup V_F(T_n). \tag{20}$$

Now by Theorem 15 one can check effectively whether or not $V_F(T_i) = V_F(T_j)$ or $V_F(T_i) = V_F(S)$, thus producing all irreducible components of $V_F(S)$. □

8.2. Algorithms for f.g. fully residually free groups

In this section we collect some results on algorithmic problems for finitely generated fully residually free groups from [13,14,17,25]. We assume below that groups from $\mathcal{F}$ are given by finite presentations. Notice that if $\langle X \mid S \rangle$ is a finite presentation of a group $G \in \mathcal{F}$ then $S(X)1$ can be viewed as a finite irreducible coefficient-free system of equations over F and $R(S) = ncl(S)$ where the radical $R(S)$ is taken in the free group $F(X)$ (without coefficients from F). This allows one to apply the algorithmic results from the Section 8.1 to groups from $\mathcal{F}$.

Theorem 22. *The word problem is decidable in groups from $\mathcal{F}$.*

Proof. Let $G \in \mathcal{F}$ and $g \in G$. Since G is finitely presented one can effectively enumerate all consequences of relators of G, so if $g = 1$ then g will occur in this enumeration. On the other hand, one can effectively enumerate all homomorphisms $\phi_1, \phi_2, \ldots,$ from G into a given free group F (say of rank 2). If $g \neq 1$ then, since G is residually free, there exists ϕ_i such that $\phi_i(g) \neq 1$, which can be verified effectively (by trying one by one all the images $\phi_1(g), \phi_2(g), \ldots$). This shows that the word problem is decidable in G, as required. □

Theorem 23. [17] *The conjugacy problem is decidable in groups from $\mathcal{F}$.*

Notice, that Theorem 23 also follows from [2], because finitely generated fully residually free groups are relatively hyperbolic [5].

Theorem 24. [25] *The membership problem is decidable in groups from $\mathcal{F}$. Namely, there exists an algorithm which for a group $G \in \mathcal{F}$ given by a finite presentation $\langle X \mid R \rangle$ and a finite tuple of* *words $h_1(X), \ldots, h_k(X), w(X)$ in the alphabet $X^{\pm 1}$ decides whether or not the element $w(X)$ belongs to the subgroup $\langle h_1(X), \ldots, h_k(X)\rangle$.*

Using an analog of Stallings' foldings introduced in [25] for finitely generated subgroups of $F^{\mathbb{Z}[t]}$, one can obtain the following results.

Theorem 25. [17] *Let $G \in \mathcal{F}$ and H and K finitely generated subgroups of G given by finite generating sets. Then $H \cap K$ is finitely generated, and one can effectively find a finite set of generators of $H \cap K$.*

Theorem 26. [17] *Let $G \in \mathcal{F}$ and H and K finitely generated subgroups of G given by finite generating sets. Then one can effectively find a finite family $\mathcal{J}_G(H, K)$ of non-trivial finitely generated subgroups of G (given by finite generating sets), such that*

(1) *every $J \in \mathcal{J}_G(H, K)$ is of one of the following types*

$$H^{g_1} \cap K, \qquad H^{g_1} \cap C_K(g_2),$$

where $g_1 \in G \setminus H$, $g_2 \in K$, moreover, g_1, g_2 can be found effectively;

(2) *for any non-trivial intersection $H^g \cap K$, $g \in G \setminus H$ there exists $J \in \mathcal{J}_G(H, K)$ and $f \in K$ such that*

$$H^g \cap K = J^f,$$

moreover, J and f can be found effectively.

Corollary 8. *Let H, K be finitely generated subgroups of finitely generated fully residually free group G. Then one can effectively verify whether or not K is conjugate into H, and if it is, then find a conjugator.*

Corollary 9. *Let H, K be finitely generated subgroups of finitely generated fully residually free group G, let H be abelian. Then one can effectively find a finite family $\mathcal{J}$ of non-trivial intersections $J = H^g \cap K \neq 1$ such that any non-trivial intersection $H^{g_1} \cap K$ has form J^k for some $k \in K$ and $J \in \mathcal{J}$. One can effectively find the generators of the subgroups from $\mathcal{J}$.*

Theorem 27. *Given a group $G \in \mathcal{F}$, a splitting D of G, and a finitely generated freely indecomposable subgroup H of G (given by a finite generating set Y) one can effectively find the splitting D_H of H induced from D. Moreover, one can describe all the vertex and edge groups, and homomorphisms, which occur in D_H, explicitly as words in generators Y.*

Proof. Following Lemma 2 one can construct effectively the graph of groups for the subgroup H using Theorem 26. Indeed, for every vertex $v \in X$ by Theorem 26 one can find effectively the complete finite family of non-trivial subgroups such that each intersection $H \cap G_v^g$ were g runs over G contains a conjugate of one of them in H. If all these intersections are trivial then by Lemma 2 the subgroup H is cyclic. Otherwise there exists a vertex $v \in X$ and an element $g \in G$ such that the subgroup $H_v = H \cap G_v^g$ is non-trivial. We start building the graph of groups Γ_H for H induced from G with the graph of groups Γ_1 consisting of the vertex v and the subgroup H_v

associated with it. Now for every edge e outgoing from v in the graph X we find by Corollary 9 the complete finite set (up to conjugation in H_v) of non-trivial intersections $H_e = H_v \cap G_e^g$ were g runs over G. If there are no non-trivial intersections of this type then either $H_v = H$ or H_v is a free factor of H. Since, by Theorem 24 the membership problem is decidable in G one can effectively check whether $H = H_v$ or not. Suppose that $H_e = H_v \cap G_e^g \neq 1$ for some $g \in G$ and an edge e outgoing from v. If u is the terminal vertex of e then $H_u \neq 1$ and we have reconstructed an edge in the graph Γ_H. Denote by Γ_2 the graph of groups obtained from Γ_1 by adding the edge e and the vertex u with the associated subgroups H_e and H_u. If the fundamental group $\pi(\Gamma_2)$ is equal to H then we are finished, otherwise we continue as above. In finitely many steps we will get a graph of groups Γ_k such that $\pi(\Gamma_k) = H$. This proves the theorem. □

Theorem 27 allows one to prove an effective version of Theorem 6.

Theorem 28. *Let G be a finitely generated group which is given as a finite sequence*

$$F = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G \tag{21}$$

of extensions of centralizers $G_{i+1} = G_i(u_i, t_i)$. Then given a finite set of elements $Y \subseteq G$ one can effectively construct the subgroup $\langle Y \rangle$ generated by Y in G from free groups by finitely many operations of the following type:

(1) *free products*;
(2) *free products with amalgamation along cyclic subgroups with at least one of them being maximal*;
(3) *separated HNN-extensions along cyclic subgroups with at least one of them being maximal*;
(4) *free extensions of centralizers*;

in such a way that all groups and homomorphisms which occur during this process are given explicitly as words in generators Y.

Proof. The result follows from Theorem 27 by induction on the length of sequence (21). □

Corollary 10. *There is an algorithm which for a given finitely generated fully residually free group G determines whether G is hyperbolic or not.*

Proof. Let G be a finitely generated fully residually free group. By Theorems 17 and 19 one can effectively embed G into a group G_n which is obtained from a free group F by a finite sequence of extensions of centralizers as in 21. Now, by Theorem 28 one can effectively construct G from F by finitely many operations of the types (1)–(4). It is known (see, for example, [10]) that in this event G is hyperbolic if and only if no operations of the type (4) occurred in the construction of G. The latter can be checked algorithmically when the finite sequence of operations is given. □

As a corollary of Theorem 27 one can immediately obtain the following result.

Theorem 29. *There is an algorithm which for a given NTQ system Q over F and a finitely generated subgroup $H \leqslant F_{R(Q)}$, given by a finite generating set Y, finds a finite presentation for H in the generators Y.*

Proof. By Theorem 17 one can effectively embed $F_{R(Q)}$ into $F^{\mathbb{Z}[t]}$. It follows that one can effectively embed $F_{R(Q)}$ into a finitely generated group H which is obtained from F by a finite sequence of extensions of centralizer. Now the result follows from Theorem 28. □

Theorem 30. *There is an algorithm which for a given group $G \in \mathcal{F}$ and a finitely generated subgroup $H \leqslant G$ (given by a finite generating set Y), finds a finite presentation for H in the generators Y.*

Proof. Let $G = \langle X \mid S \rangle$ be a finite presentation of G. As we have mentioned above one can view the relations $S(X) = 1$ as a coefficient-free finite irreducible system of equations over F with the coordinate group G. By Theorem 19 one can effectively embed the group G into the coordinate group $F_{R(Q)}$ of an NTQ system Q over F. Now the result follows from Theorem 29. □

Observe also, that Theorem 29 can be derived from [9, Theorem 5.8].

Theorem 31. *There is an algorithm which for a given homomorphism $\phi : G \to H$ between two groups from $\mathcal{F}$ decides whether or not*:

(1) *ϕ is an epimorphism*;
(2) *ϕ is a monomorphism*;
(3) *ϕ is an isomorphism.*

Proof. Let $G, H \in \mathcal{F}$ and $\langle X \mid S \rangle$ a given finite presentation of G.

(1) Obviously, ϕ is onto if and only if $G^\phi H$. Observe that G^ϕ is generated by a finite set X^ϕ, so by Theorem 24 one can effectively verify whether $G^\phi = H$ or not, as required.

(2) By Theorem 29 one can effectively find a finite presentation, say $\langle X \mid T \rangle$, of the subgroup G^ϕ of H with respect to the generating set X^ϕ. Clearly, ϕ is monic if and only if $R(S) = R(T)$ which can be effectively verified by Theorem 15. This proves (2) and the theorem since (3) follows from (1) and (2). □

Theorem 32. [21] *The Diophantine problem is decidable in groups from $\mathcal{F}$. Namely, there is an algorithm which for a given group $G \in \mathcal{F}$ and an equation $S = 1$ over G decides whether or not the equation $S = 1$ has a solution in G (and finds a solution if it exists).*

8.3. Effectiveness of a free decomposition

Theorem 33. [19] *There is an algorithm which for every finitely generated fully residually free group G and its finitely generated subgroup H determines whether or not G has a non-trivial free decomposition modulo H. Moreover, if G does have such a decomposition, the algorithm finds one.*

Corollary 11. [19] *There is an algorithm which for every finitely generated fully residually free group G and its finitely generated subgroup H finds a free decomposition of G modulo H*

$$G \simeq G_1 * \cdots * G_n$$

such that $G_1 \leqslant H$ and G_1 is freely indecomposable modulo H, G_i is freely indecomposable non-trivial group for every $i = 2, \ldots, n$.

8.4. Effectiveness of a JSJ decomposition modulo subgroups

Theorem 34. [19] *There exists an algorithm to obtain a cyclic and abelian JSJ decompositions for a f.g. fully residually free group modulo the subgroup of constants F.*

Corollary 12. *Let $U = 1$ be an irreducible system over F, such that $G = F_{R(U)}$ is a non-abelian non-surface group. Then one can effectively find a sufficient cyclic and abelian splitting D of G.*

Theorem 35. [19] *There exists an algorithm to obtain an abelian JSJ decomposition for a f.g. fully residually free group modulo subgroups $K_1, \ldots, K_m$.*

9. Decision algorithm for ∃∀-sentences

In this section we will describe a procedure to verify if an ∃∀-sentence with constants from a free group F is true in F. This gives the basic mechanism, as well as the basis of induction, for general verification procedure for arbitrary sentences. We start with a brief description of the algorithm in Section 9.1, then we formalize this description and construct the main tool of the proof, the so-called ∃∀-tree $T_{EA}(G)$. Finally, in Section 9.5 we show that the tree $T_{EA}(G)$ is finite, so the decision procedure for ∃∀-sentences eventually terminates. It will be clear from the construction that the described algorithm does not depend on the rank of F, i.e., if a given ∃∀-sentence (with constants from F) holds in F then it holds in every free group $F * F_1$ which is a free product of F and a free group F_1.

9.1. Brief description of the algorithm

Let Φ be an arbitrary ∃∀-sentence in the language of groups with constants from F. By Lemma 10 from [14] we may assume that Φ has the following form:

$$\Phi = \exists X \forall Y \ \big(U(X, Y) = 1 \to V(X, Y) = 1\big),$$

where U, V are words in variables $X \cup Y$ and constants from F. The task is to determine effectively if Φ holds in F or not. The verification procedure is divided into several steps. At each step we show that the decision procedure is effective and its output is the same for F and $F * F_1$ for any free group F_1.

(1) Suppose that

$$F \models \exists X \forall Y \ \big(U(X, Y) \neq 1\big). \tag{22}$$

Then Φ is true in F (since the premise of the implication is false). Observe, that

$$F \models \exists X \forall Y \ \big(U(X, Y) \neq 1\big) \quad \Leftrightarrow \quad F \not\models \forall X \exists Y \ \big(U(X, Y) = 1\big).$$

Hence to verify (22) it suffices to check whether or not

$$F \models \forall X \exists Y \ \big(U(X, Y) = 1\big).$$

The sentence above is positive, so by Merzljakov's theorem (see [14, Section 4.4]) there exists a decision algorithm to verify if this sentence hold in F or not. Moreover, this verification procedure does not depend on the rank of F, namely, it holds in F if and only if it holds in $F * F_1$. This finishes the case (1).

(2) Suppose now that (22) does not take place, so

$$F \models \forall X \exists Y \ \big(U(X,Y)=1\big).$$

Then, again, by the Merzljakov's theorem

$$F[X] \models \exists Y \ \big(U(X,Y)=1\big),$$

where elements from X are viewed as constants in $F[X]$. By Theorem 11.1 from [19] one can effectively describe all solutions $Y \in F[X]$ of the equation $U(X,Y)=1$ in terms of Hom-diagrams.

(2.1) Suppose that for any solution $Y = f(X) \in F[X]$ of the equation $U(X,Y)=1$ one has $V(X, f(X)) = 1$ in $F[X]$. We claim that in this case the formula Φ holds in F. Indeed, if Φ does not hold in F then the sentence

$$\neg\Phi = \forall X \exists Y \ \big(U(X,Y)=1 \wedge V(X,Y) \neq 1\big)$$

is true in F. By the general form of the Merzljakov's theorem [14, Theorem 4] F freely lifts $\neg\Phi$ so there is a solution $Y = f(X) \in F[X]$ of $U(X,Y)=1$ for which $V(X,Y) \neq 1$ in $F[X]$—contradiction with the assumption above. Notice that F lifts freely the sentence $\neg\Phi$ if and only if the following sentence holds in $F[X]$:

$$F[X] \models \exists Y \ \big(U(X,Y)=1 \wedge V(X,Y) \neq 1\big).$$

Since the group $F[X]$ is free, the sentence above is existential, and the existential theory of free groups is decidable, then there exists an effective procedure to check whether or not F lifts freely $\neg\Phi$. Furthermore, this procedure does not depend on the rank of F, as required.

(2.2) Suppose now that the case (2.1) does not hold, i.e., there exists a solution $Y = f(X) \in F[X]$ of $U(X,Y)=1$ such that $V(X, f(X)) \neq 1$. Hence, the set

$$S = \big\{ f(X) \in F[X] \bigm| U\big(X, f(X)\big) = 1 \text{ and } V\big(X, f(X)\big) \neq 1 \text{ in } F[X] \big\}$$

is non-empty. By the equational Noetherian property the system

$$\mathcal{V} = \big\{ V\big(X, f(X)\big) = 1 \bigm| f(X) \in S \big\}$$

(where elements from X are viewed as variables) is equivalent over F to some finite subsystem of itself, hence it is equivalent over F to a single equation, say $U_0(X)=1$.

Observe, that any specialization $\eta : X \to F$ of the variables X in the formula Φ which makes Φ true in F, i.e., any specialization η for which the formula

$$\forall Y \ \big(U\big(X^{\eta}, Y\big) = 1 \to V\big(X^{\eta}, Y\big) = 1\big)$$

holds in F, must satisfy the equation $U_0(X^\eta) = 1$. Indeed, if $U_0(X^\eta) \neq 1$ then there exists $f(X) \in S$ such that $V(X^\eta, f(X^\eta)) \neq 1$. Therefore, $Y = f(X^\eta)$ satisfies the equation $U(X^\eta, Y) = 1$ but does not satisfy the equation $V(X^\eta, Y) = 1$—contradiction. The observation above is rather crucial, it puts some new restrictions on all the specializations $\eta : X \to F$ that satisfy the formula Φ. To be able to use these restrictions in our decision procedure we need to find the subsystem $U_0(X) = 1$ effectively as follows.

One has $V(X_0, Y) = 1$ for any $Y \in V_{\mathrm{fund},i}$ if $V(X_0, Y) = 1$ for any Y obtained from some minimal small cancellation solution $Y_{0,i} = Y^\beta(X_0)$ by applying canonical automorphisms from a suitable finite family

$$\mathcal{B} = \{\phi_{L-4K,p}\phi_{2K,q}\}$$

(see [14], the discussion preceding Proposition 8) corresponding to different levels of the fundamental sequence. Both conditions are equivalent to the consistency of a system of type $\Delta(M_{\text{very short}}) = 1$ [14, Proposition 8]. This family can be effectively found. By [14, Lemma 67], one can effectively find a small cancellation solution β (provided X_0 does not satisfy some degenerate equation, which we can consider separately). The equation $U_0(X) = 1$ is equivalent to the system $\{V(X, f(X)) = 1 \mid f(X) = \phi(\beta(X)),\ \phi \in \mathcal{B}\}$.

(2.2.1) Suppose $U_0(X) = 1$ does not have solutions in F. In this case there is no specialization $\eta : X \to F$ that satisfies the equation $U_0(X^\eta) = 1$, so Φ is false in F. Notice, that given $U_0(X) = 1$ one can effectively verify whether or not it has a solution in F and, of course, this verification does not depend on the rank of F.

(2.2.2) Suppose that the equation $U_0(X) = 1$ has a solution in F. The argument in the case (2.2) shows that any specialization $\eta : X \to F$, for which the formula

$$\forall Y\ \big(U\big(X^\eta, Y\big) = 1 \to V\big(X^\eta, Y\big) = 1\big)$$

holds in F, satisfies the equation $U_0(X) = 1$ in F, so it factors through $F_{R(U_0)}$, i.e., $\eta = \pi\xi$ where $\pi : F[X] \to F_{R(U_0)}$ is the canonical epimorphism and $\xi : F_{R(U_0)} \to F$ is an arbitrary F-homomorphisms. Moreover, any F-homomorphism from $F_{R(U_0)}$ into F factors through the coordinate groups of one of the finitely many irreducible components of $U_0 = 1$. It has been shown in [19] that these irreducible components of $U_0 = 1$ can be found effectively. To keep notations simple we may assume from the beginning that U_0 is already irreducible over F (otherwise we replace U_0 by a suitable irreducible component of U_0).

(3) *Main strategy.* Our main strategy is to show that either using an argument similar to the one above one can effectively verify if the formula Φ holds in F, or the potential solution X^η of Φ satisfies a new proper equation. The latter cannot happen infinitely many times (since F is equationally Noetherian), which implies that in finitely many steps one would be able to verify if Φ holds in F or not.

(4) *Induction argument.* For induction we are going to generalize the argument from the case (2).

Recall, that the basic tool in this case is the lifting of various formulas (the formula $\exists Y\ U(X, Y) = 1$ in the beginning of the case (2), or the formula $\neg\Phi$ in the case (2.1)) into the group $F[X]$. This allows one to get solutions $Y = f(X)$ of the corresponding existential formulas as values of the words $f(X)$. To follow our main strategy we need to get the solutions $Y = f(X)$ assuming the variables X satisfy the newly obtained restrictions $U_0(X) = 1$. This amounts to the lifting of the corresponding formulas into the coordinate group $F_{R(U_0)}$, which is a proper quotient of $F[X]$. By implicit function theorem (see [14, Theorem 11]) and its converse (see [20, Theorem E]) one can lift an arbitrary equation into $F_{R(U_0)}$ if and only if U_0 is rationally equivalent to a regular NTQ system.

(4.1) Suppose U_0 is rationally equivalent to a regular NTQ system. By our assumption the case (1) does not hold so the formula

$$\forall X \exists Y \bigl(U_0(X) = 1 \to U(X, Y) = 1\bigr)$$

is true in F. Since U_0 is rationally equivalent to a regular NTQ system the equation $U(X, Y) = 1$ lifts into the coordinate group $F_{R(U_0)}$. By Theorem 15.4 from [19] one can effectively find all solutions Y of the system $U(X, Y) = 1$ in the group $F_{R(U_0)}$, where elements from X are viewed as constants from $F_{R(U_0)}$.

(4.1.1) Suppose that for any solution $Y = f(X) \in F_{R(U_0)}$ of the equation $U(X, Y) = 1$ one has $V(X, f(X)) = 1$ in $F[X]$. In this case the argument from (2.1) shows that the formula Φ holds in F. Indeed, if Φ does not hold in F then the sentence

$$\neg\Phi = \forall X \exists Y \bigl(U(X, Y) = 1 \wedge V(X, Y) \neq 1\bigr)$$

is true in F. This implies that the formula

$$\forall X \exists Y \bigl(U_0(X) = 1 \to \bigl(U(X, Y) = 1 \wedge V(X, Y) \neq 1\bigr)\bigr)$$

is also true in F. By implicit function theorem the formula $U(X, Y) = 1 \wedge V(X, Y) \neq 1$ lifts into $F_{R(U_0)}$. So

$$F_{R(U_0)} \models \exists Y \bigl(U(X, Y) = 1 \wedge V(X, Y) \neq 1\bigr),$$

contradicting the assumption above. Again, notice that the formula

$$\exists Y \bigl(U(X, Y) = 1 \wedge V(X, Y) \neq 1\bigr)$$

is existential and the group $F_{R(U_0)}$ has the same existential theory as the group F, as well as any non-abelian free group, therefore one can effectively check whether or not this formula lifts into $F_{R(U_0)}$ and the answer does not depend on the group F.

(4.1.2) Suppose now that the case (4.1.1) does not hold, i.e., there exists a solution $Y = f(X) \in F_{R(U_0)}$ of $U(X, Y) = 1$ such that $V(X, f(X)) \neq 1$. Observe, that $f(X)$ can be naturally viewed as a word from $F[X]$. It follows that the set

$$S = \bigl\{f(X) \in F[X] \bigm| U\bigl(X, f(X)\bigr) = 1 \text{ and } V\bigl(X, f(X)\bigr) \neq 1 \text{ in } F_{R(U_0)}\bigr\}$$

is not empty. By the equational Noetherian property the system

$$\mathcal{V}_1 = \bigl\{V\bigl(X, f(X)\bigr) = 1 \bigm| f(X) \in S\bigr\}$$

(where elements from X are viewed as variables) is equivalent over F to some finite subsystem of itself, hence it is equivalent over F to a single equation, say $U_1(X) = 1$. From construction $U_1(X)$ is not contained in the radical $R(U_0)$.

Precisely as in the case (2.2.1), if the equation $U_1(X)=1$ has no solutions in F then Φ is false in F. Therefore, one has to consider now the case analogues to (2.2.2) when $U_1(X)=1$ has solutions in F. Again, as in the case (2.2.2) one can show that any specialization $\eta : X \to F$, for which the formula

$$\forall Y\ \big(U\big(X^\eta, Y\big)=1 \to V\big(X^\eta, Y\big)=1\big)$$

holds in F satisfies the equation $U_1(X)=1$ in F, so it factors through $F_{R(U_1 \cup U_0)}$, which is a proper factor of $F_{R(U_0)}$. This gives new restrictions on X as required in the main strategy. Notice that in the case (4.1.2) we did not use the assumption that U_0 is rationally equivalent to a regular NTQ system, so the argument here holds for arbitrary systems U_0.

(4.2) Suppose $F_{R(U_0)}$ is not rationally equivalent to a regular NTQ system. The remark at the end of the case (4.1.2) shows that one needs to consider only the case (4.1.1) but for an arbitrary irreducible U_0. This is the hardest case.

(4.2.1) Suppose that for any solution $Y=f(X) \in F_{R(U_0)}$ of the equation $U(X,Y)=1$ one has $V(X, f(X))=1$ in $F[X]$. The argument in (4.1.1) allows us to assume that the formula

$$\forall X \exists Y\ \big(U_0(X)=1 \to \big(U(X,Y)=1 \wedge V(X,Y) \neq 1\big)\big) \tag{23}$$

is true in F. However, now we cannot lift equations into $F_{R(U_0)}$. To go around this problem we make use of the canonical embeddings Theorem 14 (Section 7.10) and parametrization theorem [14, Theorem 12]. This is a crucial point of the induction argument. Since it is technically quite demanding we explain here only the principal ideas leaving details for the next section.

We may assume that $F_{R(U_0)}$ is freely indecomposable, otherwise we can effectively split it into a free product of finitely many freely indecomposable factors (see Theorem 33 from Section 8.3) and continue with each of the factors in the place of $F_{R(U_0)}$. By the canonical embeddings theorem given the coordinate group $F_{R(U_0)}$, which is freely indecomposable (modulo F) and irreducible, one can effectively construct finitely many equations with coefficients in F

$$V_1(X)=1, \quad \ldots, \quad V_s(X)=1 \tag{24}$$

each of which does not belong to the radical $R(U_0)$ and finitely many NTQ systems

$$U_{D,1}=1, \quad \ldots, \quad U_{D,k}=1$$

over F with embeddings

$$\phi_1 : F_{R(U_0)} \to F_{R(U_{D,1})}, \quad \ldots, \quad \phi_k : F_{R(U_0)} \to F_{R(U_{D,k})}$$

such that the following holds (here we omit some properties mentioned in the canonical embeddings theorem):

$$V(U_0) = p_1\big(V(U_{D,1})\big) \cup \cdots \cup p_k\big(V(U_{D,k})\big) \cup V(U_0 \wedge V_1) \cup \cdots \cup V(U_0 \wedge V_s),$$

where the word mappings $p_i = (p_{i,1}, \ldots, p_{i,n})$ correspond to the embeddings $\phi_1, \ldots, \phi_k$, i.e., for every i and every $x_j \in X$

$$\phi_i(x_j) = p_{i,j}(z_1, \ldots, z_m, y_1, \ldots, y_t, g_1, \ldots, g_l).$$

By parametrization theorem if $U(X)=1$ is an NTQ system with a fundamental sequence of solutions $V_{\text{fund}}(U)$ and a formula

$$\Phi = \forall X\ \big(U(X)=1\big) \to \exists Y\big(W(X,Y)=1 \wedge W_1(X,Y)\neq 1\big)$$

is true in F, then one can effectively construct a finite set of NTQ systems $\mathcal{N}C_W(U) = \{\bar{U}_i \mid i \in I\}$ together with embeddings $\theta_i : F_{R(U)} \to F_{R(\bar{U}_i)}$ such that for every $\bar{U}_i \in \mathcal{N}C_W(U)$ the formula

$$\exists Y\ \big(W\big(X^{\theta_i}, Y\big) = 1 \wedge W_1\big(X^{\theta_i}, Y\big) \neq 1\big)$$

is true in the group $F_{R(\bar{U}_i)}$. Since the Diophantine problem in the groups $F_{R(\bar{U}_i)}$ is decidable one can effectively verify whether or not the formula above holds in the group. Moreover, if the formula holds the complete set of solutions can also be found effectively. Furthermore, for every fundamental solution $\phi : F_{R(U)} \to F$ there exists a fundamental solution ψ of one of the systems $\bar{U}_i = 1$, where $\bar{U}_i \in \mathcal{N}C_W(U)$ such that $\phi = \theta_i \psi$.

Recall that the set $\mathcal{N}C_W(U)$ was introduced in [14] right before parametrization theorem. The systems $\bar{U}_i$ are constructed in two steps which are called *normalizing* and *corrective* extensions, the corresponding embeddings $\theta_i : F_{R(U)} \to F_{R(\bar{U}_i)}$ are called correcting normalizing embeddings.

Returning to the system U_0, we apply, first, the canonical embeddings theorem to the system U_0 and obtain the NTQ systems $U_{D,1} = 1, \ldots, U_{D,k} = 1$ then we apply the parametrization theorem to each of the systems $U_{D,i}$.

It follows from (24) that each solution of $U_0(X)=1$ either satisfies one of the finite number of proper equations (24) or factors through a fundamental sequence of one of the canonical NTQ extensions $U_{D,1}=1, \ldots, U_{D,k}=1$. In the former case we are done with the induction argument, as required. In the latter case we continue as follows for each of the canonical NTQ extensions $U_{D,i}$. Recall that the formula (23) holds in F. The parametrization theorem applied to the systems $U_{D,i}$ and the formula (23) implies that one can lift the formula

$$\exists Y\ \big(U\big(X^{\theta_i}, Y\big) = 1 \wedge V\big(X^{\theta_i}, Y\big) \neq 1\big)$$

in the group $F_{R(\bar{U}_i)}$ for every $\bar{U}_i \in \mathcal{N}C_W(U_{D,i})$, so

$$F_{R(\bar{U}_i)} \models \exists Y\ \big(U\big(X^{\theta_i}, Y\big) = 1 \wedge V\big(X^{\theta_i}, Y\big) \neq 1\big).$$

Denote the NTQ system $\bar{U}_i$ by $S_1(X, X_1) = 1$ (it may have some extra variables X_1 since it is an extension of $U_0(X)=1$). For a solution $Y = f(X, X_1)$, $Y \in F_{R(S_1)}$ of the formula above consider an equation $V(X, f(X, X_1)) = 1$. We obtain an equation $U_1(X, X_1) = 1$ in variables X, X_1. There are four different cases, such that in each case the process goes differently.

(4.2.1) If the equation $U_1(X, X_1)=1$ is inconsistent, then no one X factoring through the fundamental sequence corresponding to the system $S_1(X, X_1)=1$, can make Φ true. In this case we have to consider another fundamental sequence corresponding to the equation $U_0(X)=1$. If for each fundamental sequence we have this case, then the sentence Φ is false.

(4.2.2) If the equation $U_1(X, X_1)=1$ belongs to the radical of $S_1(X, X_1)=1$, the sentence Φ is true, and X solving Φ can be obtained from a generic solution X, X_1 of the system

$S_1(X, X_1) = 1$. Indeed, for generic values of X all values of Y solving Φ can be obtained by specializations of Y given by the formulas $Y = f(X, X_1)$, and for them $V(X, Y) = 1$ is true.

(4.2.3) If the equation $U_1(X, X_1) = 1$ is a proper equation, namely, $F_{R(U_1)}$ is a proper quotient of $F_{R(S_1)}$, and corresponding values of X satisfy an equation $\bar{U}_0(X) = 1$ which is not in the radical of $U_0(X) = 1$, then we further will consider X satisfying $\bar{U}_0(X) = 1$.

(4.2.4) If the equation $U_1(X, X_1) = 1$ is a proper equation, but the corresponding values of X do not satisfy any proper equation, then each X solving Φ that satisfies $U_0(X) = 1$ (if exists) must also satisfy $U_1(X, X_1) = 1$, and we will consider such values of X.

We can suppose $U_1(X, X_1) = 1$ is irreducible, otherwise it is equivalent to a disjunction of finite number of irreducible equations. And again, equation $U_1(X, X_1) = 1$ is not equivalent to NTQ, therefore, we can only obtain Y given by a formula in some NTQ extension of $F_{R(U_0)}$. But this time we construct this extension taking into account that X, X_1 also satisfy $U_1(X, X_1) = 1$. This way we obtain a branching process and a tree. Simplifying the process we can say that each branch corresponds to a carefully chosen chain of block-NTQ groups considered together with fundamental sequences of homomorphisms. Each chain has the form:

$$F_{R(S_1)},\ F_{R(S_2)},\ \dots,\ F_{R(S_i)},\ \dots,$$

where for each i there is a homomorphism $\theta_i : F_{R(U)} \to F_{R(S_i)}$. Moreover, either $\theta_{i+1}(F_{R(U)})$ is a proper quotient of $\theta_i(F_{R(U)})$ or they are isomorphic and the dimension of the system $S_{i+1} = 1$ (the maximal possible rank r in an epimorphic image $F * F_r$) is less than the dimension of $S_i = 1$, or, in case of the same dimension, some other characteristic decreases. Therefore, the process is finite. If along some branch the process stops because we do not obtain a proper homomorphic image of $F_{R(S_i)}$ (namely, this will take place if $F_{R(U_i)}$ is isomorphic to $F_{R(S_i)}$), then the sentence Φ is true. If along all the branches the process stops because an inconsistent system is obtained (i.e., restrictions on X along this branch produced an inconsistent system), then the sentence Φ is false.

Notice that the process does not depend on the rank of F, therefore Φ is either true in all non-abelian free groups or false in all of them.

9.2. First step

We will now describe the construction of the ∀∃-tree. The initial step is described in Section 9.1. Suppose we came to case (2.2.2) in Section 9.1. Let $G = F_{R(U_0)}$, where $U_0 = 1$ is an irreducible system. We define now a tree $T_{EA}(G)$ oriented from the root, and assign to each vertex of $T_{EA}(G)$ some set of homomorphisms in $G \to F$. We assign a solution set $V(U_0)$ to the initial vertex $\hat{v}_0$ of $T_{EA}(G)$. Suppose that $U_0 = 1$ cannot be written in the NTQ form. We can construct a finite number of NTQ systems $S(b) = 1$ ($S(b)$: $S_1(X_1, \dots, X_k) = 1, \dots, S_k(X_k) = 1$)and corresponding fundamental sequences $V_{\mathrm{fund}}(S(b))$, each corresponding to one of the branches of $T_{CE}(G)$. For each such fundamental sequence we assign a vertex $\hat{v}_{1i}$ of the tree $T_{EA}(G)$. We draw an edge from vertex $\hat{v}_0$ to each vertex corresponding to $V_{\mathrm{fund}}(S(b))$.

In the case when $U_0 = 1$ is NTQ we just draw one edge on this step and assign to the vertex $\hat{v}_1$ the same set $V(U_0)$. Let $V_{\mathrm{fund}}(S(b))$ be assigned to $\hat{v}_{1i}$. By the parametrization theorem [14, Theorem 12], we can collect all values of Y given by formulas $Y = f(X_1, \dots, X_k)$ in variables $X_1, \dots, X_k$ of $F_{R(S(b))}$ and consider the equation $U_1(X_1, \dots, X_k) = 1$ which is equivalent to a system of all equations $V(X_1, \dots, X_k, f(X_1, \dots, X_k)) = 1$. Let $V_{\mathrm{fund}}(U_1)$ be the subset of homomorphisms from the set $V_{\mathrm{fund}}(S(b))$ going through some corrective extension of $S_{\mathrm{corr}}(b) = 1$, and satisfying the additional equation $U_1(X_1, \ldots, X_k) = 1$. We introduce a vertex $\hat{v}_{2i}$ and draw an edge connecting $\hat{v}_{1i}$ to $\hat{v}_{2i}$.

Let G_1 be a fully residually free group discriminated by the set of homomorphisms $V_{\mathrm{fund}}(U_1)$. Consider the family of canonical fundamental sequences for G_1 (constructed along the branches of the canonical embedding tree) modulo the images $P_1, \ldots, P_s$ of the factors in the free decomposition of the subgroup $H_1 = \langle X_2, \ldots, X_m \rangle$. If the dimension of such a sequence is greater than k_1, then the corresponding homomorphisms are contained in the fundamental sequence for $U = 1$ with the number k_1 greater than that for $V_{\mathrm{fund}}(S(b))$, and we do not consider this sequence here. So we only consider canonical fundamental sequences c for G_1 modulo the factors in the free decomposition of $H_1 = \langle X_2, \ldots, X_m \rangle$ (corresponding to coefficients of quadratic equations $S_1 = 1$ of the top level for $S_{\mathrm{corr}}(b) = 1$) with the following properties:

(1) they have dimension less or equal than k_1,
(2) G is embedded into the group discriminated by the fundamental sequence,
(3) c is consistent with the decompositions of quadratic equations of S_1 according to the remark in Section 7.8.

Namely, if we consider a cyclic decomposition D of G obtained from the JSJ decomposition by adding splittings corresponding to the free product in Section 7.8, then the standard coefficients on all the levels of c are images of elliptic elements in D.

Suppose a fundamental sequence c has the top dimension component k_1. If the system corresponding to the top level of the sequence c is the same as $S_1 = 1$, we extend the fundamental sequences modulo $P_1, \ldots, P_s$ by canonical fundamental sequences for H_1 modulo the factors in the free decomposition of the subgroup $\langle X_3, \ldots, X_m \rangle$. If such a sequence has dimension greater than or equal to k_2, then the corresponding solution can be factored through a fundamental sequence for $U = 1$ of the greater dimension. So we only consider such sequences of dimension less than or equal to k_2. If the sum of first two dimensions is strictly smaller than $k_1 + k_2$, we do the same as in case when the first dimension is smaller than k_1 (see below). We continue this way to construct fundamental sequences $V_{\mathrm{fund}}(S_1(b))$. We draw edges from the vertex corresponding to $V_{\mathrm{fund}}(U_1)$ to the vertices $V_{\mathrm{fund}}(S_1(b))$.

Suppose now that the fundamental sequence c for G_1 modulo $P_1, \ldots, P_s$ has dimension strictly less than k_1 or has dimension k_1, but the system corresponding to the top level of c is not the same as $S_1 = 1$. Suppose also that $G \neq F$. Then we use the following lemma (in which we suppose that $P_1, \ldots, P_s$ are non-trivial).

Lemma 23. *Either the image $G_{(p)}$ of G in the group $H_{(p)}$ appearing on some level p of sequence c is a proper quotient of G, or it is possible to take instead of $H_{(p)}$ a finite number of proper quotients of $H_{(p)}$ without loosing values of initial variables of $U = 1$.*

Proof. Consider the terminal group of c; denote it H_t. Suppose $G_{(p)}$ is isomorphic to G. Denote the abelian JSJ decomposition of H_t by D_t. Then there is an abelian decomposition of G induced by D_t. Therefore rigid (non-abelian and non-QH) subgroups and edge groups $\bar{K}_1, \ldots, \bar{K}_q$ of G are elliptic in this decomposition.

There exists a decomposition of some P_i which is induced from D_t. In this case instead of H_t we can take proper quotients of H_t corresponding to minimal solutions with respect to A_{D_t}. We do not lose values of the initial variables of $U = 1$ because this replacement corresponds to replacement of the fundamental sequence going through P_i by homomorphisms corresponding to some representatives of the equivalence classes with respect to the canonical group of automorphisms of P_i modulo those subgroups from $\{\bar{K}_1, \ldots, \bar{K}_q\}$ which are conjugated into $\hat{P}_i$ in G_b. $\square$

Suppose that $G_{(1)}, \ldots, G_{(p-1)}$ are isomorphic to G, and $G_{(p)}$ is a proper quotient of G. Consider the *block-NTQ group* $\bar{G}$ generated by the top p levels of the NTQ group corresponding to the fundamental sequence c and the group $F_{R(S_2(b_2))}$ corresponding to some branch of the tree $T_{CE}(G_{(p)})$ and amalgamated along $G_{(p)}$. In this case the fundamental sequence c considered modulo level $p-1$ does not have free variables. One can extract from c modulo level p a family of homomorphisms from G to $G_{(p)}$ induced by the fundamental sequence c. Denote this *extracted fundamental sequence* by c_2. We want the homomorphisms from different levels of c_2 to be restrictions of the homomorphisms from corresponding levels of c. Consider a fundamental sequence c_3 that consists of homomorphisms obtained by the composition of a homomorphism from c_2 and from a fundamental sequence corresponding to b_2. Assign some vertex $\hat{v}_{3j}$ of the tree $T_{EA}(G)$ to sequence c_3. We draw an edge from the vertex of $T_{EA}(G)$ corresponding to $V_{\text{fund}}(U_1)$ to $\hat{v}_{3j}$.

Using the parametrization theorem, one can lift equations compatible with equations in the variables of $U=1$ in corrective extensions of $\bar{G}$, which can be constructed as in [14, Section 7.5] (see also [20]).

9.3. Second step

We will describe the next step in the construction of $T_{EA}(G)$ which basically is general. Let $c_3 = V_{\text{fund}}(S^{(1)}(d))$ be a fundamental sequence corresponding to some vertex $\hat{v}_{3i}$ of $T_{EA}(G)$, let c be, as before, the corresponding canonical fundamental sequence for G_1 modulo $P_1, \ldots, P_s$. Consider the set of those homomorphisms from $\bar{G}$ in F which are going through the fundamental sequence c, which factor through a corrective extension $\bar{V}_{\text{corr}}(S^{(1)}(d))$, and satisfy some additional equation $U_2 = 1$. Let G_2 be a fully residually free group discriminated by this set.

The case when the natural image $G^{(2)}$ of G in G_2 is a proper quotient of G is the first "easy" case. In this case we assign to vertices $\hat{v}_{4j}$ the fundamental sequences corresponding to branches of $T_{CE}(G^{(2)})$ and draw edges from $\hat{v}_{3i}$ to all these vertices.

In all the other considerations below we suppose that $G^{(2)}$ is isomorphic to G.

Let the JSJ decomposition for the top level of c corresponds to the equation $S_{11}(X_{11}, X_{12}, \ldots) = 1$; some of the variables X_{11} are quadratic, the others correspond to extensions of centralizers. Construct a fundamental sequence $c^{(2)}$ corresponding to the branch of the tree $T_{CE}(G_2)$ modulo the factors in the free decomposition of the subgroup generated by $X_{12}, \ldots$.

Denote by N_0^1 the image of the subgroup generated by $X_1, \ldots, X_n$ in the group discriminated by c. So, $N_0^1 = \langle X_1, \ldots, X_n \rangle_c$. Denote by $N_0^2 = \langle X_1, \ldots, X_n \rangle_{c^{(2)}}$ the image of $\langle X_1, \ldots, X_n \rangle$ in the group discriminated by $c^{(2)}$. If N_0^2 is a proper quotient of N_0^1 and c is not the same as the top level of $S(b)$, we have another "easy" case. In this case we do what we did at the previous step taking N_0^2 instead of G_2 and we do not consider vertices corresponding to NTQ systems with the same top level as $S^{(1)}(d)$.

In all the cases below we suppose that N_0^1 is isomorphic to N_0^2.

Case 1. The top levels of c and $c^{(2)}$ are the same, then we go to the second level of c and consider it the same way as the first level.

Case 2. The top levels of c and S_1 are the same (therefore c has only one level). We work with $c^{(2)}$ the same way as we did for c. Then either the image of G on some level k of $c^{(2)}$ is a proper quotient of G, or we can take proper quotients of the group on this level instead of this groups using Lemma 23. If at some point the sum of dimensions for $c^{(2)}$ is not maximal, we amalgamate the top part of $c^{(2)}$ above level k and this quotient.

Case 3. The top levels of c and S_1 are not the same and the top levels of c and $c^{(2)}$ are not the same, then there is a level k of $c^{(2)}$ such that we can suppose that the image of either G or N_0^1 on this level is a proper quotient. Consider fundamental sequences for this quotient. Denote them by f_i. Construct fundamental sequences for the subgroup generated by $X_1, \ldots, X_n$ with the top part being extracted from the top part of $c^{(2)}$ (above level k) and bottom part being some f_i, but not the sequence with the same top part as c.

9.4. General step

We now describe the nth step of the construction. Denote by N_i the block-NTQ group constructed on the ith step, and by N_i^j, $j > i$, the image of it on the jth step. Let $\{j_k,\ k = 1, \ldots, s\}$ be all the indices for which the top level of N_{j_k+1} is different from the top level of N_{j_k}.

If G or some of the groups $N_{j_k}^{j_k+1}$ is not embedded into N_{n-1}^n we replace the first such group by its proper quotient in N_{n-1}^n and consider only the fundamental sequences that have the top level different from N_{j_k}. In all other cases we can suppose that the groups G and all $N_{j_k}^{j_k+1}$ are embedded into N_{n-1}^n.

Case 1. The top levels of $c^{(n)}$ and $c^{(n-1)}$ are the same. In this case we go to the second level and consider it the same way as the first level.

If going from the top to the bottom of the block-NTQ system, we do not obtain the case considered above or Cases 2, 3 and all the blocks of N_{n-1} and N_n are the same, we do the following. Consider the group G_{corr} which was constructed for the fundamental sequence corresponding to the homomorphisms from G going through N_{n-1}, and the image of this group G' in N_n, which is a proper quotient of G_{corr}. Then $G < G'$ and we take G' instead of G.

Case 2. The top levels of c^{n-1} and $c^{n-2}, \ldots, c^{n-i}$ are the same, and the top levels of c^{n-1} and c^n are not the same. Then on some level p of $c^{(n)}$ we can suppose that the image of N_{n-i-1}^{n-i} is a proper quotient (or the fundamental sequence goes through another branch constructed on the previous step). Consider fundamental sequences f_i for this quotient with the top level different from $c^{(n-i-1)}$. Construct N_n as a block-NTQ group with the top part being $c^{(n)}$ above level p and the bottom part f_i.

Case 3. The top levels of $c^{(n-2)}$ and $c^{(n-1)}$ are not the same and the top levels of $c^{(n-1)}$ and $c^{(n)}$ are not the same. Then on some level p of $c^{(n)}$ the image of N_{n-2}^{n-1} is a proper quotient (or we can take a finite family of proper quotients of the group on this level such that the image of N_{n-2}^{n-1} in each such group is a proper quotient). Construct a block-NTQ group as in the previous case.

In this way we continue the construction of the tree $T_{EA}(G)$.

9.5. The ∀∃-tree is finite

A crucial fact is the following.

Theorem 36. *The tree $T_{EA}(G)$ is finite.*

Proof. We begin with the characterization of the case when the fundamental sequence c has only one level. If R is an NTQ system, denote by $ab(R)$ the sum of ranks of abelian vertex groups in R minus the sum of the ranks of the edge groups for the edges from them.

Let an NTQ system $Q(X_1, \dots, X_n) = 1$ have the form

$$Q_1(X_1, \dots, X_n) = 1,$$
$$\vdots$$
$$Q_n(X_n) = 1,$$

where $Q_1 = 1$ corresponds to the top level of JSJ decomposition for $F_{R(Q)}$, variables from X_1 are either quadratic or correspond to extensions of centralizers. Consider this system together with a fundamental sequence $V_{\text{fund}}(Q)$ defining it. Let $V_{\text{fund}}(U_1)$ be the subset of $V_{\text{fund}}(Q)$ satisfying some additional equation $U_1 = 1$, and G_1 a group discriminated by this subset. Consider the family of those canonical fundamental sequences for G_1 modulo the images $P_1, \dots, P_s$ of the factors in the free decomposition of the subgroup $\langle X_2, \dots, X_m \rangle$, which have the same dimension modulo them as $Q_1 = 1$. Denote this free decomposition by H_1*.

Denote such a fundamental sequence by c, and corresponding NTQ system $S = 1$ (mod H_1*), where $S = 1$ has form

$$S_1(X_{11}, \dots, X_{1m}) = 1,$$
$$\vdots$$
$$S_m(X_{1m}) = 1.$$

Denote by D_Q a canonical decomposition corresponding to the group $F_{R(Q)}$. Non-QH, non-abelian subgroups in this decomposition are $P_1, \dots, P_s$. Abelian and QH-subgroups correspond to the system $Q_1(X_1, \dots, X_n) = 1$. For each i there exists a canonical homomorphism

$$\eta_i : F_{R(Q)} \to F_{R(S_i, \dots, S_m)}$$

(see Theorem 13) such that $P_1, \dots, P_s$ are mapped into rigid subgroups in the canonical decomposition of $\eta_i(F_{R(Q)})$.

Each QH-subgroup in the decomposition of $F_{R(S_i, \dots, S_m)}$ as an NTQ group is a QH-subgroup of $\eta_i(F_{R(Q)})$. By Lemma 10, for each QH-subgroup Q_1 of $\eta_i(F_{R(Q)})$ there exists a QH-subgroup of $F_{R(Q)}$ that is mapped into a subgroup of finite index in Q_1. The size of this QH-subgroup is, obviously, greater or equal to the size of Q_1. Those QH-subgroups of $F_{R(Q)}$ that are mapped into QH-subgroups of the same size by some η_i are called stable.

Lemma 24. *In the conditions above there are the following possibilities*:

(i) *the set of homomorphisms going through c is generic for each regular quadratic equation in $Q_1 = 1$ and $ab(c) = ab(V_{\text{fund}}(Q))$ (in this case c has only one level identical to Q_1)*;
(ii) *it is possible to reconstruct system $S = 1$ so that size$(S) <$ size(Q_1)*;
(iii) *size$(S) =$ size(Q_1), $ab(c) < ab(V_{\text{fund}}(Q))$.*

Proof. The fundamental sequence c modulo the decomposition H_1* has the same dimension as $Q_1 = 1$. The dimension of $Q_1 = 1$ is the sum of the following four numbers:

(1) the dimension of a free factor $F_1 = F(t_0, \ldots, t_{k_0})$ in the free decomposition of $F_{R(Q)}$ corresponding to an empty equation in $Q_1 = 1$;
(2) the number of abelian factors;
(3) the sum of dimensions of surface groups factors;
(4) the number of free variables of quadratic equations with coefficients in $Q_1 = 1$ corresponding to the fundamental sequence $V_{\text{fund}}(Q)$.

Because c has the same dimension, the free factor F_1 is unchanged. By the remark in Section 7.9, abelian and surface factors are sent into different free factors.

Let $Q_{1i} = 1$ be one of the standard quadratic equations in the system $Q_1 = 1$. If the set of solutions of $Q_{1i} = 1$ over $F_{R(Q_2,\ldots,Q_n)}$ that factor through the system $S = 1$ and through the generalized equation Ω_{Q_1} is a generic set for $Q_{1i} = 1$, then by [14, Theorem 9], we conclude that $S = 1$ can be reconstructed so that it contains only one quadratic equation as a part of the system $S_m = 1$. Indeed, suppose a QH-subgroup $\bar{Q}_{1i}$ corresponding to $Q_{1i} = 1$ mapped on some level s of $S = 1$ onto a subgroup of the same size. Then it is stable. Suppose also that a QH-subgroup of $F_{R(Q)}$ that is a subgroup of $\bar{Q}_{1i}$ is projected on some level k above s into a QH-subgroup $\bar{Q}_k$. Then this projection is a monomorphism. On all the levels above s we can join the image of a subgroup of $\bar{Q}_{1i}$ to a non-QH-subgroup adjacent to it (and not count it in the size). We can join the image of $\bar{Q}_{1i}$ to a non-QH-subgroup on all the levels above m, and replace the image of it on level m by the isomorphic copy of $\bar{Q}_{1i}$.

Is all QH-subgroups corresponding to $Q_1 = 1$ are stable, then the regular size of $S = 1$ is the same as a regular size of $Q_1 = 1$ and if $ab(c) = ab(V_{\text{fund}}(Q))$, then reconstructed $S = 1$ has only one level.

The lemma is proved. □

To finish the proof of the theorem, notice that by Lemma 24, every time we apply the transformation of Case 3 (we refer to the cases from Section 9.4) in the construction of AE-tree we either (i) decrease the dimension in the top block, or (ii) replace the NTQ system in the top block by another NTQ system of the same dimension but with smaller size, or (iii) $ab(c)$ decreases. Hence Case 3 cannot be applied infinitely many times to the top block. If we apply Case 2, we consider the second block for proper quotients of a finite number of groups. Hence, starting from some step, we come to a situation, when the fundamental sequences factor through the same block-NTQ system, and the image of G in the last level of these systems is a proper quotient of G. If we apply all the time Case 1 at all the levels of block-NTQ system and obtain the same block-NTQ system, then every time we replace G by G' (the image of G_{corr}). The intersection of the images of G' in the fundamental sequence for G' with different levels of N_n on some level will be different from the intersection of the corresponding image of G. Either this intersection contains less factors in the free decomposition, or (if it contains the same number of factors) we can apply Lemma 24 to the fundamental sequences for G_{corr} and G'. This shows that Case 1 cannot appear infinitely many times. □

10. Elimination schemes

Now we refine the description of algebraic sets over free group F which was given in [13]. For this we need the following

Definition 24. Let Q be an NTQ system over G and

$$\begin{aligned} &Q_1(Z_m, \ldots, Z_1) = 1, \\ &Q_2(Z_{m-1}, \ldots, Z_1) = 1, \\ &\quad\vdots \\ &Q_m(Z_1) = 1 \end{aligned}$$

be a partition of Q into subsystems $Q_1, \ldots, Q_m$ (equations in Q_i inherit the order from Q). Let

$$\begin{aligned} &P_1 = (p_{11}, \ldots, p_{1k_1}), \\ &\quad\vdots \\ &P_m = (p_{m1}, \ldots, p_{mk_m}) \end{aligned}$$

be tuples of words, where $p_{ij} \in G[Z_i]$.

We denote by $Q[X_m; \ldots; X_1 \mid Z_m; \ldots; Z_1] = 1$ the following system of equations:

$$\begin{aligned} &Q_1(Z_m, \ldots, Z_1) = 1, \\ &Q_2(Z_{m-1}, \ldots, Z_1) = 1, \\ &\quad\vdots \\ &Q_m(Z_1) = 1, \\ &x_{11} = p_{11}(Z_1), \quad \ldots, \quad x_{1k_1} = p_{1k_1}(Z_1), \\ &x_{21} = p_{21}(Z_2), \quad \ldots, \quad x_{2k_2} = p_{2k_2}(Z_2), \\ &\quad\vdots \\ &x_{m1} = p_{m1}(Z_m), \quad \ldots, \quad x_{mk_m} = p_{mk_m}(Z_m), \end{aligned}$$

where $X_1 = (x_{11}, \ldots, x_{1k_1}), \ldots, X_m = (x_{m1}, \ldots, x_{mk_m})$.

The system $Q[X_m; \ldots; X_1 \mid Z_m; \ldots; Z_1] = 1$ together with a fixed set of solutions $\mathcal{V}$ of the system $Q = 1$ in G is called an *elimination scheme*.

Let a pair $Q[X_m;\ldots;X_1 \mid Z_m;\ldots;Z_1]=1$, $\mathcal{V}$ be an elimination scheme. Put

$$Sh_{\mathcal{V}}(Q)=\{U \mid U=P(Z^{\phi}),\ \phi\in\mathcal{V}\}.$$

Theorem 37. *For every finite system of equations $S(X_1,\ldots,X_m)=1$ over a free group F, one can find effectively a finite family of schemes*

$$U_i[X_m;\ldots;X_1 \mid Z_m;\ldots;Z_1]=1,$$

such that $V_F(S)=\bigcup_i Sh_F(U_i)$.

Proof. We construct fundamental sequences for $S=1$ modulo the subgroup $\langle X_{m-1},\ldots,X_1\rangle$ as in the construction of $T_{CE}(S)$ in Section 7.6. We obtain a finite family of terminal groups. For each of these consider proper quotients such that all reducing solutions factor through them. For each quotient consider a fundamental sequence modulo the subgroup $\langle X_{m-1},\ldots,X_1\rangle$. We consider each terminal group independently. As a result we will have a finite number of terminal groups of different levels. Denote such a terminal group by T_1. For a fundamental sequence with terminal group T_1, denote by $Q_1(Z_m,T_1)=1$ the corresponding NTQ system over T_1, where Z_m are variables of this NTQ system. Considering fundamental sequences for T_1 modulo the subgroup $\langle X_{n-2},\ldots,X_1\rangle$ designed as in the construction of T_{CE} for the system corresponding to this group, we again obtain a finite family of terminal groups. For each of these we consider proper quotients such that all reducing solutions factor through them. For each quotient we consider a fundamental sequence modulo the subgroup $\langle X_{m-2},\ldots,X_1\rangle$, and consider each terminal group independently. As a result we will have a finite number of terminal groups of different levels. Denote such a terminal group by T_2. Denote by $Q_2(Z_{m-1},T_2)=1$ the corresponding NTQ system over T_2. We continue construction for T_2 in the same way as above. As a result we will have a finite number of fundamental sequences and corresponding NTQ groups. Mappings $P_1,\ldots,P_m$ express variables $X_1,\ldots,X_m$ in terms of $Z_1,\ldots,Z_m$. Each solution of $S(X_1,\ldots,X_m)=1$ factors through one of these fundamental sequences and so belongs to $Sh_F(U_i)$ for the corresponding scheme. This completes the proof. □

11. Projective images

In Section 9 we described the process which is used to verify an ∃∀-sentence. In this section we describe some construction which will be later used to work with general sentences and formulas.

11.1. Scheme of the construction

In this section we will describe a construction of a *projective tree* $T_p(U)$ for an irreducible system of equations $U(X)=1$ over F relative to a collection of *Diophantine conditions*. To each branch we assign a decreasing sequence of families of solutions of $U=1$ and, in particular, a chain in the form:

$$F_{R(W_1)},\ F_{R(P_1)},\ \ldots,\ F_{R(W_i)},\ F_{R(P_i)},\ \ldots,$$

where the systems $W_i(X,\bar{X}_i)=1$ are block-NTQ, there are homomorphisms $\theta_i : F_{R(U)}\to F_{R(W_i)}$, each group $\theta_{i+1}(F_{R(U)})$ is a quotient of $\theta_i(F_{R(U)})$, systems $P_i(X,\bar{X}_i,Z_i)=1$ contain variables $X, \bar{X}_i$. This tree has the property that every solution of $U(X)=1$ that factors through the fundamental sequence for $W_i(X, \bar{X}_i)=1$ and for which there are values of Z_i such that the system $P_i(X, \bar{X}_i, Z_i)=1$ holds, factors through the fundamental sequence for one of the next step block-NTQ systems $W_{i+1}(X, \bar{X}_{i+1})=1$. The tree depends on the initial system $U=1$ and the systems $P_i=1, i=1, \ldots$.

11.2. Projective tree: construction of the first level edges

Since we are going to deal with many different systems of equations we need an alternative notation for a system. A system of equations $W(X)=1$ sometimes will be denoted by $\mathcal{W}$.

Let $G=F_{R(U)}$ be the coordinate group of the system $U=1$ and $J<G$ a fixed subgroup of G. Let

$$F_{R(U)}=F_{R(\bar{U}_1)} * \cdots * F_{R(\bar{U}_k)} * F(Z)$$

be an irreducible free decomposition of $F_{R(U)}$ modulo J, where $J \leqslant F_{R(\bar{U}_1)}$ and $F_{R(\bar{U}_1)}$ is freely indecomposable modulo J, $F_{R(\bar{U}_2)}, \ldots, F_{R(\bar{U}_k)}$ are freely indecomposable non-cyclic groups, and $F(Z)$ is a free group with basis Z. Denote

$$F_{R(\bar{U})}=F_{R(\bar{U}_1)} * \cdots * F_{R(\bar{U}_k)}.$$

Let $W=1$ be the canonical NTQ system corresponding to a branch of the canonical embedding tree $T_{CE}(F_{R(U)})$ of the system $U=1$. Suppose $P=1$ is a given system of equations with coefficients in the group $F_{R(W)}$ which has a solution in some extension of $F_{R(W)}$ (so $N_1=F_{R(W)}$ canonically embeds into $F_{R(P)}$), but does not have a solution in $F_{R(W)}$. Consider a finite family of terminal groups of fundamental sequences for $\mathcal{P}$ modulo factors in the free decomposition of $F_{R(U)}$. If the image of N_1 in the terminal group is a proper quotient, consider this image as in the construction of an $\forall\exists$-tree. Below we consider only terminal groups in which N_1 embeds as a subgroup. Let $F_{R(T)}$ be one of them. If it is freely decomposable so that the decomposition is compatible with factors in the free decomposition of $F_{R(U)}$ we consider different factors separately. Therefore we can further assume that is freely indecomposable modulo these factors.

Recall from the canonical embedding construction that we represent $F_{R(W)}=F(Z) * F_{R(K_1)}$ (observe that $F(Z)$ is still a free factor) as the coordinate group of the NTQ system in the following form:

$$\begin{aligned} &S_1(X_1, \ldots, X_n)=1,\\ &S_2(X_2, \ldots, X_n)=1,\\ &\qquad\vdots\\ &S_n(X_n)=1. \end{aligned}$$

The system $\mathcal{K}_1$ is constructed as follows. We take a free decomposition of $F_{R(\bar{U})}$ into factors $F_{R(\bar{U}_1)}, \ldots, F_{R(\bar{U}_k)}$, and let

$$F_{R(K_1)}=F_{R(M_1)} * \cdots * F_{R(M_k)},$$

where $\mathcal{M}_i : M_i = 1$ is a NTQ system corresponding to a branch of $T_{CE}(\bar{U}_i)$ (observe that all of them except $F_{R(M_1)}$ are systems without coefficients). Then $\mathcal{S}_1$ is the union of the first levels of the systems $\mathcal{M}_i$ and empty equation in variables Z, $\mathcal{S}_2$ is the union of the second levels of $\mathcal{M}_i$, and so on. Recall that the first level of each $\mathcal{M}_i$ is constructed from the abelian JSJ decomposition of $F_{R(\bar{U}_i)}$. Each variable from X_1 is either quadratic or corresponds to an extension of a centralizer. The group $F_{R(M_i)}$ does not split either as a free product or as a centralizer extension modulo $F_{R(\bar{U}_i)}$, and if it has a splitting, we consider only solutions of $F_{R(M_i)}$ minimal with respect to the group of canonical automorphisms corresponding to this splitting (see Section 7.11).

We construct the family of block-NTQ groups for each $F_{R(T)}$ with respect to this family of solutions as described below.

(1) We construct a canonical NTQ system $\mathcal{L}_1$ for $F_{R(U)}$ and a fundamental sequence of solutions modulo factors in a free decomposition of the subgroup generated by $X_2, \dots, X_n$. We call this fundamental sequence an *enveloping fundamental sequence* for S_1 and call the group $F_{R(L_1)}$ an *enveloping NTQ group*.

(2) We construct a *tight enveloping NTQ group*. This is done as follows.

(a) We take an NTQ group induced by the image of $F_{R(U)}$ from $F_{R(L_1)}$ as described in Section 7.12. This does not increase the dimension, because we add only elements from abelian subgroups. Denote this group by $Ind(F_{R(U)})$. Then we do the following.
(b) We first add to $Ind(F_{R(U)})$ all the QH-subgroups of $F_{R(L_1)}$ that have non-trivial intersection with $Ind(F_{R(U)})$, do not have free variables, the corresponding level of $Ind(F_{R(U)})$ intersects non-trivially some of their adjacent vertex groups, and their addition decreases the dimension.
(c) We add also those QH-subgroups Q of the group $F_{R(L_1)}$ that intersect $Ind(F_{R(U)})$ in a subgroup of finite index (in Q) and have less free variables than the subgroup in the intersection.
(d) We add all the elements that conjugate different QH-subgroups (abelian vertex groups) of $Ind(F_{R(U)})$ into the same QH-subgroup of $\mathcal{L}_1$ if this decreases the dimension.
(e) We add edge groups of abelian subgroups of the enveloping group that have non-trivial intersection with $Ind(F_{R(U)})$ if this does not increase the dimension.

We can do these steps (which we call adjustment) by levels from the top to the bottom (considering on level $i+1$ the image of the group extended on level i) of the system corresponding to $Ind(F_{R(U)})$ and denote the obtained group by $Adj(F_{R(U)})$. We repeat the adjustment iteratively as many times as possible. Then we add once more those QH-subgroups of different levels of the enveloping group that contain QH-subgroups of the tight enveloping NTQ group as subgroups of finite index and whose addition decreases the dimension. We call the constructed NTQ group the tight enveloping NTQ group. We will also call the corresponding system (fundamental sequence) the tight enveloping system (fundamental sequence) and use the same notation ($TEnv(S_1)$). As a size of a QH-subgroup Q in the tight enveloping NTQ group we consider the size of the QH-subgroup in the enveloping group containing Q as a subgroup of a finite index.

The dimension of the tight enveloping fundamental sequence ($TEnv(S_1)$) is less than or equal to the dimension of $\mathcal{S}_1$ modulo free factors of $\langle X_2, \dots, X_n\rangle$. (Because we consider only fundamental sequences compatible with the free factorization of the subgroup $\langle X_2, \dots, X_n\rangle$ used in Remark 1 and compatible with the splitting of quadratic equations as discussed in Section 7.8.) If the dimensions are the same, we can always reorganize the levels of the enveloping system $\mathcal{L}_1$ into another $Env(S_1)$ so that they have the same fundamental solutions and $size(TEnv(S_1)) \leqslant size(S_1)$. If all the parameters (dimension, size, ab) are the same, then

TEnv(S_1) has one level the same as S_1. (Notice, that the dimension of the tight enveloping NTQ fundamental sequence (*TEnv*(S_1)) is the same as the maximal dimension of the corresponding subgroup in the terminal group in the enveloping fundamental sequence modulo free factors of $\langle X_2, \ldots, X_n \rangle$.)

(3) Then we consider an induced NTQ system $\bar{\mathcal{S}}_1 : \bar{S}_1 = 1$ for $F_{R(\bar{U})}$ with respect to *TEnv*(S_1).

Lemma 25. *Either the image H of $F_{R(U)}$ in the terminal group of Env(S_1) is a proper quotient of $F_{R(U)}$ or this terminal group does not have a sufficient splitting modulo J.*

Proof. Suppose H is isomorphic to $F_{R(U)}$. Consider the terminal group T_1 of $\bar{\mathcal{S}}_1$. Let D_{T_1} be the abelian JSJ decomposition of T_1 modulo J (we take the Grushko decomposition followed by the abelian decompositions of free factors). Non-active subgroups of $F_{R(\bar{U})}$ are elliptic in D_{T_1}. The free factors of $\langle X_2, \ldots, X_n \rangle$ cannot be elliptic in D_{T_1} and their QH-subgroups either do not intersect with conjugates of QH-subgroups of T_1 or intersect in finite index subgroups. Since for each M_i we consider only solutions of $F_{R(M_i)}$ minimal with respect to the group of canonical automorphisms corresponding to the splitting of $F_{R(M_i)}$ modulo $F_{R(U_i)}$, D_{T_1} cannot be a sufficient splittings of free factors of $\langle X_2, \ldots, X_n \rangle$ modulo non-active subgroups of $F_{R(\bar{U})}$. □

To obtain H effectively, we first construct the block-NTQ groups for $F_{R(T)}$ without the requirement that the solutions are minimal with respect to the group of canonical automorphisms of $F_{R(M_i)}$ modulo $F_{R(\bar{U}_i)}$ and then replace the terminal group of *Env*(S_1) by the maximal fully residually free standard quotients with respect to J (if exist) and repeat this step as many times as possible. The quotient of H in each last maximal standard quotient is proper.

Case 1. If *TEnv*(S_1) has smaller dimension than $\mathcal{S}_1$ or has the same dimension but not identical to $\mathcal{S}_1$ (therefore has smaller size), then we consider an induced NTQ system $\bar{\mathcal{S}}_1 : \bar{S}_1 = 1$ for $F_{R(U)}$ with respect to *TEnv*(S_1). We can actually obtain a proper quotient of $F_{R(U)}$ not on the terminal level of $\bar{\mathcal{S}}_1$ but earlier, on some level k. Denote by $\hat{\mathcal{S}}_1$ the top part of $\bar{\mathcal{S}}_1$ before level k, and by $\bar{H}$ the image of $F_{R(U)}$ in level k, which is a proper quotient of $F_{R(U)}$. Amalgamate $F_{R(\hat{S}_1)}$ and a NTQ group corresponding to a branch of $T_{CE}(\bar{H})$ along $\bar{H}$. This gives a block-NTQ system $\mathcal{W}_2$ and the coordinate group $N_2 = F_{R(W_2)}$. Consider a fundamental sequence obtained by taking a fundamental sequence for $\hat{\mathcal{S}}_1$ and pasting to it a fundamental sequence corresponding to $\bar{H}$. Denote the obtained sequence by $V_{\text{fund}}(W_2)$ and consider the set of solutions of $U = 1$ having form $\alpha V_{\text{fund}}(W_2)$, where α expresses variables from $\mathcal{U}$ in terms of variables from $\mathcal{W}_2$.

Case 2. If *TEnv*(S_1) (and, therefore, $\bar{\mathcal{S}}_1$) is identical to $\mathcal{S}_1$ then consider terminal groups of the system $\bar{\mathcal{S}}_1$. Construct a NTQ system $\mathcal{L}_2$ modulo free factors of the subgroup generated by $X_3, \ldots, X_n$, then construct an induced NTQ system $\bar{\mathcal{S}}_2$ modulo free factors of the subgroup generated by $X_3, \ldots, X_n$, and so on until we either stop or obtain that *TEnv*$(\mathcal{S}_i)$ is not identical to $\mathcal{S}_i$, then apply the transformation of Case 1.

Denote by $\mathcal{W}_2$ the obtained block-NTQ system. Denote the coordinate group of the enveloping system by E_2 and of the tight enveloping system by TE_2.

If we have not applied the transformation of Case 1, we finally obtain the block-NTQ system $\mathcal{W}_1'$ which is the same as $\mathcal{W}_1$. Then we denote $\mathcal{W}_2 = \mathcal{W}_1'$.

A family consisting of a system $\mathcal{W}_2 = 1$, groups $N_2 = F_{R(\mathcal{W}_2)}$ and TE_2, the set of homomorphisms from $F_{R(U)}$ to F having form $\alpha V_{\text{fund}}(\mathcal{W}_2)$, where $V_{\text{fund}}(W_2)$ is a fundamental sequence constructed above, and the enveloping fundamental sequence for $V_{\text{fund}}(Env(\mathcal{W}_2))$ composed from the enveloping sequences for different levels, is called a *projective image of* $U = 1$ *of the second type*. A projective image is proper if $\alpha V_{\text{fund}}(\mathcal{W}_1)$ is not contained in $\alpha V_{\text{fund}}(\mathcal{W}_2)$.

11.3. Projective tree: step n of the construction

Suppose we have already constructed proper projective images of type $n-1$. In particular, we constructed groups $N_1, \dots, N_{n-1}$; $E_2, \dots, E_{n-1}$; and $TE_2, \dots, TE_{n-1}$. Here we describe the nth step of the construction.

Consider some equation $P_{n-1} = 1$ with coefficients in N_{n-1} that does not have a solution in N_{n-1}. We obtain a finite family of extensions of $F_{R(U)}$ which do not have sufficient splittings modulo $F_{R(U)}$ and which are terminal groups for fundamental sequences for $P_{n-1} = 1$ modulo $F_{R(U)}$. If the image of some $TE_i \in \{TE_2, \dots, TE_{n-1}\}$ or $E_i \in \{E_2, \dots, E_{n-1}\}$ in this terminal group is a proper quotient, suppose i is the smallest number with this property. Consider fundamental sequences for this quotient of TE_i, as we did in the construction of an $\forall\exists$-tree, denote the corresponding groups by TE_n. In this case each $\mathcal{W}_n$ is a system induced by $F_{R(U)}$ from a fundamental sequence corresponding to TE_n. The groups $N_n = F_{R(W_n)}$ and TE_n define projective images of nth type.

Below we suppose that the terminal group contains all $E_2, \dots, E_{n-1}$ as subgroups. Consider each such terminal group separately. Denote such a terminal group by $F_{R(T^{(n)})}$. We construct the family of block-NTQ groups for each $F_{R(T^{(n)})}$ in the following manner.

We construct a NTQ system $\mathcal{L}_1^{(n)}$ modulo the rigid subgroups in the decomposition of the top level of $F_{R(L_1^{(n-1)})}$.

Case 1. We consider the NTQ system corresponding to the tight enveloping fundamental sequence corresponding to G. Suppose it has smaller dimension than the top level of $TEnv(\mathcal{W}_{n-1})$ or has the same dimension but not identical to the top level of $TEnv(\mathcal{W}_{n-1})$. The number of factors in the free decomposition of the terminal level of the tight enveloping fundamental sequence is not more than the number of factors in the free decomposition of the second level of $TEnv(\mathcal{W}_{n-1})$ induced from the free decomposition of the enveloping fundamental sequence. If this number is the same, then using Lemma 11 we can make its size to be smaller than $size(TEnv(\mathcal{W}_{n-1}))$. Similarly to Lemma 25 one can prove that the image H of E_{n-1} in the terminal group of the enveloping sequence is a proper quotient of E_{n-1} or this terminal group does not have a sufficient splitting modulo J.

Let m be the minimal such number that the image of E_m in the terminal group of the enveloping sequence is a proper quotient of E_m. Suppose it is a proper quotient on level k (k is minimal with this property) of the enveloping sequence.

Denote by $\bar{\mathcal{S}}_1^{(n)}$ the block-NTQ system induced from the enveloping system until level k by E_m. Denote by $\hat{\mathcal{S}}_n$ the top part of $\bar{\mathcal{S}}_1^{(n)}$ before level k, and by $\bar{H}^{(n)}$ the image of E_m in level k, which is a proper quotient of E_m. Consider a fundamental sequence obtained by extracting a fundamental sequence for a subgroup E_m from $\hat{\mathcal{S}}_n$ and pasting to it a fundamental sequence corresponding to $\bar{H}^{(n)}$. We construct a fundamental sequence for $\bar{H}^{(n)}$ as we did for $F_{R(L_1^{(m)})}$. We can consider in this case only fundamental sequences that either do not have a maximal dimension or (if the dimension is maximal) do not have a maximal size. The image of E_{m-1} on the terminal level of such a sequence is a proper quotient of E_{m-1}. We consider now instead of $\bar{\mathcal{S}}_1^{(n)} = 1$ the fundamental sequence induced by E_{m-1} from the corresponding enveloping fundamental sequence until the terminal level. We repeat the same procedure for the fundamental sequence for $\bar{H}^{(n)}$ and a proper quotient of E_k ($k < m$ and k is minimal with this property) at its terminal level by induction. This gives a block-NTQ group N_n and a system $\mathcal{W}_n$ such that $N_n = F_{R(W_n)}$. Denote the corresponding fundamental sequence $V_{\text{fund}}(W_n)$ and consider the set of solutions of $U = 1$ having form $\alpha_n V_{\text{fund}}(W_n)$, where α_n express variables from $U = 1$ in terms of variables from W_n.

Case 2. We consider again the tight enveloping fundamental sequence corresponding to G. Suppose it is the same as the top level of $TEnv(N_{n-1})$. Denote by $\bar{\mathcal{S}}_1^{(n)}$ the fundamental sequence induced from the tight enveloping fundamental sequence. We consider terminal groups of the system $\bar{\mathcal{S}}_1^{(n)}$. Construct a NTQ system $\mathcal{L}_2^{(n)}$ modulo the variables of the next level of $\mathcal{L}_1^{(n-1)}$ then construct the induced NTQ system modulo rigid subgroups of the group generated by the variables of this level of ${\mathcal{L}_1}^{(n-1)}$, and so on until we either apply the transformation of Case 1 or (and this will be Case 3) we construct a block-NTQ system the same as $\mathcal{W}_{n-1}$ with the same tight enveloping fundamental sequence. The group $N_{n-1} = F_{R(W_{n-1})}$ is embedded into the group of the enveloping system. Denote by TE_n the coordinate group of the tight enveloping system.

Case 3. If there were no level on which the transformation of Case 1 is applied, N_n is the same as N_{n-1}.

Definition 25. A family consisting of a system $W_n = 1$, groups $N_n = F_{R(W_n)}$ and TE_n, set of homomorphisms from $F_{R(U)}$ to F having form $\alpha_n V_{\text{fund}}(W_n)$, where $V_{\text{fund}}(W_n)$ is a fundamental sequence constructed above, and the enveloping fundamental sequence for N_n is called a *projective image of* $U = 1$ *of the* n*th type*. A projective image is *proper* provided that $\alpha_{n-1} V_{\text{fund}}(W_{n-1})$ is not contained in $\alpha_n V_{\text{fund}}(W_n)$.

The tree $T_p(U)$ is a rooted tree. We assign the solution set of $U = 1$ to the root vertex w_0. We assign solutions of $U = 1$ factored through the fundamental sequences corresponding to the branches of $T_{CE}(G)$ to vertices w_{1i}. We assign projective images to vertices. If a projective image is obtained on the nth step from the projective image on the $(n-1)$st step, we connect corresponding vertices by an edge.

11.4. Infinite branches of the tree

The following theorem describes infinite proper descending chains of projective images.

Theorem 38. *Let $W = 1$ be a NTQ system corresponding to a branch of $T_{CE}(F_{R(U)})$ and $V_{\text{fund}}(W)$ a fundamental sequence for it. For every infinite proper descending chain $V_{\text{fund}}(W)$, $\alpha_1 V_{\text{fund}}(W_1), \ldots, \alpha_n V_{\text{fund}}(W_n), \ldots$ of projective images of $V_{\text{fund}}(W)$ there is a number m, such that all the systems $W_n = 1, \ldots$ for $n \geqslant m$ coincide, the systems $TEnv(W_n)$ for $n \geqslant m$ coincide, and $\alpha_n V_{\text{fund}}(W_n)$ are obtained from $\alpha_m V_{\text{fund}}(W_m)$ by first assigning to sets of free variables for W_m values from the enveloping NTQ groups which are either correspond to free variables or to hyperbolic elements in some level decomposition or to infinite index subgroups of some MQH-subgroups, and then specializing them.*

Moreover, for each enveloping system there is a generic family of solutions that induces a family of solutions of $TEnv(W_n)$ such that each solution has rank that is equal to $\dim(TEnv(W_n))$.

Proof. Suppose that constructing $\mathcal{W}_{i+1}$ from $\mathcal{W}_i$ we have Case 1. Since $\dim(TEnv(W_{i+1})) \leqslant \dim(TEnv(W_i))$, either the dimension decreases or it is the same. We have to consider the case when the dimension is the same. In this case the dimension of the tight enveloping fundamental sequence obtained from $\mathcal{L}_1^{i+1}$ is the same as the dimension of the top level of $TEnv(W_i)$. The number of factors in the free decomposition of the terminal level of the tight enveloping fundamental sequence is not more than the number of factors in the free decomposition of the second level of $TEnv(W_i)$ induced from the free decomposition of the enveloping fundamental sequence. Either this number is decreasing or it is the same. Suppose it is the same.

Each QH-subgroup corresponding to the tight enveloping fundamental sequence by Lemma 11 contains the image of some QH-subgroup corresponding to the top level of $TEnv(W_i)$ as a subgroup of finite index. Therefore the size of each QH-subgroup (quadratic equation) of the tight enveloping fundamental sequence is less or equal than the size of some QH-subgroup corresponding to the top level of $TEnv(W_i)$. If Q_{1i} is one of the stable QH-subgroups corresponding to the top level of $TEnv(W_i)$, then we can move it to the second level. Therefore the size of the tight enveloping fundamental sequence can be made less or equal than the size of the top level of $TEnv(W_i)$. Since we have Case 1, the size must decrease. Every time when Case 1 appears in the process, either the dimension or the number of factors decreases or (if they do not decrease) the size decreases. Therefore after finite number of steps we obtain a proper quotient of one of the finite number of groups TE_i. Therefore Case 1 can appear in the process only a finite number of times. Similarly, Case 2 can only appear a finite number of times.

The infinite process is only possible if beginning at some step we always have Case 3. In this case we refine systems $L_i = 1$ for $i = 1, \dots, n$ (this means that we assign to free variables values from different NTQ groups) or we increase centralizers.

The first statement of the theorem can be now proved analogously to Theorem 36. The only infinite process possible is when the fundamental sequences factor through the same block-NTQ system $TEnv(W_n)$ (and, therefore, W_n) and on some levels of this system there are free factors, and variables from these free factors can take values from different NTQ systems. Since $TEnv(W_n) = TEnv(W_m)$, free variables can appear either from infinite index subgroups of MQH-subgroups or from hyperbolic elements of different levels of the enveloping group.

To prove the last statement we need the following lemma.

Lemma 26. *Let G be the coordinate group of a consistent regular quadratic equation $S(X) = 1$ with coefficients over $F(A)$. Let H be a finitely generated infinite index subgroup of the MQH-subgroup Q of G such that H does not intersect cyclic subgroups generated by conjugates of the coefficients of the equation. Then there is a positive unbounded family of solutions Φ of $S(X) = 1$ such that the restriction of each solution from Φ on H is an isomorphism.*

Proof. Consider an orientable equation. One can represent Q as a fundamental group of a graph of groups with elliptic boundary subgroups and vertex groups being free of rank 2. (The first vertex group is generated by $y_1, y_1^{x_1}, [x_1, y_1]$, and so on.) One can choose a solution ϕ of $S(X) = 1$ such that the restriction of ϕ on each vertex group is a monomorphism because any two non-commuting elements in a free group freely generate a free subgroup. There exists a number N and positive N-large family of automorphisms Ψ of G such that for any $\psi \in \Psi$ all elements of H^{ψ} are hyperbolic with respect to the above decomposition of Q. Therefore $\psi\phi$ is an isomorphism on H. $\square$

The last statement can be now proved by induction on the number of levels of $TEnv(W_n)$. $\square$

12. The proof of Theorems 1 and 2

12.1. Base of induction, ∃∀-sentences

By Lemma 11 from [14], for any sentence Θ in the language L_A of group theory with constants from A one can effectively find a sentence

$$\Phi = \exists X_1 \forall Y_1 \dots \exists X_k \forall Y_k \ \big(U(X_1, Y_1, \dots, X_k, Y_k) = 1 \to V(X_1, Y_1, \dots, X_k, Y_k) = 1\big),$$

where U and V are non-trivial elements in the free group $F(X_1, Y_1, \dots, X_k, Y_k)$, such that either Θ or its negation is equivalent to Φ modulo the theory of the class of all torsion-free non-abelian CSA groups satisfying the Vaught conjecture (non-abelian fully residually free groups belong to this class).

Let $F_{R(S)}$ be the coordinate group of a regular NTQ system $S = 1$, in particular, it can be a free group. Let us consider the sentence Φ in $F_{R(S)}$. We construct a tree $T_{F_{R(S)}}(\Phi)$ with directed edges, such that the vertices of $T_{F_{R(S)}}(\Phi)$ correspond to some schemes constructed in Section 10. The initial vertex v_0 corresponds to the empty scheme.

Suppose first that $k = 1$, and Φ has coefficients in F. We have

$$\Phi = \exists X \forall Y \ \big(U(X, Y) = 1 \to V(X, Y) = 1\big).$$

In this case the tree $T = T_{F_{R(S)}}(\Phi)$ is constructed according to the decision algorithm for ∃∀-sentences described in Section 8. We will remind the construction but this time we will use the language of schemes. There are two possibilities.

(1) The sentence

$$\tilde{\Phi} = \exists X \forall Y \ (U \neq 1)$$

is true in $F_{R(S)}$. The negation of this sentence is equivalent to a positive sentence, and in this case the tree consists of one vertex v_0, which is the initial and final vertex, and Φ is true. The vertex v_0 is called a stop-vertex.

(2) The sentence $\tilde{\Phi}$ is false in $F_{R(S)}$. Then clearly

$$F \models \forall X \exists Y \ (U = 1),$$

and we can repeat the argument of case (2), Section 9.1 for the group $F_{R(S)}$ in the place of F. Namely, by the implicit function theorem [14, Theorem 11], or (in case $F_{R(S)}$ is free) by Merzljakov's theorem [14, Theorem 4]

$$F_{R(S)} \models \big(\exists Y \ \big(U(X, Y) = 1\big)\big).$$

There is a scheme for $U = 1$, having the form

$$Q^{1,0}\big[Y; X \mid Y^{(1)}; X\big] = 1,$$

and by Theorem 2 in [13] every solution of $U = 1$ in which Y is given by a formula of X can be obtained from a finite collection $\mathcal{F}_1$ of such schemes. Each scheme consists of a NTQ system

$$q^{1,0}\big(Y^{(1)}, X\big) = 1$$

and word mappings $Y = p^{1,0}(Y^{(1)})$ so that $F_{R(S,q^{1,0})} \models U(X, p^{1,0}(Y^{(1)})) = 1$. For the family $\mathcal{F}_1$ we assign a vertex v of the tree T and an edge from v_0 to v. This vertex is said to be a vertex of level $(1, 0)$. For sentence Φ to be true, X must exist such that for all Y given by the schemes from $\mathcal{F}_1$ one has $V(X, Y) = 1$. This gives an infinite system of equations on X which, by Noetherian property (see Section 1.1), is equivalent to a finite system $U_0(X) = 1$. If this equation is empty ($V(X, Y) = 1$ for every Y described by a scheme in $\mathcal{F}_1$), then v is called a *stop-vertex*. If this system is inconsistent, Φ is false, and v is called a *dead-end* vertex. Suppose that $U_0 = 1$ is consistent. The solution set of this system is the union of solutions of irreducible systems. Therefore, for sentence Φ to be true, X must be a solution of one of these systems. We will consider each possibility independently. To keep notations simple we assume that $U_0 = 1$ is irreducible. Let X satisfy the equation $U_0(X) = 1$. By Theorem 37 with $m = 1$, there exists a finite family of schemes $\hat{\mathcal{F}}_1$ for $U_0 = 1$ of the form

$$Q^{1,1}\bigl[X \bigm| X^{(1)}\bigr] = 1$$

such that each $X \in V_{F_{R(S)}}(U_0)$ satisfies one of these schemes. Each of these schemes consists of a word mapping $X = p^{1,1}(X^{(1)})$ and NTQ system

$$q^{1,1}\bigl(X^{(1)}\bigr) = S(b)\bigl(X^{(1)}\bigr) = 1$$

and corresponds to a branch b of the tree $T_{CE}(F_{R(U_0)})$ constructed in Section 7.6. To the NTQ system corresponding to branch b of the tree $T_{CE}(F_{R(W_1)})$ we assign the fundamental sequence $V_{\mathrm{fund}}(S(b))$. Assign a vertex of T to each scheme from $\hat{\mathcal{F}}_1$. The vertices of T corresponding to the schemes from $\hat{\mathcal{F}}_1$ are said to be vertices of level $(1, 1)$. We draw an edge from a vertex of level $(1, 0)$ to each vertex of level $(1, 1)$ corresponding to it. Vertices of level $(1, 1)$ of T correspond to vertices of level 1 of $T_{EA}(F_{R(U_0)})$. And, later, vertices of level (i, i) of T will correspond to vertices of level i of $T_{EA}(F_{R(U_0)})$.

Let vertex v of level $(1, 1)$ correspond to a scheme $Q^{1,1}[X \mid X^{(1)}] = 1$. Variables X satisfying the scheme $Q^{1,1} = 1$ can give new Y's such that $U(Y, X) = 1$. Apply Theorem 13, [14] with sentence of the form

$$\forall X^{(1)} \exists Y \ \bigl(q_1^{1,1}\bigl(X^{(1)}\bigr) = 1 \to U\bigl(p^{1,1}\bigl(X^{(1)}\bigr), Y\bigr) = 1\bigr).$$

Then there is a finite number of corrective extensions $\bar{q}_1^{1,1}(X^{(1)}) = 1$ such that equation $U(p^{1,1}(X^{(1)}), Y) = 1$ has a solution in each group $F_{R(S,\bar{q}_1^{1,1})}$. Construct the outgoing edges from v corresponding to schemes for the equation $U = 1$:

$$Q^{2,1}\bigl[Y; X \bigm| Y^{(2)}; X^{(1)}\bigr] = 1.$$

The NTQ system for such a scheme has the form

$$q^{2,1}\bigl(Y^{(2)}, X^{(1)}\bigr) = 1,$$
$$\bar{q}_1^{1,1}\bigl(X^{(1)}\bigr) = 1.$$

By [19, Theorem 14.1] there is a finite number of such schemes, describing all formula solutions Y of $X^{(1)}$. Let $\mathcal{F}_2$ be the family of all formula solutions Y. Assign to this family a vertex of level $(2, 1)$ and draw an edge from the vertex of level $(1, 1)$ with the scheme $Q^{1,1} = 1$.

To make formula Φ true, the values of $X^{(1)}$, if they were obtained from the fundamental sequence for $\bar{q}_1^{1,1}(X^{(1)})$, must satisfy equation $V=1$ together with each solution Y satisfying $Q^{2,1}=1$. If each value of $X^{(1)}$ belonging to the fundamental sequence $V_{\mathrm{fund}}(\bar{q}_1^{1,1}(X^{(1)}))$ automatically satisfies $V=1$ together with each solution Y, then we draw a vertex of level $(2,2)$, assign the system $\bar{q}_1^{1,1}(X^{(1)})$ to this vertex and this is a stop-vertex. If we obtained a stop-vertex, the sentence Φ is true. Otherwise, this condition will give a finite number of irreducible equations $W_1(X^{(1)})=1,\dots,W_k(X^{(1)})=1$, such that $X^{(1)}$ satisfying $\bar{q}_1^{1,1}=1$ must satisfy one of them.

If such $X^{(1)}$ does not exist, vertex v of level $(2,1)$ is a dead-end vertex. There are no edges going from a dead-end vertex and the sentence cannot be proved along the branch ending with such a vertex. If such $X^{(1)}$ exists, then we consider each equation independently. Consider $X^{(1)}$ satisfying the equation $W_1=1$. If this condition gives an additional equation $U_1(X)=1$ on X then we consider a vertex v of the tree $T_{F_{R(S)}}(\Phi)$, draw an edge to this vertex from the vertex of level $(1,0)$ corresponding to the scheme $Q^{2,1}$ and glue to v the tree $T_{F_{R(S)}}(\Phi_1)$ corresponding to the formula

$$\Phi_1=\exists X\forall Y\ \big(\big(U(X,Y)=1\wedge U_1(X)=1\big)\to V(X,Y)=1\big).$$

Due to the Noetherian property, the number of steps when we add a proper equation on X is finite.

Suppose that the equation $U_1(X^{(1)})=1$ does not give an additional equation on X. Then there is a finite number of schemes

$$Q^{2,2}\big[X \bigm| X^{(2)}\big]=1$$

with the block-NTQ system $q_1^{2,2}(X^{(2)})=1$ constructed from the fundamental sequences of the tree $T_{EA}(F_{R(S,W_1)})$. For a vertex corresponding to the scheme $Q^{2,1}=1$ we construct outgoing edges to the vertices corresponding to these schemes.

We continue this process. If on some step, when the initial X satisfy an equation $U_1(X)=1$, the system $q_1^{j+1,j+1}(X^{(j)})=1$ implies a new equation $U_2(X)=1$ on the initial X, we continue the construction of the tree as for the formula

$$\forall X\exists Y\ \big(\big(U_1(X)=1\wedge U_2(X)=1\wedge U(X,Y)=1\big)\to V(X,Y)=1\big).$$

By the Noetherian property, there can be only a finite number of such steps. Suppose therefore that $F_{R(S,W)}$ is embedded into $F_{R(S,q_1^{j,j})}$. Then we follow the construction of $T_{EA}(F_{R(S,W)})$. By Theorem 36 this tree is finite.

If we obtained dead-end vertices along all the branches of $T(\Phi)$, then $F_{R(S)}\not\models\Phi$.

Suppose there is some stop-vertex. This means that all Y obtained by the formula from X satisfying $q_1^{j,j}(X^{(j)})=1$ and solving the equation $U=1$ together with X, automatically solve the equation $V=1$. In this case the formula Φ is true. Indeed, if Φ were false, $\neg\Phi$ could be lifted into the group $F_{R(S,q_1^{j,j})}$, therefore the formula solution Y would exist not solving $V=1$.

We can find X solving Φ effectively. It is expressed in terms of $X^{(j)}$ solving the system $q_1^{j,j}(X^{(j)})=1$. Suppose this system has the form

$$S_1(X_1,\dots,X_n)=1,$$

$$\vdots$$

$$S_n(X_n) = 1.$$

Construct $X^{(j)} = X_1 \cup \cdots \cup X_n$ inductively, as a positive N-large solution of $S_i = 1$, where N depends on $S_i = 1$ (see Section 7.14). All values of Y solving $U(X, Y) = 1$ together with such X will be specializations of values expressing Y by a formula in $X^{(j)}$. Since we are in a stop-vertex, these values of Y will also satisfy $V(X, Y) = 1$.

12.2. *Induction step, T_{X_k}*

Consider now the general sentence

$$\Phi = \exists X_1 \forall Y_1 \ldots \exists X_k \forall Y_k \bigl(U(X_1, Y_1, \ldots, X_k, Y_k) = 1 \to (X_1, Y_1, \ldots, X_k, Y_k) = 1\bigr).$$

We will first show that this sentence is true if and only if some boolean combination of sentences with less alternations of quantifiers is true, and that the reduction does not depend on the rank of the free group F and even on the group $F_{R(S)}$. This proves Theorem 1. Simultaneously we will show that the analog of the tree $T_{F_{R(S)}}(\Phi)$ which is going to be constructed for the general case, is finite. Since each step of the construction of the tree is effective, this will imply Theorem 2.

For the sentence Φ there are two possibilities.

(1) The sentence

$$\tilde{\Phi} = \exists X_1 \forall Y_1 \ldots \exists X_k \forall Y_k \ (U \neq 1)$$

is true in $F_{R(S)}$. In this case Φ is true. The negation of this sentence is equivalent to a positive sentence, therefore, it is simultaneously true or false in free groups of all ranks $\geqslant 2$ and in $F_{R(S)}$.

(2) The sentence $\tilde{\Phi}$ is false in $F_{R(S)}$. Then

$$F_{R(S)} \models \forall X_1 \exists Y_1 \ldots \forall X_k \exists Y_k \ (U = 1).$$

Using the Merzljakov theorem we can consider a complete family $\mathcal{F}_1$ of those schemes for $U = 1$ which have form

$$Q^{1,0}\bigl[Y_k; X_k; Y_{k-1}; X_{k-1}; \ldots; Y_1; X_1 \bigm| Y_k^{(1)}; X_k; Y_{k-1}^{(1)}; X_{k-1}; \ldots; Y_1^{(1)}; X_1\bigr] = 1.$$

We say that these schemes have level $(1, 0)$. In all the schemes of level $(1, 0)$ variables from Y_k are functions of X_k and $Y_{k-1}^{(1)}$. Each such scheme contains a fundamental sequence for Y_k (of level $(1, 0)$), namely, a fundamental sequence for $F_{R(U)}$ modulo $\langle X_1, Y_1, \ldots, X_k\rangle$. Those solutions of $U = 1$ in $F_{R(S)} * F(X_k)$, that correspond to $\mathcal{F}_1$ and are minimal with respect to these fundamental sequences, discriminate a finite number of groups $F_{R(U_{1,i})} * F(X_k)$, where $U_{1,i} = U_{1,i}(X_1, Y_1, \ldots, X_{k-1}, Y_{k-1}, Y_{k-1}^{(1)}) = 1$, such that the disjunction $U_{1,i} = 1$ has solution for those and only those values of $X_1, Y_1, \ldots, Y_{k-1}$ for which for any X_k there exists Y_k such that $U = 1$. Either Φ is false and can be disproved on level $(1, 0)$ or for each $Q^{1,0} = 1$ from $\mathcal{F}_1$ we have $V = 1$. The equation $V = 1$ gives a finite number of additional equations on $F_{R(U(1,i))} * F(X_k)$. Let G_i, $i \in J$, be corresponding fully residually free groups. Either this implies a proper equation on $F_{R(U(1,i))}$ (we consider this situation separately), or $V = 1$ holds for every solution of the corresponding scheme $Q^{1,0} = 1$ (and Φ is verified for any $X_1, Y_1, \ldots, X_{k-1}$ satisfying this scheme and not satisfying any other scheme of level $(1,0)$), or we have to consider a finite number of fundamental sequences for groups G_i modulo the subgroup $F_{R(U(1,i))}$.

We consider fundamental sequences for groups G_i, $i \in J$, modulo $F_{R(U_{1,i})}$, which are fundamental sequences for X_k at level $(1,1)$. As we did in case $k=1$, for each such group we construct the tree T_{X_k} which is the analog of $T_{EA}(G_i)$ considering X_k as variables and $X_1, Y_1 \dots, Y_{k-1}, Y_{k-1}^{(1)}$ as parameters. We obtain for each branch of the tree T_{X_k} a finite number of finite sequences of groups $F_{R(U_{1,i})}, F_{R(V_{2,i})}, \dots, F_{R(V_{r,i})}, F_{R(U_{r,i})}$ such that

$$U_{1,i} = U_{1,i}\big(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}\big),$$
$$U_{m,i} = U_{m,i}\big(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(m)}, Y_{k-1}^{(m)}\big), \quad m = 2, \dots, r,$$

which are terminal groups of fundamental sequences for Y_k in schemes of level $(m, m-1)$, and

$$V_{m,i} = V_{m,i}\big(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(m)}\big), \quad m = 2, \dots, r,$$

which are terminal groups of fundamental sequences for X_k in schemes of level (m,m). They correspond to vertices of T_{X_k} that have distance m to the root.

For each m the group $F_{R(U_{m,i})}$ does not have a sufficient splitting modulo the subgroup generated by $X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(m)}$, and the group $F_{R(V_{m,i})}$ does not have a sufficient splitting modulo the subgroup generated by $X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}$. On each step we consider terminal groups of all levels. Below we will sometimes skip index i and write U_m, V_m instead $U_{m,i}, V_{m,i}$.

12.3. Schemes for Y_{k-1}

We will show that modulo $\langle X_1, Y_1, \dots, X_{k-1}\rangle$ there is only a finite number of possible schemes for Y_{k-1}. The solution set $\mathcal{V}$ associated with each scheme will consist of compositions $\phi\psi$, where ϕ is a solution from a reduced fundamental sequence modulo $\langle X_1, Y_1, \dots, X_{k-1}\rangle$ (with terminal group, say, K) and ψ is a K-algebraic solution from a $\sim_{MAX}$-equivalence class. We first consider $m=1$. We want to describe modulo $X_1, Y_1, \dots, X_{k-1}$ the schemes for Y_{k-1} which occur when Y_k solving $U=1$ is on level $(2,1)$, namely, is expressed by a formula in the variables from the schemes of level $(1,1)$ in the tree T_{X_k}. The restricted fundamental sequences that are parts of these schemes for Y_{k-1} will be called *fundamental sequences of level* $(2,1)$ *for* Y_{k-1}. It is clear that schemes for Y_{k-1} of level $(2,1)$ depend on the schemes for Y_k of level $(2,1)$. Schemes for Y_k depend on $Z_{k-1}^{(2)}$ and $Y_{k-1}^{(2)}$. There exists a bounded number of different non-$\sim_{MAX}$-equivalent classes of $Y_{k-1}^{(2)}$ modulo $F_{R(V_2)}$ and $Z_{k-1}^{(2)}$ modulo $F_{R(U_1)}$, therefore there is a number M such that for each value of $X_1, Y_1, \dots, X_{k-1}, Y_{k-1}$ there exists at most M schemes for Y_k.

The main problem is that when we add restrictions on variables that are necessary to satisfy $V=1$, some of the algebraic solutions $Z_{k-1}^{(2)}$ or $Y_{k-1}^{(2)}$ may become reducing, some may become equivalent, and new algebraic solutions may appear. The second problem is that the number of them, being bounded by Theorem 11, can, nevertheless, change. Each new value of $Z_{k-1}^{(2)}$ will produce a new projective image of the fundamental sequence for Y_{k-1} and to each new value we assign a copy of the variable $Z_{k-1}^{(2)}$, denote it $Z_{k-1}^{(2,i)}$. New values of $Z_{k-1}^{(2)}$ appear in some order in the process, but we will not consider them in this order; the first several variables $Z_{k-1}^{(2,1)}, \dots$ will correspond not to solutions $Z_{k-1}^{(2)}$ that appear first but to algebraic solutions $Z_{k-1}^{(2)}$.

Our goal is to show first that there is only a finite number of (restricted) fundamental sequences of level (2, 1) for Y_{k-1} modulo $\langle X_1, Y_1, \dots, X_{k-1}\rangle$. Simultaneously we will obtain a description of values of $X_1, Y_1, \dots, X_{k-1}$ for which the sentence cannot be either proved or disproved on levels less than (2, 1). We will define initial fundamental sequences of level (2, 1) for Y_{k-1} and then obtain from them other fundamental sequences of level (2, 1).

Definition 26. Suppose groups $F_{R(U_{2,m})}$ have been constructed in the process of construction of the tree $T_{R(X_k)}$, and they all do not have sufficient splitting modulo $F_{R(V_2)}$. Consider fundamental sequences for the equations

$$U_{2,m}\bigl(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2)}, Y_{k-1}^{(2)}\bigr) = 1$$

modulo the subgroup $\langle X_1, Y_1, \dots, X_{k-1}\rangle$. Then consider a sequence of proper projective images of these fundamental sequences obtained the following way. To obtain the projective image on the ith step we add i copies of variables $Z_{k-1}^{(2)}$ (denote them $Z_{k-1}^{(2,i)}$), $Y_{k-1}^{(2)}$ (denote them $Y_{k-1}^{(2,i)}$) and consider the fundamental sequences modulo the subgroup $\langle X_1, Y_1, \dots, X_{k-1}\rangle$ for the group discriminated by i solutions of the systems

$$U_{2,m}\bigl(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,j)}, Y_{k-1}^{(2,j)}\bigr) = 1, \quad j = 1, \dots, i,$$

with the following properties:

(1) $Y_{k-1}^{(1)}, Z_{k-1}^{(2,j)}, Y_{k-1}^{(2,j)}$ are algebraic;
(2) $Z_{k-1}^{(2,j)}$ not *MAX*-equivalent to $Z_{k-1}^{(2,p)}$, $p \neq j$, $p, j = 1, \dots, i$;
(3) for any of the finite number of values of $Z_{k-1}^{(2)}$ fundamental sequences for $V_2(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2)}) = 1$ are contained in the union of corrective extensions of fundamental sequences for $U_2(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,j)}, Y_{k-1}^{(2,j)}) = 1$ for different values of $Y_{k-1}^{(2,j)}$;
(4) there is no non-equivalent algebraic $Z_{k-1}^{(2,i+1)}$ solving $V_2 = 1$;
(5) the solution $X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}$ does not satisfy an equation which implies $V = 1$ for any value of X_k.

We say that these sequences are *initial sequences* for Y_{k-1} of level (2, 1) and *width* i related to $F_{R(V_2)}$. The possible width of fundamental sequences is bounded by Theorem 11. To express (3) we have to add variables for primitive roots of specializations of edge groups and abelian vertex groups in the decomposition of $F_{R(V_2)}$.

Similarly to the initial fundamental sequences of level (2, 1) and width i related to $F_{R(V_2)}$, we can construct initial fundamental sequences of level (2, 1) and *width* $i = i_1 + \cdots + i_t$.

Definition 27. Let $F_{R(V_{2,1})}, \dots, F_{R(V_{2,t})}$ be the whole family of groups on level (1, 1). To construct the *initial fundamental sequences of level* (2, 1) and *width* $i = i_1 + \cdots + i_t$, we consider the fundamental sequences modulo the subgroup $\langle X_1, Y_1, \dots, X_{k-1}\rangle$ for the group discriminated by solutions of the systems

$$U_{2,m_s}\bigl(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,j,s)}, Y_{k-1}^{(2,j,s)}\bigr) = 1, \quad j = 1, \dots, i_s,\ s = 1, \dots, t,$$

with properties:

(1) $Y_{k-1}^{(1)}, Z_{k-1}^{(2,j,s)}, Y_{k-1}^{(2,j,s)}$ are algebraic;
(2) $Z_{k-1}^{(2,j,s)}$ not *MAX*-equivalent to $Z_{k-1}^{(2,p,s)}$, $p \neq j,\ p, j = 1, \ldots, i_s,\ s = 1, \ldots, t$;
(3) for any of the finite number of values of $Z_{k-1}^{(2)}$ fundamental sequences for $V_{2,s}(X_1, Y_1, \ldots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2)}) = 1$ are contained in the union of fundamental sequences for $U_{2,m_s}(X_1, Y_1, \ldots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,j)}, Y_{k-1}^{(2,j)}) = 1$ for different values of $Y_{k-1}^{(2,j,s)}$;
(4) there is no non-equivalent $Z_{k-1}^{(2,i_s+1,s)}$ algebraic solving $V_{2,s} = 1, s = 1, \ldots, t$;
(5) the solution $X_1, Y_1, \ldots, Y_{k-1}, Y_{k-1}^{(1)}$ does not satisfy a proper equation which implies $V = 1$ for any value of X_k.

Let $H = F_{R(W)}$ be the group discriminated by an initial fundamental sequence of level (2, 1) and width i (with generators

$$X_1, Y_1, \ldots, X_{k-1}, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,j,s)}, Y_{k-1}^{(2,j,s)}, \quad j = 1, \ldots, i_s,\ s = 1, \ldots, t).$$

12.4. Fundamental sequences of level (2, 1)

For any value of $X_1, Y_1, \ldots, X_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,j,s)}, Y_{k-1}^{(2,j,s)}$ factoring through the fundamental sequence of level (2, 1) there are the following possibilities.

(1) It satisfies properties (1)–(5).

(2) It does not satisfy one of the properties (1), (2), or (5). The family of values of variables for which one of the properties (1), (2) or (5) fails, can be described by a finite number of fundamental sequences.

(3) It satisfies (1), (2), (5), but does not satisfy (3) or (4), therefore there exists algebraic $Z_{k-1}^{(2,i_s+1,s)}$ not equivalent to others $Z_{k-1}^{(2,j,s)}$, such that

$$V_2\big(X_1, Y_1, \ldots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,i_s+1,s)}\big) = 1.$$

We consider groups discriminated by solutions of $W = 1$ and extended automorphically minimal values of $Z_{k-1}^{(2,i_s+1,s)}$. Denote such a group by W_{surplus}.

The following lemma is just a consequence of the parametrization theorem in [14].

Lemma 27. *Consider an initial fundamental sequence of level* (2, 1) *and width i with the group denoted by $H = F_{R(W)}$ discriminated by this sequence* (*with generators*

$$X_1, Y_1, \ldots, X_{k-1}, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,j,s)}, Y_{k-1}^{(2,j,s)}, \quad j = 1, \ldots, i_s,\ s = 1, \ldots, t).$$

For each value of $X_1, Y_1, \ldots, X_{k-1}$ factoring through this sequence one and only one of the following possibilities holds:

(i) *for this fundamental sequence there exists a generic family of values of the variables $Y_{k-1}^{(1)}, Z_{k-1}^{(2,j,s)}, Y_{k-1}^{(2,j,s)}$, $j = 1, \ldots, i_s$, $s = 1, \ldots, t$, $Z_{k-1}^{(3,k)}$ with properties* (1)–(5) *from Definition* 27;
(ii) *for this fundamental sequence there is no values of the variables with properties* (1)–(5);

(iii) *statements* (i) *and* (ii) *do not hold, and any value of the variables with properties* (1)–(5) *from this fundamental sequence can be extended to a solution of one of* W_{surplus} *and going through one of the finite number of fundamental sequences that describe the variables for which solution* $Z_{k-1}^{(2,i_s+1,s)}$ *is reducing or equivalent to one of the* $Z_{k-1}^{(2,j,s)}$*,* $j \leqslant i$*. Each such a fundamental sequence is a proper projective image of the initial fundamental sequence.*

These proper projective images will also appear in the process of verification of sentence Φ and are called *fundamental sequences of level* (2, 1)*, width* i*, depth* 2. For each such fundamental sequence consider again groups discriminated by this sequence and extended automorphically minimal values of $Z_{k-1}^{(2,i_s+2,s)}$ such that

$$V_2\big(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,i_s+2,s)}\big) = 1.$$

Variables for which the solution $Z_{k-1}^{(2,i_s+2,s)}$ is reducing or equivalent to one of the $Z_{k-1}^{(2,j,s)}$, $j \leqslant i$, belong to a projective image of the second fundamental sequence. This is a new fundamental sequence of level (2, 1), width i and depth 3. Denote this sequence of proper projective images by $N_1 = F_{R(\mathcal{W}_1)}, \dots, N_n = F_{R(\mathcal{W}_n)}$ as we did in Section 7. Fundamental sequences corresponding to $\mathcal{W}_1, \dots, \mathcal{W}_n$ are fundamental sequences modulo the subgroup $P = \langle X_1, Y_1, \dots, X_{k-1}\rangle$. If the sequence of proper projective images stabilizes, then by Theorem 38 free variables from N_n take values from different NTQ systems. Moreover, for the enveloping system there is a generic family of solutions that induces a family of solutions of $\mathit{TEnv}(W_n)$ such that each solution has rank that is equal to $\dim(\mathit{TEnv}(W_n))$. In this case we consider, instead of the fundamental sequences induced by the first principal group $H = F_{R(W)}$, the fundamental sequences induced by the subgroup of the enveloping group generated by $X_1, Y_1, \dots, X_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,j,s)}, Y_{k-1}^{(2,j,s)}$, $j = 1, \dots, i_s, n$, where $Z_{k-1}^{(2,n,s)}$ is a new not-*MAX*-equivalent to $Z_{k-1}^{(2,j,s)}$, $j = 1, \dots, i_s$, extended automorphically minimal algebraic solution (if it exists). We call this group a second principal group and consider projective images for these fundamental sequences. Denote these systems of equations by $\mathcal{W}_{n,2}$ and denote $N_{n,2} = F_{R(W_{n,2})}$. These are *fundamental sequences of level* (2, 1)*, width* i*, thickness* 2 *and depth* 1. When the sequence of fundamental sequences of thickness 2 and corresponding NTQ systems $\mathcal{W}_{r,2}$, $r \geqslant n$, stabilizes, we add another solution $Z_{k-1}^{2,j,s}$ (if exists), denote these systems of equations by $\mathcal{W}_{j,3}$. These are *fundamental sequences of level* (2, 1)*, width* i*, thickness* 3 *and depth* 1. When the chain of projective images of these sequences of increasing depth (and fixed thickness) stabilizes, we add new variables, increase thickness, and so on.

We can similarly define fundamental sequences of level $(m, m-1)$.

12.5. Finiteness results

In this subsection we will prove another finiteness result.

Lemma 28. *Only a bounded number of fundamental sequences of level* $(m, m-1)$ *and width* i *can appear in the process.*

Proof. We will prove the lemma for fundamental sequences of level (2, 1). Each chain of fundamental sequences of level (2, 1) and width i is a chain of proper projective images of the fundamental sequences corresponding to $H = F_{R(W)}$ and later corresponding to other principal groups. In each step of the process we have a new extended automorphically minimal not-*MAX*-equivalent algebraic solution $Z_{k-1}^{(2,j)}$, and some of the minimal solutions considered previously become reducing solutions. If the process goes infinitely, we can suppose that for an infinite number of indices t, k and $p(t) \leqslant k$, groups $N_{n_1,t} = F_{R(\mathcal{W}_{n_1,t})}, \ldots, N_{n_{p(t)},t+k} = F_{R(\mathcal{W}_{n_{p_t},t+k})}$ are embedded into the group $N_{n_{p(t)},t+k}$. By Theorem 38 we can suppose that we are at the step when $N_{n_{i+1},t}$ is obtained from $N_{n_i,t}$ by assigning to free variables minimal values from some NTQ system. Some of the algebraic solutions may become reducing solutions. Our goal is to get a contradiction. In particular, we will also show that one can "replace" extended automorphisms of the group $F_{R(V_2)}$ modulo $F_{R(U_1)}$ corresponding to the edges of the JSJ decomposition in this situation by canonical automorphisms. This can be done by showing that it is enough to consider only solutions for which the images of edge group generators become only bounded powers. We can adjust the proof of Theorem 11 for this situation. Extended automorphisms corresponding to the JSJ decomposition of the group $F_{R(V_2)}$ that are not canonical automorphisms modulo $\langle X_1, Y_1, \ldots, Y_{k-1}, Y_{k-1}^{(1)}\rangle$ are the following: (1) conjugation of non-QH-subgroups of $F_{R(V_2)}$ by some elements that commute with generators of edge groups; (2) multiplication of HNN stable letters by some elements that commute with generators of edge groups, (3) extended automorphisms of abelian and QH-subgroups. The discussion after the proof of Theorem 11 shows that extended automorphisms of the first type can be replaced by canonical automorphisms.

Suppose for an infinite number of indices r, each solution $Z_{k-1}^{(2,n_1)}, \ldots, Z_{k-1}^{(2,n_r)}$, $r = 1, \ldots, p$ is extended automorphically minimal in the free group assigned to $N_{n_{p(t)},t+k}$. We can also assume they all are solutions of the same system $V_2 = 1$ modulo $F_{R(U_1)}$ and do not satisfy any proper equation (if a subsequence does satisfy, we replace $V_2 = 1$ by this proper equation). We can return to the notation of Theorem 11 and denote $K = F_{R(V_2)}$ and $H = F_{R(U_1)}$. Let D_K be a JSJ decomposition of K modulo H.

Let G_{v_0} be the subgroup defined at the end of Section 6. We add to the graph of groups corresponding to G_{v_0} all the edges e such that for a subsequence of indices $\{t\}$ and for some generic family of solutions for $\mathcal{W}_{n_{p(t)},t+k}$ the image of the generator of G_e in F only becomes a bounded power of some elements in F. We also add vertex groups that are connected by these edges. Add to K elements d_j, $j \in J$, that commute with the edge groups of the JSJ decomposition D_K of K modulo H, and consider groups discriminated by minimal (with respect to $\sim_{MAX}$) solutions and generators of the images in F of the edge groups in place of variables d_j, $j \in J$, that induced from the chosen generic families. Denote any of the finite family of such fully residually free groups by K_E. Denote the edge groups of K by G_{e_j}, $j \in J$. Let $\{e_i, i \in I\}$ be the set of edges either joining G_{v_0} with abelian vertex groups or edges with group $\langle G_{e_i}, d_i\rangle$ abelian of rank 2. Let $P = \langle X_1, Y_1, \ldots, X_{k-1}\rangle \leqslant K_E$. The following remark is obvious.

Remark 2. Suppose values of elements from P in $F(A)$ are fixed. Let G_{e_i} generated by one or several elements g_{e_i} be embedded in $N_{n_1,t}$ as a subgroup of $F_{R(V_2)}$. Let $\{g_{e_i}^{\phi_s}\}$ be a family of specializations of g_{e_i} corresponding to a generic family of solutions $\{\phi_s\}$ for $\mathcal{W}_{n_1,t}$ in the group $F(A) * F_1$. Then the values of $g_{e_i}^{\phi_s}$ that correspond to algebraic solutions of $V_2 = 1$ modulo $F_{R(U_1)}$ and algebraic solutions of $F_{R(U_1)}$ modulo P are in the subgroup of $F(A) * F_1$ conjugated to $F(A)$.

As in the proof of Theorem 11, we can obtain a finite number of cut equations $\bar{\Pi}$ and corresponding systems (11) for K_E and H. Suppose there is a sequence of solutions of $\bar{\Pi}$, w_{tN} such that for each w_{tN} there are at least N distinct solutions of $\bar{\Pi}$: $(y_{1N}^{(i)}, \ldots, y_{nN}^{(i)})$, $i = 1, \ldots, N$. Let

$x_{1N}, \dots, x_{nN}$ be some fixed solution in this family. As we did in the proof of Theorem 11 we construct for K_E and H groups $F_{R(P)}$ and $F_{R(U)}$, such that K_E is not conjugated into a fundamental group of a proper subgraph of the JSJ decomposition D of $F_{R(U)}$ modulo $\langle x_1, \dots, x_n\rangle$. The decomposition D induces a decomposition D_{v_0} of G_{v_0} in which all the subgroups $G_{e_i}, i \in I$, are elliptic. This decomposition can be extended to a decomposition $\bar{D}_{K_E}$ of K_E. The growing variables Z from $F_{R(P)}$ and $F_{R(U)}$ cannot grow due to extended automorphisms corresponding to QH-subgroups of D_{K_E} because solutions were taken minimal with respect to extended automorphisms of D_{K_E} and minimal with respect to splittings of $N_{n_{p(t)},t+k}$ modulo $N_{n_1,t}$. Therefore the quadratic system of $\bar{D}_{K_E}$ has smaller size than the quadratic system of D_{K_E} (this is equivalent to saying that either the regular size or the number of edges between non-abelian subgroups is smaller). We can consider $\bar{D}_{K_E}$ instead of D_{K_E}. Either the number of edge groups with unboundedly deep roots becomes larger for this decomposition or we can apply induction. The number of edge groups with unboundedly deep roots cannot increase infinitely and the size of decompositions of the same group cannot decrease infinitely. This gives a contradiction with the assumption that the process goes infinitely.

We can similarly show that for each $m = 2, \dots, r$ there is only a finite number of fundamental sequences of level $(m, m-1)$. $\square$

Denote terminal groups of fundamental sequences of level (2, 1) by $F_{R(U_{i,\mathrm{coeff}})}$, where

$$U_{i,\mathrm{coeff}} = U_{i,\mathrm{coeff}}(X_1, Y_1, \dots, X_{k-1}, \hat{Z}_{k-1}) = 1.$$

12.6. Fundamental sequences of level (m, m)

Definition 28. Let $F_{R(V_{2,1})}, \dots, F_{R(V_{2,t})}$ be the whole family of groups on level $(1, 1)$. To construct the initial fundamental sequences of level $(2, 2)$ and *width* $i = i_1 + \cdots + i_t$, we consider the fundamental sequences modulo the subgroup $\langle X_1, Y_1, \dots, X_{k-1}\rangle$ for the group discriminated by solutions of the systems

$$U_{2,m_s}\big(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,j,s)}, Y_{k-1}^{(2,j,s)}\big) = 1, \quad j = 1, \dots, i_s,\ s = 1, \dots, t,$$

that belong to the initial fundamental sequence of level (2, 1) and width $i = i_1 + \cdots + i_t$ and

(6) for some s a solution $X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,1,s)}, Y_{k-1}^{(2,1,s)}$ can be extended to a solution of some

$$V_3\big(X_1, Y_1, \dots, Y_{k-1}, Y_{k-1}^{(1)}, Z_{k-1}^{(2,1,s)}, Y_{k-1}^{(2,1,s)}, Z_{k-1}^{(3,1,s)}\big) = 1.$$

We add extra variables $Z_{k-1}^{(3,1,s)}$.

Similarly we can define fundamental sequences of level $(2, 2)$ derived from the initial ones and define fundamental sequences of level (m, m).

There is a finite number of fundamental sequences of level (m, m).

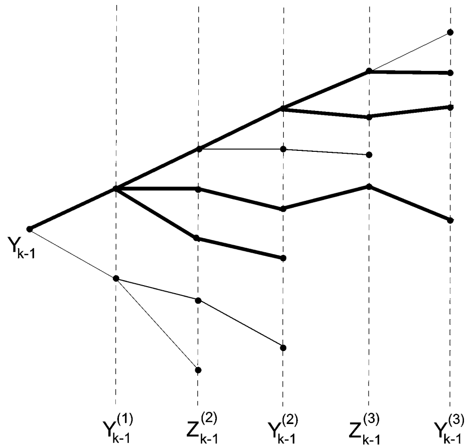

Fig. 4. Example of $\bar{T}_{X_k}$ and a false subtree.

Lemma 29. *Consider an initial fundamental sequence of level* $(2,2)$ *and width* i *with the group denoted by* $H = F_{R(W)}$ *discriminated by this sequence* (*with generators*

$$X_1, Y_1, \dots, X_{k-1}, Y_{k-1}, Y^{(1)}_{k-1}, Z^{(2,j,s)}_{k-1}, Y^{(2,j,s)}_{k-1}, \quad j = 1, \dots, i_s,\ s = 1, \dots, t).$$

For each value of $X_1, Y_1, \dots, X_{k-1}$ *factoring through this sequence one and only one of the following possibilities holds*:

- (i) *for this fundamental sequence there exists a generic family of values of the variables* $Y^{(1)}_{k-1}, Z^{(2,j,s)}_{k-1}, Y^{(2,j,s)}_{k-1}$, $j = 1, \dots, i_s$, $s = 1, \dots, t$, $Z^{(3,k)}_{k-1}$ *with properties* (1)–(6) *from Definition* 28;
- (ii) *for this fundamental sequence there is no values of the variables with properties* (1)–(6);
- (iii) *statements* (i) *and* (ii) *do not hold, and any value of the variables with properties* (1)–(6) *from this fundamental sequence can be extended to a solution of one of* W_{surplus} *and going through one of the finite number of fundamental sequences that describe the variables for which solution* $Z^{(2,i_s+1,s)}_{k-1}$ *is reducing or equivalent to one of the* $Z^{(2,j,s)}_{k-1}$, $j \leqslant i$. *Each such a fundamental sequence is a proper projective image of the initial fundamental sequence.*

12.7. Proof of Theorem 1

For each value $\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1}$ of variables $X_1, Y_1, \dots, X_{k-1}$ we can construct a finite tree $\bar{T}_{X_k}(\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1}, \bar{Y}_{k-1})$ such that the branches are labeled by sequences

$$\bar{Y}_{k-1},\ \bar{Y}^{(1)}_{k-1},\ \bar{Z}^{(2)}_{k-1},\ \dots,\ \bar{Z}^{(m)}_{k-1},\ \dots,$$

such that $V_{n,i}(\bar{X}_1, \dots, \bar{Z}^{(n)}_{k-1}) = 1$ and $U_{n,i}(\bar{X}_1, \dots, \bar{Z}^{(n)}_{k-1}, \bar{Y}^{(n)}_{k-1}) = 1$. Moreover, all specializations $\bar{Y}^{(1)}_{k-1}, \bar{Z}^{(2)}_{k-1}, \dots, \bar{Z}^{(m)}_{k-1}$ are algebraic extended automorphically minimal solutions of corresponding systems. An example of such tree is shown on Fig. 4.

Two sequences are considered the same if corresponding specializations are equivalent. We label each vertex of this tree except the root vertex by two labels: the system $V_{n,i}(\bar{X}_1, \dots,$

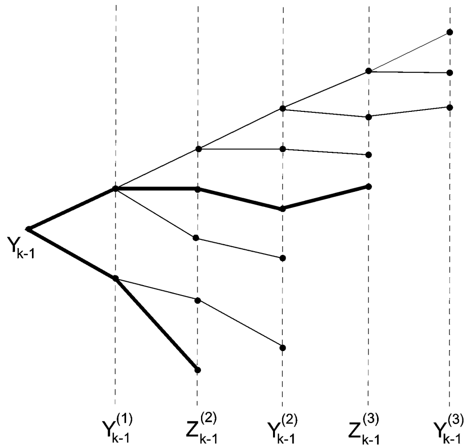

Fig. 5. Example of $\bar{T}_{X_k}$ and a true subtree.

$\bar{Z}_{k-1}^{(n)}) = 1$ or $U_{n,i}(\bar{X}_1, \ldots, \bar{Z}_{k-1}^{(n)}, \bar{Y}_{k-1}^{(n)}) = 1$ and the representative of the equivalence class $\bar{Z}_{k-1}^{(n)}$ or $\bar{Y}_{k-1}^{(n)}$. We say that for a given value of $X_1, Y_1, \ldots, X_{k-1}$ the formula

$$\Psi = \forall Y_{k-1} \exists X_k \forall Y_k \, \big(U(X_1, Y_1, \ldots, X_k, Y_k) = 1 \to V(X_1, Y_1, \ldots, X_k, Y_k) = 1\big) \tag{25}$$

can be proved on level (m, m) if for each value of Y_{k-1} there is a truth subtree in $\bar{T}_{X_k}$ (a subtree along which we prove the sentence for this value of $X_1, Y_1, \ldots, X_{k-1}, Y_{k-1}$) having branches of length less or equal than $2m$. Figure 5 shows an example of a truth subtree (in bold), that corresponds to a value of $X_1, Y_1, \ldots, X_{k-1}, Y_{k-1}$ for which the formula Ψ can be proved on level $(2, 2)$.

It can be disproved on level $(m, m-1)$ if there exists a value of Y_{k-1} such that the sentence $\exists X_k \forall Y_k$ $(U = 1 \to V = 1)$ is false and there is no false subtree in $\bar{T}_{X_k}$ having branches longer than $2m - 1$. Figure 4 shows an example of a false subtree (in bold), that corresponds to a value of $X_1, Y_1, \ldots, X_{k-1}$ for which the formula Ψ can be disproved on level $(3, 2)$.

For a given value of $X_1, Y_1, \ldots, X_{k-1}$ formula (25) cannot be proved on level less than $(2, 1)$ if and only if the following conditions are satisfied.

(a) There exist algebraic solutions for some $U_{i,\mathrm{coeff}} = 1$ corresponding to the terminal group of a fundamental sequence $V_{i,\mathrm{fund}}$ satisfying possibility (i) in Lemma 27.
(b) There is no algebraic solutions for equations corresponding to the terminal groups of fundamental sequences that describe solutions from $V_{i,\mathrm{fund}}$ that do not satisfy one of properties (1)–(5). There is a finite number of such fundamental sequences.
(c) There is no algebraic solutions for equations corresponding to the terminal groups of fundamental sequences of level $(2, 1)$ and greater depth derived from $V_{i,\mathrm{fund}}$.
(d) $(X_1, Y_1, \ldots, Y_{k-1}, Y_{k-1}^{(1)})$ cannot be extended to a solution of $V = 1$ by arbitrary X_k (X_k of level 0) and Y_k of level $(1, 0)$.

For a given value of $X_1, Y_1, \ldots, X_{k-1}$ formula (25) can be disproved on level $(2, 1)$ if and only if it cannot be proved on level less than $(2, 1)$ and there is no algebraic solutions for equations corresponding to the terminal groups of fundamental sequences of level (2, 2) corresponding to $V_{i,\text{fund}}$.

These conditions can be described by the conjunction of sentences of the following two forms.

(1) Disjunction of sentences

$$\exists Z_{k-1} \forall B, C \; \big(U_{i,\text{coeff}} = 1 \wedge V_{\text{alg}}(X_1, \ldots, X_{k-1}, Z_{k-1}, B, C) \neq 1\big),$$

where the non-equation $V_{\text{alg}}(X_1, \ldots, X_{k-1}, Z_{k-1}, B) \neq 1$, says that either variables from B do not commute with edge groups in the JSJ decomposition D for $F_{R(U_{\text{coeff}})}$ modulo $\langle X_1, Y_1, \ldots, X_{k-1}\rangle$ or conjugation by them of the corresponding vertex groups does not give a reducing solution even after replacing the generators of the MQH-subgroups of D by a minimal solution C.

(2) $$\forall Z_{k-1} \exists B \; \big(U'_{i,\text{coeff}} = 1 \rightarrow V'_{\text{alg}}(X_1, \ldots, X_{k-1}, Z_{k-1}, B) = 1\big).$$

For a given value of $X_1, Y_1, \ldots, X_{k-1}$ formula (25) can be proved if and only if it cannot be disproved at any level.

Therefore Φ can be written as a boolean combination of sentences with less alternations of quantifiers. The process of reduction of Φ to the boolean combination of these sentences does not depend on $F_{R(S)}$. One can apply induction and show that every sentence can be reduced the same way for any regular NTQ group $F_{R(S)}$ to a boolean combination of $\forall\exists$-sentences. Theorem 1 is proved.

The argument presented above also proves the following results.

Theorem 39. [33] *Every formula in the language of a free group is equivalent to a boolean combination of* $\forall\exists$*-formulas.*

Theorem 40. *A coordinate group of a regular NTQ system has the same elementary theory as a non-abelian free group.*

From the other hand, Theorem F from [20] states that every finitely generated group which is $\forall\exists$-equivalent to a free non-abelian group is isomorphic to the coordinate group of a regular NTQ system. Theorem 40 and this result imply the following theorem.

Theorem 41. *A finitely generated group G has the same elementary theory as a non-abelian free group if and only if G is isomorphic to the coordinate group of a regular NTQ system.*

12.8. Proof of Theorem 2

We will now prove Theorem 2. Lemma 28 shows that all possible values of Y_{k-1} considered in the process are obtained by a substitution from generic families corresponding to a finite number of fundamental sequences modulo the subgroup $\langle X_1, Y_1, \ldots, X_{k-1}\rangle$. We will use this fact for verification of a sentence. This is a branching process, and we will construct a tree $T_{R(S)}(\Phi)$ and will prove (using induction) that it is finite.

To describe the construction of the tree $T_{R(S)}(\Phi)$ we need a generalization of the implicit function theorem.

Theorem 42 *(Generalized implicit function theorem). Suppose that* $S[X_k, Y_{k-1}, \dots, X_1] = 1$ *is a non-degenerate normalized triangular quasi-quadratic system of the following form*:

$$\begin{aligned}
&S_k[X_k, Y_{k-1}, \dots, X_1] = 1,\\
&R_{k-1}[Y_{k-1}, \dots, X_1] = 1,\\
&S_{k-1}[X_{k-1}, Y_{k-2}, \dots, X_1] = 1,\\
&\quad\vdots\\
&R_1[Y_1, X_1] = 1,\\
&S_1[X_1] = 1,
\end{aligned}$$

where $S_k = 1, \dots, S_1 = 1$ *are normalized triangular quasi-quadratic systems, in variables* $X_k, \dots, X_1$, *respectively, and* $R_{k-1} = 1, \dots, R_1 = 1$ *are triangular quasi-quadratic systems in variables* $Y_{k-1}, \dots, Y_1$, *respectively.*

Suppose

$$\alpha = \forall X_1 \exists Y_1 \dots \forall X_k \exists Y_k \bigl(S = 1 \to v(X_1, Y_1, \dots, X_k, Y_k) = 1\bigr)$$

is true in F. *Then there exists a finite number of corrective extensions* $\bar{S} = 1$ *of* $S = 1$ *obtained by taking corrective extensions of systems* $S_i = 1$ *such that each homomorphism from* $F_{R(S)}$ *to* F *factors through one of these extensions*; *and for each of them there is a finite number of schemes* $T[Y_k, X_k, \dots, Y_1, X_1 \mid Y_k^{(1)}, X_k, \dots, Y_1^{(1)}, X_1] = 1$ (*which can be constructed effectively*) *for a system* $\bar{S}(X_1, Y_1, \dots, X_k, Y_k) = 1 \wedge v(X_1, Y_1, \dots, X_k, Y_k) = 1$, *with triangular quasi-quadratic system in the following form*

$$\begin{aligned}
&P_k\bigl[Y_k^{(1)}, X_k, Y_{k-1}^{(1)}, \dots, X_1\bigr] = 1,\\
&\bar{S}_k[X_k, Y_{k-1}, \dots, X_1] = 1,\\
&\quad\vdots\\
&P_2\bigl[Y_2^{(1)}, X_2, Y_1^{(1)}, X_1\bigr] = 1,\\
&\bar{S}_2[X_2, Y_1, X_1] = 1,\\
&P_1\bigl[Y_1^{(1)}, X_1\bigr] = 1,\\
&\bar{S}_1[X_1] = 1,
\end{aligned}$$

and $Y_i = \tilde{p}_i(Y_i^{(1)})$, *such that the following property is satisfied*:

If X_i *is generic for each* $S_i = 1$, $i = 1, \dots, k$, *then* Y_i *that satisfy* $R_i = 1$, $i = 1, \dots, k-1$, *and* Y_k *solving* α *either satisfy one of these schemes or one of the finite number of proper equations depending on the system* $S = 1$.

Proof. By [12, Lemma 8] and by Theorem 37, for a positive unbounded family of solutions of equations $S_1 = 1, \ldots, S_k = 1$ (see Section 12.1) we can consider a complete finite family of schemes for $S = 1 \wedge v = 1$, say $T_1 = 1, \ldots, T_t = 1$. Let

$$T_i = T_i\big[Y_k, X_k, \ldots, Y_1, X_1 \mid Y_k^{(1)}, X_k^{(1)}, \ldots, Y_1^{(1)}, X_1^{(1)}\big] = 1$$

such that for X's in this family Y's can be determined by $T_i = 1$. Then by [14, Theorem 12] there is a finite number of corrective extensions of systems $S_1 = 1, \ldots, S_k = 1$ such that each solution of $S = 1$ factors through one of them, and each element of $Y_k^{(1)}$ can be expressed in terms of X_k solving $\bar{S}_k = 1$ and arbitrary elements Y_k' solving equations in "short variables" together with $Y_{k-1}, X_{k-1}, \ldots, X_1$ (these Y_k' correspond to short Y_k's, see the proof of Theorem 5 in [14]). The family Y_k' can be taken as an algebraic minimal solution of one of the equations corresponding to a group with no sufficient splitting modulo the subgroup generated by $X_1, Y_1, \ldots, X_{k-1}, Y_{k-1}$ in $F_{R(S)}$.

Variables Y_k', Y_{k-1} can be expressed in terms of X_{k-1} solving $\bar{S}_{k-1} = 1$ and arbitrary elements Y_{k-1}' solving together with $Y_{k-2}, X_{k-2}, \ldots, X_1$ system of equations in "short variables." Finally we obtain a finite number of schemes as in the formulation of the theorem. Theorem 34 provides effectiveness. □

According to Section 12.7, the sentence Φ is true and the verification of the sentence stops after considering Y_{k-1} from the schemes of level (2, 1) if and only if a sentence of the following form with less alternations of quantifiers is true:

$$\begin{aligned} \Phi_k = {} & \exists X_1 \forall Y_1 \ldots \exists X_{k-1} \exists Z_{k-1} \\ & \left(\bigwedge_{j=1}^{m} \left(\forall B_j \exists C_j \exists D_j \left(\bar{U}_j = 1 \to \bigwedge_i [t_{ij}, w_{ij}] = 1 \wedge \bar{V}_j = 1 \right)\right) \right. \\ & \left. \wedge \bigvee_{j=m+1}^{n} \left(\forall C_j \forall D_j \left(\bar{U}_j = 1 \wedge \left(\bigvee_i [t_{ij}, w_{ij}] \neq 1 \vee \bar{V}_j \neq 1 \right)\right)\right)\right), \end{aligned}$$

where $\bar{U}_j(X_1, Y_1, \ldots, X_{k-1}, Z_{k-1}, B_j) = 1$ and $\bar{V}_j(X_1, Y_1, \ldots, X_{k-1}, Z_{k-1}, B_j, C_j, D_j) = 1$ are equations ($\bar{V}_i = 1$ corresponds to a maximal reducing quotient for $F_{R(\bar{U}_i)}$), $\{t_{ij}\} = C_j$ and w_{ij} are some words in $X_1, Y_1, \ldots, X_{k-1}, Z_{k-1}, B_j$. Notice, that variables from the families C_j, D_j do not appear in $\bar{U}_j$. Similarly, Φ is true and the verification of the sentence Φ stops after considering Y_{k-1} from the schemes of level $(m, m-1)$ if and only if a sentence of the same form as Φ_k is true.

Using Lemma 6 from [14] we can transform Φ_k to the following form:

$$\begin{aligned} \Phi_k = \exists X_1 \forall Y_1 \ldots \exists X_{k-1} \exists Z_{k-1} & \left(\bigwedge_{j=1}^{m} \left(\forall B_j \exists C_j \exists D_j \left(\bar{U}_j = 1 \to \bigwedge_i [t_{ij}, w_{ij}] = 1 \wedge \bar{V}_j = 1 \right)\right) \right. \\ & \left. \wedge \big(\forall C\ (\bar{U} = 1 \to \bar{V} = 1)\big) \right). \end{aligned}$$

We can now construct the analog of the tree T_{X_k} for the variables X_{k-1}, Z_{k-1} modulo $\langle X_1, Y_1, \dots, X_{k-2}, Y_{k-2}\rangle$. Denote this tree by $T_{X_{k-1}}$. This tree is finite and can be effectively constructed provided one knows the sentence Φ_k. It is not the exact analog of the $\exists\forall$-tree, because we have variables from C_j, D_j. It is the tree of projective images. Therefore at each step of the construction we either obtain a proper quotient of the block-NTQ group from the previous step, or decrease the number of free factors in the projective image, or the dimension, or the size of the block-NTQ system or obtain stabilization. The number of consecutive steps when the sequence of proper projective images stabilizes can be bounded as in Lemma 28. The bound can be found effectively as in the proof of Theorem 11 provided Φ_k is known.

The X_k*-situation* for $\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1}, \bar{Y}_{k-1}$ is the tree $\bar{T}_{X_k}$ with second labels and the label of the root removed. We denote it by $\mathcal{S}it(\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1}, \bar{Y}_{k-1})$. By Theorem 11 there is a finite number of possible X_k-situations. The *global* X_k*-situation* for $\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1}$ is the finite collection of different X_k-situations for different $\bar{Y}_{k-1}$. We denote it $\mathcal{GS}it(\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1})$. All values of $X_1, Y_1, \dots, X_{k-1}$ with the same global X_k-situation are partitioned into a finite number of families: S_i, $i \in I$. There is a finite number of schemes for Y_{k-1} modulo $X_1, Y_1, \dots, X_{k-1}$ and generic families for the corresponding fundamental sequences such that for any $(\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1}) \in S_i$ there exists $\bar{Y}_{k-1}$ described by one of these schemes such that for all X_k corresponding to this global X_k-situation there exists Y_k corresponding to this global X_k-situation. The number of such families is finite because by Lemma 28 there is only a finite number of fundamental sequences for Y_{k-1} of each level $(m, m-1)$.

The Y_{k-1}*-configuration* for $\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1}$ is a pair consisting of $\mathcal{GS}it(\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1})$ and the family of corresponding schemes for Y_{k-1}. We denote it $\mathcal{C}onf(\bar{X}_1, \bar{Y}_1, \dots, \bar{X}_{k-1})$. There is a finite number of possible Y_{k-1}-configurations.

We can now describe the construction of the decidability tree $T_{F_{R(S)}}(\Phi)$. This tree is a rooted tree oriented from the root. We assign to its vertices different levels: $(0,0)$, $(1,0)$, $(1,1)$, $(2,1)$ and so on. These levels are not the same as levels for fundamental sequences for Y_{k-1} considered above, which correspond to levels of vertices in T_{X_k}. To avoid confusion we will call levels of vertices in the tree $T_{F_{R(S)}}(\Phi)$ *d-levels* (the levels in the decidability tree).

If the formula

$$\tilde{\Phi} = \exists X_1 \forall Y_1 \dots \exists X_k \forall Y_k \ (U \neq 1)$$

is true in $F_{R(S)}$, the initial vertex v_0 is a stop-vertex, and Φ is true.

If the formula $\tilde{\Phi}$ is false in $F_{R(S)}$, then

$$F_{R(S)} \models \forall X_1 \exists Y_1 \dots \forall X_k \exists Y_k \ (U = 1).$$

Using Theorem 42 we consider a complete family $\mathcal{F}_1$ of those elimination schemes for $U = 1$ which have the form

$$Q^{1,0}\big[Y_k; X_k; \dots; Y_1; X_1 \bigm| Y_k^{(1)}; X_k; \dots; Y_1^{(1)}; X_1\big] = 1,$$

assign $\mathcal{F}_1$ to a vertex $v_{1,0}$ of d-level $(1,0)$ and draw an edge from v_0 to $v_{1,0}$.

To make variables satisfy equation $V = 1$, for each $Q^{1,0} = 1$ from $\mathcal{F}_1$ we obtain a family of schemes using Theorem 42 with $Q^{1,0}$ in place of S:

$$Q^{1,1}\big[X_k; \dots; Y_1; X_1 \bigm| X_k^{(1)}; \dots; Y_1^{(1)}; X_1^{(1)}\big] = 1.$$

This family can be obtained effectively. If one such family is empty, Φ is false, and vertex $v_{1,0}$ is a dead-end vertex. We draw edges from $v_{1,0}$ to vertices of d-level $(1,1)$ and assign to each such vertex $v_{1,1,i}$ a scheme $Q^{1,1}=1$.

If $v_{1,1,i}$ corresponds to a scheme $Q^{1,1}=1$ for which $X_k^{(1)}=X_k$, we can construct, by induction (on k), the tree $T_{R(S)}(\Phi)$ considering X_k as free variables (denote this tree by $T'_{R(S)}(\Phi)$). We assign the tree $T'_{R(S)}(\Phi)$ to the vertex $v_{1,1,i}$. If the sentence Φ is true when variables from X_k are considered as free variables, it is true in $F_{R(S)}$, because for a generic value of X_k all possible Y_k will be given by a formula in X_k. If the sentence Φ is false when variables from X_k are considered as free variables, one has to consider X_k satisfying some schemes modulo $X_1, Y_1, \ldots, X_{k-1}, Y_{k-1}$, in other words, one has to go further along the branches of T_{X_k} for X_k and replace $Q^{1,1}=1$ by the corresponding schemes.

We apply Theorem 42 to each scheme $Q^{1,1}=1$ independently and obtain for each scheme $Q^{1,1}=1$ a family of schemes

$$Q^{2,1}\big[Y_k; X_k; \ldots; Y_1; X_1 \bigm| Y_k^{(2)}; X_k^{(1)}; \ldots; Y_1^{(2)}; X_1^{(1)}\big]=1$$

and assign to each such family a vertex of d-level $(2,1)$. We can continue this process and have to be sure that it stops. Each fundamental sequence of level $(m,m-1)$ for Y_{k-1} corresponds to one of the finite number of $\mathit{Conf}(X_1, Y_1, \ldots, X_{k-1})$. The process is organized in such a way that if for the values $X_1, Y_1, \ldots, X_{k-1}$ described by a scheme of some d-level (j,j) Y_{k-1}-configuration has been changed, then this configuration wont appear again when we add more restrictions on $X_1, \ldots, X_{k-1}$. Indeed, every time when the X_k-situation changes in the process and than changes back, some algebraic solution $Z_{k-1}^{(i)}$ or $Y_{k-1}^{(i)}$ becomes a reducing solution or becomes equivalent to another one. Even if some new algebraic solution occurs, and the global X_k-situation does not change, we obtain a deeper fundamental sequence of level $(m,m-1)$ for Y_{k-1}. Suppose the Y_{k-1}-configuration had not been changed when we went from a vertex $v_{s,s,i}$ of d-level (s,s) to a vertex $v_{s+1,s+1,i}$ of d-level $(s+1,s+1)$. One can fix a family of elimination schemes $\mathcal{F}$ for X_k corresponding to a branch of T_k and corresponding to the given situation at $v_{s,s,i}$. One can verify by induction the sentence Φ for X_k corresponding to a disjunction of generic families for the fundamental sequences of the schemes from $\mathcal{F}$. Indeed, suppose the given situation contains only the schemes up to some level (m,m) for X_k. Considering only X_k corresponding to the given situation, one can construct another auxiliary tree $T''_{R(S)}(\Phi)$ and assign it to the vertex $v_{s+1,s+1,i}$. It is a tree $T_{R(S)}(\Theta)$ for some sentence Θ in the form of Φ_k. This sentence Θ depends on the fundamental sequences for Y_{k-1}. The tree $T''_{R(S)}(\Phi)$ is finite and induction can be applied because $T_{X_{k-1}}$ is finite. If Θ is true, then Φ is true. If Θ is false, this means that more restrictions must be put on X_k. In this case there are two possibilities:

(1) the vertex $v_{s+2,s+1,i}$ in the tree $T_{R(S)}(\Phi)$ is a dead-end vertex;
(2) the Y_{k-1}-configuration changes.

We put these restrictions on X_k and continue the process. Since the Y_{k-1}-configuration can only be changed finitely many times, eventually we either obtain that the sentence Φ is false or obtain that Θ is true for some Y_{k-1}-configuration and, therefore, Φ is true. A desired X_1 will satisfy a disjunction of schemes, it will be a function of a suitable chosen positive N-large solution $X_1^{(j)}$ of the block-NTQ system corresponding to one of these schemes. After X_1 is chosen, for each possible scheme for Y_1 we can similarly chose X_2 and so on.

All the steps in the construction of the tree $T_{R(S)}(\Phi)$ are effective, and the construction does not depend on the group $F_{R(S)}$. This proves Theorem 2.

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