Solutions to Final Exam, April/May 2006

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Q1. (a) $f\circ g = f(g(x)) = \sqrt{g(x) - 1}$

$$= \sqrt{1+\left(\frac{x}{1+x^2}\right)^2 - 1}$$

$$= \sqrt{\left(\frac{x}{1+x^2}\right)^2}$$

$$= \left|\frac{x}{1+x^2}\right| = \frac{|x|}{1+x^2}$$

$$g\circ f = g(f(x)) = 1+\left(\frac{f(x)}{1+[f(x)]^2}\right)^2$$

$$= 1+\left(\frac{\sqrt{x-1}}{1+(\sqrt{x-1})^2}\right)^2$$

$$= 1+\frac{x-1}{(1+x-1)^2}$$

$$= 1+\frac{x-1}{x^2}$$

$$= \frac{x^2+x-1}{x^2}$$

$$f\circ f = f(f(x)) = \sqrt{f(x)-1}$$

$$= \sqrt{\sqrt{x-1} - 1}$$

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(b) $y = f(x) = \ln(1+x^3)$

$\downarrow \qquad \downarrow$

$x = \ln(1+y^3)$ (inverse function)

$\Rightarrow e^x = e^{\ln(1+y^3)} = 1+y^3$

$\Rightarrow y^3 = e^x - 1$

$\Rightarrow \boxed{y = \sqrt[3]{e^x - 1}} =: f^{-1}(x)$, inverse function

- For $f(x) = \ln(1+x^3)$, the domain is

$D = \{x \mid 1+x^3 > 0\} = \{x \mid x > -1\} = (-1, \infty)$

the range is:

$R = (-\infty, \infty)$

- For $f^{-1}(x) = e^x - 1$, the domain is

$D = (-\infty, \infty)$,

the range is:

$R = (-1, \infty)$

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Q2 (a) $\lim_{x \to 2} \frac{\sqrt{x^2+5} - 3}{2x^2 - 8} = \lim_{x \to 2} \frac{\sqrt{x^2+5} - 3}{2(x-2)(x+2)}$

$= \lim_{x \to 2} \frac{(\sqrt{x^2+5} - 3)(\sqrt{x^2+5} + 3)}{2(x-2)(x+2)(\sqrt{x^2+5} + 3)}$

$= \lim_{x \to 2} \frac{(\sqrt{x^2+5})^2 - 3^2}{2(x-2)(x+2)(\sqrt{x^2+5} + 3)}$

$$= \lim_{x \to 2} \frac{x^2+5-9}{2(x-2)(x+2)(\sqrt{x^2+5}+3)}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{2\cancel{(x-2)}\cancel{(x+2)}(\sqrt{x^2+5}+3)}$$

$$= \lim_{x \to 2} \frac{1}{2(\sqrt{x^2+5}+3)}$$

$$= \frac{1}{2(\sqrt{2^2+5}+3)} = \boxed{\frac{1}{12}}$$

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(b) $$\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{x+1} = \lim_{x \to -\infty} \frac{\sqrt{2x^2+1}\,/x}{(x+1)/x}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{\frac{2x^2+1}{x^2}}}{\frac{x+1}{x}}$$

$$\left(\because \frac{1}{x} = -\sqrt{\frac{1}{x^2}} \text{ for } x < 0 \right)$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{1+\frac{1}{x}}$$

$$= \frac{-\sqrt{2+0}}{1+0} = \boxed{-\sqrt{2}}$$

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<u>Q3</u> (a) The points where the function $f(x) = \frac{|x+1|}{x^2+x}$ is undefined are the zeros of the denominator x^2+x, namely, $x^2+x=0$

$$\Rightarrow \boxed{x_1 = 0} \ \& \ \boxed{x_2 = -1}$$

For $x_1 = 0$, the sided limits are:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x+1|}{x(x+1)} = \frac{|0+1|}{0^+(0^+ +1)} = \frac{1}{0^+}$$

$$= \boxed{+\infty}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|0^- +1|}{0^-(0^- +1)} = \frac{1}{0^-} = \boxed{-\infty.}$$

For $x_2 = -1$, the sided limits are:

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{|x+1|}{x(x+1)}$$

$$= \lim_{x\to -1^+} \frac{\cancel{x+1}}{x\cancel{(x+1)}} = \lim_{x\to -1^+} \frac{1}{x} = \boxed{-1.}$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{|x+1|}{x(x+1)}$$

$$= \lim_{x\to -1^-} \frac{-\cancel{(x+1)}}{x\cancel{(x+1)}} = \lim_{x\to -1^-} \frac{-1}{x} = \boxed{1}$$

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(b) $\begin{cases} f(0^-) = f(0^+) \\ f(2^-) = f(2^+) \end{cases}$

$\begin{cases} -1 = a\cdot 0 + b = b \\ 2a + b = \frac{2}{2} \end{cases} \Rightarrow \boxed{\begin{cases} a = 1 \\ b = -1 \end{cases}}$

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Q4 (a) $f'(x) = (x^3+2x+5)' \sin 2x$

$$+ (x^3+2x+5)(\sin 2x)'$$

$$= (3x^2+2)\sin 2x$$

$$+ (x^3+2x+5)(\cos 2x)\cdot(2x)'$$

$$= (3x^2+2)\sin 2x + 2(x^3+2x+5)\cos 2x$$

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(b) $f'(x) = \left(\ln^2(1+\cos^2 5x)\right)'$

$$= 2\ln(1+\cos^2 5x)\cdot\frac{1}{1+\cos^2 5x}\cdot(1+\cos^2 5x)'$$

$$= 2\ln(1+\cos^2 5x)\ \frac{2\cos 5x\cdot(\cos 5x)'}{1+\cos^2 5x}$$

$$= 2\ln(1+\cos^2 5x)\cdot\frac{2\cos 5x\cdot(-5\sin 5x)}{1+\cos^2 5x}$$

$$= \frac{-20\ln(1+\cos^2 5x)\cdot\cos 5x\sin 5x}{1+\cos^2 5x}$$

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(c) $f'(x) = \dfrac{(\arccos^3 x)'\sqrt{1-x^2} - \arccos^3 x\,(\sqrt{1-x^2})'}{(\sqrt{1-x^2})^2}$

$$= \frac{3\arccos^2 x\cdot(\arccos x)'\sqrt{1-x^2} - \arccos^3 x\cdot(\sqrt{1-x^2})'}{1-x^2}$$

$$= \frac{3\arccos^2 x \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right) \cdot \sqrt{1-x^2} - \arccos^3 x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (1-x^2)'}{1-x^2}$$

$$= \frac{-3\arccos^2 x - \arccos^3 x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)}{1-x^2}$$

$$= \frac{-3\arccos^2 x + \frac{x \cdot \arccos^3 x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{-3\sqrt{1-x^2}\arccos^2 x + x \cdot \arccos^3 x}{(1-x^2)^{3/2}}$$

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(d) $f(x) = (1+x^2)^{\arctan x}$

$$\ln f(x) = \ln (1+x^2)^{\arctan x}$$

$$= \arctan x \cdot \ln(1+x^2)$$

$$\frac{1}{f(x)} \cdot f'(x) = \left(\arctan x \cdot \ln(1+x^2)\right)'$$

$$= (\arctan x)' \ln(1+x^2)$$

$$+ \arctan x \cdot \left(\ln(1+x^2)\right)'$$

$$= \frac{1}{1+x^2} \ln(1+x^2)$$

$$+ \arctan x \cdot \frac{1}{1+x^2} \cdot (1+x^2)'$$

$$= \frac{\ln(1+x^2)}{1+x^2} + \arctan x \cdot \frac{2x}{1+x^2}$$

So, $$f'(x) = f(x)\left[\frac{\ln(1+x^2)}{1+x^2} + \frac{2x \cdot \arctan x}{1+x^2}\right]$$

$$= (1+x^2)^{\arctan x} \cdot \left[\frac{\ln(1+x^2) + 2x \cdot \arctan x}{1+x^2}\right]$$

Q5. (a) $$f'(x) = \left(\sqrt{x^2+24}\right)' = \left((x^2+24)^{\frac{1}{2}}\right)'$$

$$= \frac{1}{2}(x^2+24)^{\frac{1}{2}-1} \cdot (x^2+24)'$$

$$= \frac{1}{2}(x^2+24)^{-\frac{1}{2}}(2x+0)$$

$$= x(x^2+24)^{-\frac{1}{2}}$$

$$= \frac{x}{\sqrt{x^2+24}}$$

(b) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+24} - \sqrt{x^2+24}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{(x+h)^2+24} - \sqrt{x^2+24}\right)\left(\sqrt{(x+h)^2+24} + \sqrt{x^2+24}\right)}{h\left(\sqrt{(x+h)^2+24} + \sqrt{x^2+24}\right)}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{(x+h)^2+24}\right)^2 - \left(\sqrt{x^2+24}\right)^2}{h\left(\sqrt{(x+h)^2+24} + \sqrt{x^2+24}\right)}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 \cancel{+24} - (x^2 \cancel{+24})}{h\left(\sqrt{(x+h)^2+24} + \sqrt{x^2+24}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h\left(\sqrt{(x+h)^2+24} + \sqrt{x^2+24}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}\left(\sqrt{(x+h)^2+24} + \sqrt{x^2+24}\right)}$$

$$= \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+24} + \sqrt{x^2+24}}$$

$$= \frac{2x}{2\sqrt{x^2+24}} = \frac{x}{\sqrt{x^2+24}}$$

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(c) $$f(x) \approx f(1) + f'(1)(x-1)$$

$$= \sqrt{1^2+24} + \frac{1}{\sqrt{1^2+24}}\cdot(x-1)$$

$$= 5 + \frac{x-1}{5}$$

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(d) $\sqrt{28} = \sqrt{2^2+24} = f(2)$

$$\approx f(1) + f'(1)(2-1)$$

$$= 5 + \frac{2-1}{5} = \boxed{5.2}$$

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Q6. (a) $(0,-1)$ is on the curve

$$y^2 \cos x = x y^5 + y + 2$$

because:

the left-hand-side $= (-1)^2 \cos 0 = 1 \cdot 1 = 1$

the right-hand-side $= 0 \cdot (-1)^5 + (-1) + 2 = 1$

Taking the derivative with respect to x to the equation, we have:

$$(2y\cos x)\, y' + y^2(-\sin x)$$
$$= y^5 + x\, 5y^4 y' + y'$$

$$\Rightarrow (2y\cos x - 5xy^4 - 1)\, y' = y^5 + y^2 \sin x$$

$$\Rightarrow y' = \frac{y^5 + y^2 \sin x}{2y\cos x - 5xy^4 - 1}$$

So, the slope of the tangent line is:

$$m = y'\Big|_{(0,-1)} = \frac{y^5 + y^2\sin x}{2y\cos x - 5xy^4 - 1}\Big|_{(0,-1)} = \frac{1}{3}$$

the equation of the tangent line:

$$\frac{y-(-1)}{x-0} = m = \frac{1}{3}$$

$$\frac{y+1}{x} = \frac{1}{3} \Rightarrow \boxed{y = \frac{1}{3}x - 1}$$

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(b)

$$f(x) = \frac{12 + x^3}{2x^3} = 6x^{-3} + \frac{1}{2}$$

$$f'(x) = 6\cdot(-3)\,x^{-4} = -18\,x^{-4} = (-1)^1 6\cdot 3\,x^{-3-1}$$

$$f''(x) = 6\cdot(-3)\cdot(-4)\,x^{-5} = 6\cdot(-1)^2 3\cdot 4\,x^{-5}$$

$$f'''(x) = 6\,(-3)\cdot(-4)\cdot(-5)\,x^{-6}$$

$$= (-1)^3\, 6\cdot 3\cdot 4\cdot 5\,x^{-6}$$

$$= (-1)^3\, 6\cdot 3\cdot 4\cdot 5\,x^{-3-3}$$

$\vdots$

$$f^{(n)}(x) = (-1)^n\, 6\cdot 3\cdot 4\cdot \cdots (3+n-1)\,x^{-3-n}$$

$$= (-1)^n\cdot 6\cdot 3\cdot 4\cdot 5\cdots(n+2)\,x^{-3-n}$$

$$= (-1)^n\, 6\cdot \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(n+2)\,x^{-3-n}}{1\cdot 2}$$

$$= (-1)^n\, 3\,(n+2)!\,x^{-3-n}$$

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(c) $$\lim_{x\to 0} \frac{\sin^2 x}{x\ln(1+3x)}$$

$$= \lim_{x\to 0} \frac{(\sin^2 x)'}{(x\ln(1+3x))'}$$

$$= \lim_{x\to 0} \frac{2\sin x\cos x}{\ln(1+3x) + x \cdot \frac{1}{1+3x}(3x)'}$$

$$= \lim_{x\to 0} \frac{2\sin x\cos x}{\ln(1+3x) + \frac{3x}{1+3x}}$$

$$= \lim_{x\to 0} \frac{2(1+3x)\sin x\cos x}{(1+3x)\ln(1+3x)+3x}$$

$$= \lim_{x\to 0} \frac{\sin x}{(1+3x)\ln(1+3x)+3x} \cdot 2(1+3x)\cos x$$

$$= \lim_{x\to 0} \frac{\sin x}{(1+3x)\ln(1+3x)+3x} \cdot \lim_{x\to 0} 2(1+3x)\cos x$$

$$= \lim_{x\to 0} \frac{(\sin x)'}{((1+3x)\ln(1+3x)+3x)'} \cdot \lim_{x\to 0} 2(1+3x)\cos x$$

$$= \lim_{x\to 0} \frac{\cos x}{3\ln(1+3x) + \frac{(1+3x)}{1+3x}\cdot 3 + 3} \cdot 2(1+0)\cos 0$$

$$= \frac{\cos 0}{3\ln(1+0)+3+3} \cdot 2 \cdot \cos 0$$

$$= \frac{1}{3 \cdot 0 + 6} \cdot 2 \cdot 1 = \boxed{\frac{1}{3}}$$

Q7 (a) for $x=-1$, we have

$x'(t) = 5$ cm/sec and y satisfies:

$y^2 - 6(-1)^4 = y$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3$ or $y = -2$

because y is negative, so, we get $\boxed{y=-2}$

Taking the derivative to the equation $y^2 - 6x^4 = y$ with respect to t (time), we have:

$2y\,y'(t) - 6 \cdot 4x^3\,x'(t) = y'(t)$

so, $y'(t) = \dfrac{24x^3\,x'(t)}{2y-1}$

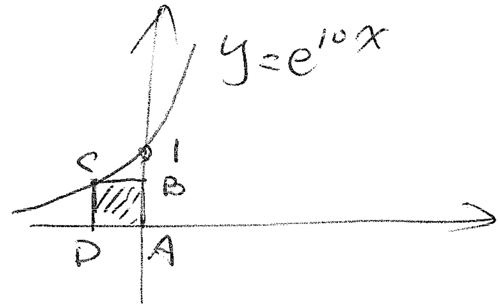
So $y'(t)\Big|_{(x,y)=(-1,-2)} = \dfrac{24\cdot(-1)^3\cdot 5}{2\cdot(-2)-1} = \boxed{24\ \text{cm/sec}}$

Thus, y-coordinate is increasing.

(b) Let point C be:

$C(x,y) = C(x, e^{10x})$

because it is on the curve $y=e^{10x}$



Area: $A(x) = |x|\cdot|y|$

$= |x|\cdot|e^{10x}|$ $\quad(\because x<0)$

$= -x\cdot e^{10x}$

Its critical number is:

$0 = A'(x) = -(xe^{10x})' = -e^{10x} - 10xe^{10x}$

$= -(1+10x)e^{10x}$

i.e. $1+10x=0 \Rightarrow \boxed{x=-\frac{1}{10}}$

Since $A''(x)\Big|_{x=-\frac{1}{10}} = -\left(10e^{10x} + 10(1+10x)e^{10x}\right)\Big|_{x=-\frac{1}{10}}$

$= -10e^{10x}(2+10x)\Big|_{x=-\frac{1}{10}}$

$= -10e^{-1} < 0$

applying the second derivative test, then $A(x)$ reaches the maximum at $x=-\frac{1}{10}$, and the maximum is:

$$A\left(-\tfrac{1}{10}\right) = -\left(-\tfrac{1}{10}\right)e^{10\cdot\left(-\frac{1}{10}\right)} = \tfrac{1}{10}e^{-1}$$

$$= \boxed{\frac{1}{10e}}$$

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Q8 (a) Domain $=\{x \mid x^2-4 \neq 0\}$

$=\{x \mid x \neq 2 \ \& \ x \neq -2\}$

$= \boxed{(-\infty,-2)\cup(-2,2)\cup(2,\infty)}$

- Symmetry: $f(-x) = \dfrac{(-x)^2}{(-x)^2-4} = \dfrac{x^2}{x^2-4} = f(x)$

  So $f(x)$ is $\boxed{\text{even}}$

- ~~V~~ertical asymptotes:

  $\boxed{x=2}$ $\boxed{x=-2}$

- horizontal asymptote:

  $$\lim_{x\to\pm\infty} \frac{x^2}{x^2-4} = 1$$

  So $\boxed{y=1}$

(b) $f'(x) = \left(\frac{x^2}{x^2-4}\right)' = \frac{(x^2)'(x^2-4) - x^2(x^2-4)'}{(x^2-4)^2}$ (15)

$$= \frac{2x(x^2-4) - x^2 \cdot 2x}{(x^2-4)^2}$$

$$= \frac{2x(x^2-4-x^2)}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}$$

- Critical number: $f'(x) = 0$

$\boxed{x=0}$

- For $x<0$, i.e. $x \in (-\infty, -2) \cup (-2, 0)$

$f'(x) = \frac{-8x}{(x^2-4)^2} > 0$

So, $f(x) \nearrow$

For $x>0$, i.e, $x \in (0, 2) \cup (2, \infty)$

$f'(x) = \frac{-8x}{(x^2-4)^2} < 0$

So, $f(x) \searrow$

Thus, $f(0) = 0$ is a local maximum

$(\nearrow \cdot \searrow)$

(c) $f''(x) = \left(\frac{-8x}{(x^2-4)^2}\right)' = -8 \cdot \frac{(x)'(x^2-4)^2 - x((x^2-4)^2)'}{(x^2-4)^4}$

$$= -8 \cdot \frac{(x^2-4)^2 - 4x^2(x^2-4)}{(x^2-4)^4} = \frac{8(3x^2+4)}{(x^2-4)^3}$$

It can be checked that:

$$f''(x) \neq 0 \quad \text{for all } x.$$

So, $f(x)$ doesn't have an inflection point.

It can also seen that:

$$3x^2+4>0$$

and $(x^2-4)^3 > 0$ for $x^2 > 4$, i.e. $x>2$ or $x<-2$

and $(x^2-4)^3 < 0$ for $x^2-4<0$, i.e. $-2<x<2$

Thus,

$$f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3} > 0 \quad \text{for } x>2 \text{ or } x<-2$$

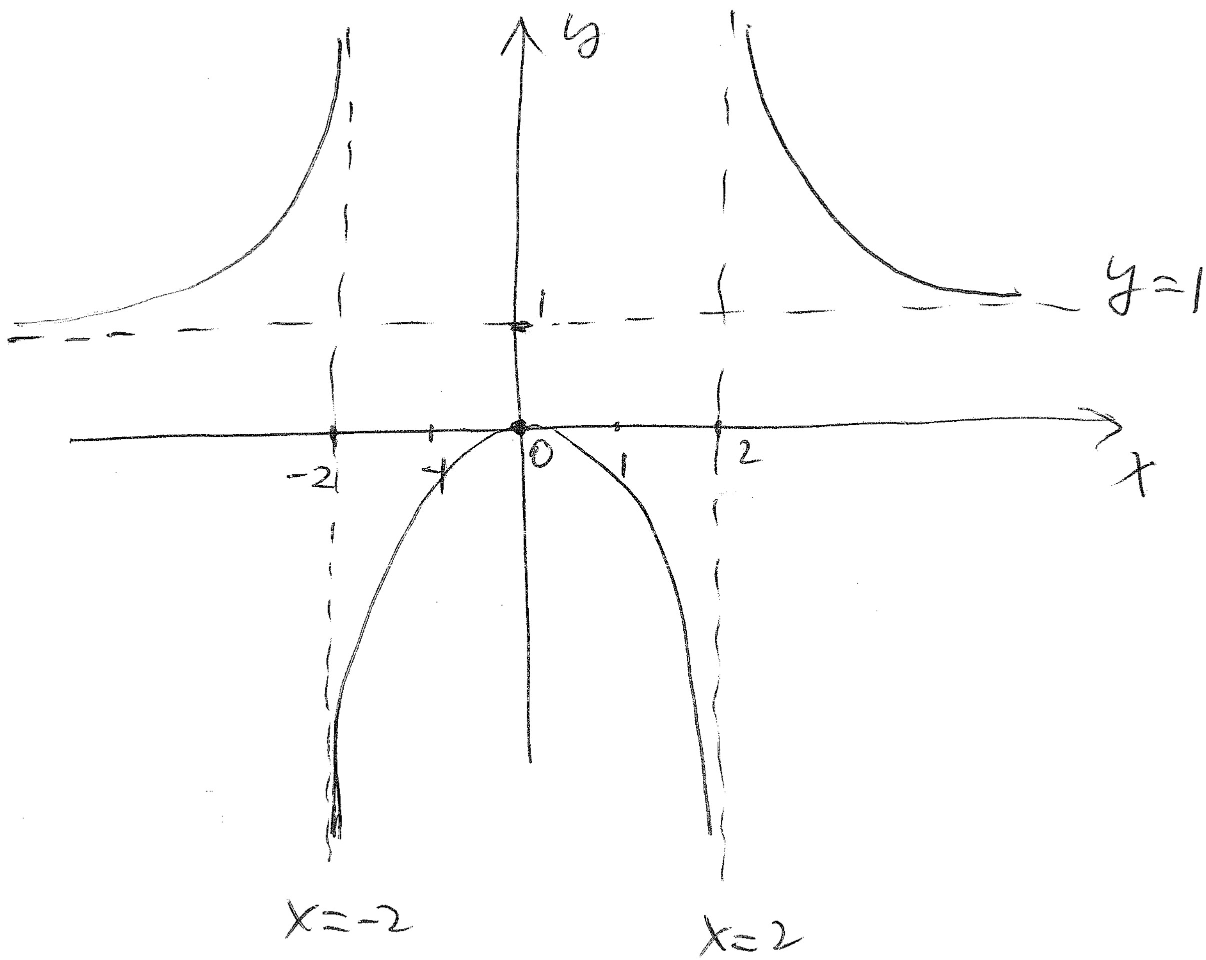
So, f: concave upward

and $$f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3} < 0 \quad \text{for } -2<x<2$$

So f: concave downward

(d) Table

x	$f'(x)$	f''	f
$(-\infty, -2)$	+	+	↗ + ∪ = ↗
$(-2, 0)$	+	−	↗ + ∩ = ↗
$x = 0$	0	−	local max
$(0, 2)$	−	−	↘ + ∩ = ↘
$(2, \infty)$	−	+	↘ + ∪ = ↘



Bonus Question

(a) For the equation $10x^3 + x = 10$

Let $f(x) = 10x^3 + x - 10$

Select: $a = \frac{1}{2}$, $b = 1$

and we have

$$f(\tfrac{1}{2}) = 10 \cdot (\tfrac{1}{2})^3 + \tfrac{1}{2} - 10 = -\frac{33}{4} < 0$$

$$f(1) = 10 \cdot 1^3 + 1 - 10 = 1 > 0$$

So $f(\frac{1}{2}) \cdot f(1) < 0$, by applying the Intermediate Value Theorem, the continuous function $f(x)$ has at least one zero between $\frac{1}{2}$ and 1. (i.e. one root)

(b) Since $f'(x) = (10x^3 + x - 10)' = 30x^2 + 1 > 0$

So $f(x)$ is increasing for $x \in (-\infty, \infty)$.

Thus, the root such that $f(x) = 0$ is unique.

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