MATH 334 and 354. Midterm 2 – Fall 2003/2004.

Solutions.

Problem 1. Neville's method is used to interpolate a function using points

$$x_0 = 0, \; x_1 = 1, \; x_2 = 2, \; x_3 = 3$$

and it is known that

$$P_{0,1}(1.5) = -2, \; P_{1,2}(1.5) = -3/2, \; P_{0,1,2,3}(1.5) = -31/16\,.$$

Find $P_{1,2,3}(1.5)$.

Solution. Neville's cone gives the value of Lagrange polynomial at a given point $x^* = 1.5$. Given $P_{0,1}(x^*)$ and $P_{1,2}(x^*)$ we can find by Neville's formula

$$P_{0,1,2}(x^*) = \frac{(x^* - x_0)\,P_{1,2}(x^*) - (x^* - x_2)\,P_{0,1}(x^*)}{x_2 - x_0}$$

$$= \frac{(1.5 - 0)\,(-3/2) - (1.5 - 2)\,(-2)}{2 - 0} = \frac{-13}{8} = -1.625$$

so, we conclude that

$$P_{0,1,2}(1.5) = \frac{-13}{8} = -1.625\,.$$

Again by using Neville's formula we have

$$-\frac{31}{16} = P_{0,1,2,3}(x^*) = \frac{(x^* - x_0)\,P_{1,2,3}(x^*) - (x^* - x_3)\,P_{0,1,2}}{x_3 - x_0}$$

$$= \frac{1}{3}\left((1.5 - 0)\,P_{1,2,3}(1.5) - (1.5 - 3)\,\frac{-13}{18} \right)$$

and from here

$$\mathbf{P_{1,2,3}(1.5) = -\frac{9}{4} = -2.25}.$$

Problem 2. We interpolate a function f by using nodes

$$x_0 = -2, \; x_1 = -1, \; x_2 = 0, \; x_3 = 1, \; x_4 = 2, \; x_5 = 3$$

and Newton's divided difference formula method. The interpolation by using the nodes

$$\{\, x_0, \, x_1, \, x_2, \, x_3 \,\}$$

is

$$P_3(x) = 4 - 2(x+2) + \frac{1}{2}(x+2)(x+1) + \frac{1}{6}(x+2)(x+1)x\,.$$

(a) Find interpolation P_5 using all the points if

$$f[x_1, x_2, x_3, x_4] \;=\; f[x_2, x_3, x_4, x_5] \;=\; -\frac{1}{3}\,.$$

(b) Estimate the interpolation error at $x \,=\, 2.5$ if it is known that

$$\sup_{-2\,\le\,x\,\le\,3} |f^{(5)}(x)| \;\le\; 17 \quad \text{and} \qquad \sup_{-2\,\le\,x\,\le\,3} |f^{(6)}(x)| \;\le\; 25\,.$$

(c) If $f(2)$ changes by $+1$ and $f(3)$ changes by -3 how will the interpolation value of $f(2.5)$ change?

Solution. (a) The basic idea here is that if we construct Lagrange interpolating polynomial in Newton's divided difference form, then: If we like to add an additional interpolation node, we have to add only an additional term to the interpolating polynomial previously obtained. Here we have to add two more nodes x_4 and x_5 so we need $f[x_0, x_1, x_2, x_3, x_4]$ and $f[x_0, x_1, x_2, x_3, x_4, x_5]$. Note that this the main **numerical advantage** of Newton's divided difference formula compared to the Lagrange's formula of the unique interpolating polynomial. We have

$$f[x_0] = 4,\; f[x_0, x_1] = -2,\; f[x_0, x_1, x_2] = 1/2,$$

$$f[x_0, x_1, x_2, x_3] = 1/6\,.$$

By using the cone of divided differences we calculate

$$f[x_0, x_1, x_2, x_3, x_4] \;=\; -\frac{1}{8}$$

and

$$f[x_1, x_2, x_3, x_4, x_5] \;=\; 0\,.$$

Then applying the recursion formula for the divided differences

$$f[x_0, x_1, x_2, x_3, x_4, x_5] \;=\; \frac{1}{40}\,.$$

Hence, the interpolation polynomial of degree at most 5 based on 6 interpolation nodes (conditions) $x_0, x_1, x_2, x_3, x_4, x_5$ in Newton's form is:

$$P_5(x) \;=\; \left(4 \,-\, 2(x+2) \,+\, \frac{1}{2}(x+2)(x+1) \,+\, \frac{1}{6}(x+2)(x+1)x\right)$$

$$+\, \left(-\frac{1}{8}\,(x+2)(x+1)x(x-1) \,+\, \frac{1}{40}\,(x+2)(x+1)x(x-1)(x-2)\right)\,.$$

The second row of the above formula contains the two additional terms that we add to the interpolating polynomial $P_3(x)$, based on the interpolation nodes $\{x_0, x_1, x_2, x_3\}$, in order to obtain the interpolating polynomial $P_5(x)$, based on the interpolation nodes $\{x_0, x_1, x_2, x_3, x_4, x_5\}$..

(b) An upper bound for the interpolating error is

$$\frac{\sup_{-2 \le x \le 3} |\mathbf{f^{(6)}(x)}|}{\mathbf{6\,!}} \, \mathbf{(2.2^2 \,-\, 4)(2.5^2 \,-\, 1)(2.5)(3 - 2.5)}$$

$$\le \frac{\mathbf{25}}{\mathbf{6\,!}} \, \mathbf{(2.5^2 \,-\, 4)(2.5^2 \,-\, 1)(2.5)(3 - 2.5) \,=\, 0.5126953125}\,.$$

(c) **To solve (c) we observe that the interpolation nodes are equally spaced with a starting node** $x_0 = -2$**, and a step** $h = 1$ **so, we can use Newton's forward finite difference formula.** Denote by $\tilde{f}$ the changed **(perturbed)** function. Then

$$\tilde{f}(x_0) - f(x_0) \,=\, 0,\ \tilde{f}(x_1) - f(x_1) \,=\, 0,\ \tilde{f}(x_2) - f(x_2) \,=\, 0,$$

$$\tilde{f}(x_3) - f(x_3) \,=\, 0,\ \tilde{f}(x_4) - f(x_4) \,=\, 1,\ \tilde{f}(x_5) - f(x_5) \,=\, -3\,.$$

After we calculate **the cone of finite differences** for $\tilde{f} - f$ at the points $x_0.x_1, x_2, x_3, x_4, x_5$ and use it to construct Newton's forward finite difference formula:

$$N(\tilde{f} - f\,; x_0 + th) \,=\, \Delta^4\,(\tilde{f} - f)_0 \binom{t}{4} \,+\, \Delta^5\,(\tilde{f} - f)_0 \binom{t}{5}$$

where

$$\Delta^4\, f_0 \,=\, 1 \quad \text{and} \quad \Delta^5\, f_0 \,=\, -8\,.$$

Here $h = 1$, $x_0 \,=\, -2$ and to $x* = 2.5$ corresponds $t^* \,=\, 4.5$:

$$x_0 \,+\, t^*\, h \,=\, 2.5 \,=\, x^* \quad \to \quad t^* \,=\, 4.5\,.$$

Finally, ($t^* \,=\, 2.5$) by using the linear property of the interpolating polynomial:

$$N(\tilde{f}; 2.5) \,-\, N(f; 2.5) \,=\, N(\tilde{f} - f; 2.5) \,=\, N(\tilde{f} - f; x_0 + t^*\, h)$$

$$= \, N(\tilde{f} - f; -2 \,+\, 4.5 \,\cdot\, 1) \,=\, 1 \binom{t^*}{4} \,-\, 8 \binom{t^*}{5}$$

$$= \frac{t^*(t^* - 1)(t^* - 2)(t^* - 3)}{4!} \,-\, 8\, \frac{t^*(t^* - 1)(t^* - 2)(t^* - 3)(t^* - 4)}{5!}$$

$$= \, 2.4609375 \,-\, 1.96875 \,=\, \mathbf{0.4921875}\,.$$

Second solution by using Newton's divided difference formula and the cone of divided difference for $\tilde{f} - f$**.** We have

$$(\tilde{f} - f)[x_0] = 0,\ (\tilde{f} - f)[x_0, x_1] = 0,\ (\tilde{f} - f)[x_0, x_1, x_2] = 0,$$

$$(\tilde{f} - f)[x_0, x_1, x_2, x_3] = 0,\ \ (\tilde{f} - f)[x_0, x_1, x_2, x_3, x_4] \,=\, \frac{1}{24},$$

$$(\tilde{f} - f)[x_0, x_1, x_2, x_3, x_4, x_5] \,=\, -\frac{1}{15}\,.$$

Hence, $(x^* = 2.5)$

$$N_5(\tilde{f};2.5) - N_5(f;2.5) = N(\tilde{f}-f;2.5)$$

$$= \frac{1}{24}(x^*+2)(x^*+1)x^*(x^*-1) - \frac{1}{15}(x^*+2)(x^*+1)x^*(x^*-1)(x^*-2)$$

$$= \frac{1}{24}(2.5+2)(2.5+1)2.5(2.5-1) - \frac{1}{15}(2.5+2)(2.5+1)2.5(2.5-1)(2.5-2)$$

$$= 2.4609375 - 1.96875 = \mathbf{0.4921875}\,.$$

Note that in the case of equally spaced nodes Newton's finite difference formulas are much more simpler than Newton's divided difference formula. The finite difference cone is easier to be calculated than the corresponding divided difference cone!

Problem 3. Solve the equation $x^3 + 3\,x = 5$ using Neville's method in inverse interpolation based on the data:

x	-1	0	1	2
$x^3 + 3x$	-4	0	4	14

Solution. We start with organizing the data for the inverse function of the function $y(x) = x^5 + 3x - 5$. The first derivative

$$y^{'}(x) = 3\,x^2 + 3 > 0 \quad (\,x \in (-\infty,\infty))$$

so f is one-to-one, i.e., the inverse function is well defined. Also, $y(1) = -1 < 0$ and $y(2) = 9 > 0$ so $f(x) = 0$ has precisely one root on $(-\infty,\infty)$ by the Intermediate Value Theorem and the fact that $y(x)$ is increasing function. We have:

$y = x^3 + 3x - 5$	-9	-5	-1	9
$x(y)$	-1	0	1	2

and the equation $x^* = x(0)$ is equivalent to $y(x^*) = 0$ so, to get an approximation for the unique root of $y(x) = 0$ we have to calculate the interpolating polynomial for the inverse function $x(y)$ at the point $y = 0$. Then

$$P_0(0) = -1,\; P_1(0) = 0,\; P_2(0) = 1,\; P_3 = 2\,.$$

$$P_{0,1}(0) = \frac{-x_0\,P_1 + x_1\,P_0}{x_1 - x_0} = \frac{5}{4}\,.$$

$$P_{1,2}(0) = \frac{-x_1\,P_2 + x_2\,P_1}{x_2 - x_1} = \frac{5}{4}\,.$$

$$P_{2,3}(0) \,=\, \frac{-x_2\,P_3 \,+\, x_3\,P_2}{x_3 - x_2} \,=\, \frac{11}{10}\,.$$

$$P_{0,1,2}(0) \,=\, \frac{-x_0\,P_{1,2} \,+\, x_2\,P_{0,1}}{x_2 - x_0} \,=\, \frac{9(5/4) - 1(5/4)}{-1+9} \,=\, \frac{5}{4}.$$

$$P_{1,2,3}(0) \,=\, \frac{-x_1\,P_{2,3} \,+\, x_3\,P_{1,2}}{x_3 - x_1} \,=\, \frac{5(11/10) + 9(5/4)}{14} \,=\, \frac{67}{56}.$$

$$P_{0,1,2,3}(0) \,=\, \frac{-x_0\,P_{1,2,3} \,+\, x_3\,P_{0,1,2}}{x_3 - x_0} \,=\, \frac{9(67/56) + 9(5/4)}{18} \,=\, \frac{137}{112}.$$

Hence, an approximation by inverse interpolation of the unique root x^* of the equation $y(x) = 0$ **based on the given discrete data** is

$$\tilde{\mathbf{x}} \,=\, \frac{\mathbf{137}}{\mathbf{112}} \,=\, \mathbf{1.223214} \,\approx\, \mathbf{x}^*\,.$$

Problem 4. Hermite interpolating polynomial is constructed using Newton's divided difference table. We know that $f^{'}(0) = -1$ and $f^{'}(1) = 1$. Some entries from the table has been erased:

a) Find all values replaced by ?#.

b) Estimate the interpolation error at $x = 1.5$ if it is known that

$$\sup_{x\,\in\,[0,2]} |f^{(6)}(x)| \,\leq\, 7 \quad \text{and} \quad \sup_{x\,\in\,[0,2]} |f^{(7)}(x)| \,\leq\, 13\,.$$

Solution. a) Let $x_0 = 0$, $x_1 = 1$, $x_2 = 2$. This is Hermite interpolation based on 6 interpolation data:

$$f(x_0), f^{'}(x_0), f(x_1),\, f^{'}(x_1),\, f(x_2),\, f^{'}(x_2)$$

so, the Hermite interpolating polynomial is of degree at most 5. From the cone of divided differences we have:

$$f[x_0, x_1] \,=\, -1\, f[x_1, x_1] \,=\, f^{'}(1) \,=\, 1.$$

Also

$$\mathbf{f}[\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_2}] \,=\, \frac{\mathbf{f}^{'}(\mathbf{x_2}) \,-\, \mathbf{f}[\mathbf{x_1}, \mathbf{x_2}]}{\mathbf{x_2} - \mathbf{x_1}}$$

or

$$\mathbf{2} \,=\, \mathbf{f}^{'}(\mathbf{x_2}) - (-\mathbf{2}) \quad \Rightarrow \quad \mathbf{f}^{'}(\mathbf{x_2}) \,=\, \mathbf{f}[\mathbf{x_2}, \mathbf{x_2}] \,=\, \mathbf{0}\,.$$

$$f[x_0, x_0, x_1] \,=\, \frac{f[x_0, x_1] \,-\, f[x_0, x_0]}{x_1 \,-\, x_0} \,=\, (-1 - (-1))/(1 - 0) \,=\, 0\,.$$

$$f[x_0, x_0, x_1, x_1] \,=\, \frac{f[x_0, x_1, x_1] \,-\, f[x_0, x_0, x_1]}{x_1 - x_0} \,=\, (2 - 0)/(1 - 0) \,=\, 2$$

and

$$f[x_0,x_0,x_1,x_1,x_2,x_2] \;=\; \frac{f[x_0,x_1,x_1,x_2,x_2] \;-\; f[x_0,x_0,x_1,x_1,x_2]}{x_2 - x_0}$$

$$= \; (15/4 - (-9/4))/(2-0) \;=\; 3\,.$$

b) The Hermite interpolant is based 6 interpolation conditions and it is a polynomial of degree 5. So, according to the error analysis concerning Hermite Interpolation Formula, for $x^* = 1.5$ we have:

$$|f(x^*) \;-\; H_5(f;x^*)| \;=\; \frac{|f^{(6)}(\xi)|}{6!}(x^* - x_0)^2(x^* - x_1)^2(x^* - x_2)^2,$$

where the point ξ belongs to the interval $[0,2]$. Hence,

$$|\mathbf{f(1.5)} \;-\; \mathbf{H_5(f;1.5)}| \;\leq\; \frac{\mathbf{7}}{\mathbf{6!}}\mathbf{1.5^2\,0.5^4} \;= \mathbf{0.001367188}\,.$$

Problem 5. In determining a free cubic spline interpolant $S(x)$ on

$$a = x_0 \;<\; x_1 \;<\; x_2 \;<\cdots<\; x_n = b$$

which of the following must be true?

I. $S(x_j) \;=\; f(x_j)$ for each $j = 0,1,\ldots,n$.

II. $S_j^{'}(x_{j+1}) \;=\; S_{j+1}^{'}(x_j)$ for each $j = 0,1,\ldots,n-1$.

III. $S^{''}(x_j) \;=\; f^{''}(x_j)$ for each $j = 0,1,\ldots,n$.

Solution. Of course I must hold because the unique cubic free spline interpol ant interpolates f at the interpolation nodes x_j for $j = 0,\ldots,n$.

As we know, $S_j(x)$ denotes the free spline interpolant on the interval $[x_j, x_{j+1}]$ and $S_{j+1}(x)$ denotes the free spline interpolant on the interval $[x_{j+1}, x_{j+2}]$ and we must have

$$S_j^{'}(x_{j+1}) \;=\; S_{j+1}^{'}(x_{j+1})$$

because the free cubic spline interpolant is two times continuously differentiable **but this not the condition given by II**. There is also one technical reason which shows that II is not a property of the unique cubic free spline interpolant. We have no part denoted by $S_n(x)$. As we mentioned the free cubic spline interpolant on $n+1$ nodes consists of n cubic polynomials $S_j(x)$ for $[x_j, x_{j+1}]$) and $j = 0,1,\ldots,n-1$. So, there is no $S_n(x)$.

The third III is also not a property of the unique free cubic spline interpolant: The unique free cubic spline interpolant on $n+1$ nodes consists of n cubic polynomials ($S_j(x)$ for $[x_j, x_{j+1}]$) and $j = 0,1,\ldots,n-1$ which are smoothly connected up to second derivative so,

$$S_j^{''}(x_{j+1}) \;=\; S_{j+1}^{''}(x_{j+1}) \quad j = 0,1,\ldots,n-2$$

and in general the second derivative of S is not equal to the second derivative of f. Moreover, the function f could be only continuous (moreover only bounded) on $[a, b]$. In other words the unique free cubic spline interpolant needs and uses only the functional value of f on $[a, b]$. Moreover, at the end points $x_0 = a$ and $x_n = b$ we have free boundary conditions:

$$S^{''}(a) \,=\, S^{''}(b) \,=\, 0$$

which evidently are not the conditions given by III.

So, the correct answer is E) I only.

Problem 6. Given a function $f(x) \,=\, a\,x^2 \,+\, b\,x \,+\, c$ and interpolation nodes

$$0 = x_0 \,<\, x_1 \,<\, \cdots \,<\, x_{n-1} \,<\, x_n \,=\, 1\,.$$

a) For which a, b, c the unique clamped cubic interpolant of f based on the data $(x_i, f(x_i))$ $i = 0, 1, \ldots, n$ coincides with f.

b) For which a, b, c the unique natural (free) cubic spline interpolant of f based on the data $(x_i, f(x_i))$, $i = 0, 1, \ldots, n$ coincides with f.

Solution. First we observe that the data

$$\mathbf{(x_i, f(x_i))\;\; i = 0, 1, \ldots, n \quad (1)}$$

belong to the polynomial of degree at most 2, or in other words to the graph of the function $\mathbf{f(x) = a\,x^2 \,+\, b\,x \,+\, c}$.

a) **Then the unique clamped cubic spline interpolant $\mathbf{S_{cl,f}(x)}$ for (to) $\mathbf{f(x) = a\,x^2 \,+\, b\,x \,+ c}$ based on the interpolating data (1) coincides with the function (is the function by itself) for each real numbers $\mathbf{a,\, b,\, c}$. So, the answer is: For each $\mathbf{a,\; b,\; c}$ real and $\mathbf{f(x) = a\,x^2 \,+\, b\,x \,+ c}$ we have**

$$\mathbf{S_{cl,f}(x) \,=\, f(x)\,.}$$

To show this we have only to prove that each polynomial of degree two is a cubic spline. However this is trivial because: In each interval $\mathbf{[x_i, x_{i+1}]}$ the function $\mathbf{f(x) = a\,x^2 \,+\, b\,x \,+\, c}$ is a cubic (polynomial of degree at most 3). Also, being infinitely many time differentiable, in particular, it has second continuous derivative on $\mathbf{[x_0, x_n]}$. And what is left to check that the cubic spline $\mathbf{S_{cl,f}(x)}$ defined by

$$\mathbf{Scl, f(x) \,=\, f(x)\;\; x \,\in\, [x_0, x_n] \quad (2)}$$

satisfies the boundary conditions:

$$\mathbf{S^{'}(x_0) \,=\, f^{'}(x_0),\;\; S^{'}(x_n) \,=\, f^{'}(x_n)\,.}$$

However this is trivially obtained by differentiating both sides of the equality (2).

b) Analogously, we have that the function

$$\mathbf{f(x) = a\,x^2 + b\,x + c}$$

is a CUBIC SPLINE by itself. So, it is also a spline interpolant for (to) f because it is passing true the points

$$\mathbf{(x_i, f(x_i)) \;\; i = 0, 1, \ldots, n}$$

In order

$$\mathbf{S_{N,f}(x) = f(x)}$$

to be the UNIQUE FREE (NATURAL) SPLINE INTERPOLANT to (for) f we must have that

$$\mathbf{S''_{N,f}(x_0) = 0 \quad \text{and} \quad S''_{N,f}(x_n) = 0}$$

which is equivalent to

$$\mathbf{f''(x_0) = 2a = f''(x_n) = 0\,.}$$

So, the answer is: Only for $a = 0$ and each real numbers b and c, the UNIQUE NATURAL CUBIC SPLINE INTERPOLANT TO

$$\mathbf{f(x) = a\,x^2 + b\,x + c = b\,x + c}$$

coincides with (is the function) f.

Please, try to understand the above solution. It will give you more information and more confidence to WHAT IS A SPLINE INTERPOLANT.

Problem 7. Construct linear and quadratic least square approximation for the given data:

x_i	3001	3002	3003	3004	3005
y_i	-2	0	1	-1	0

Solution. Let $x_i = 3001 + i$, $i = 0, 1, 2, 3, 4$. In order to minimize computational complexity we shall look for the solution in the form of a Taylor's basis about the point a:

$$A\,(x - a) + B\,,$$

where

$$a = \frac{x_0 + x_1 + x_2 + x_3 + x_4}{5} = 3003\,.$$

For convenience we write a new data table:

$x_i - a$	-2	-1	0	1	2
y_i	-2	0	1	-1	0

and calculate:

$$\sum_{i=0}^{5} (x_i - a) = 0, \ \sum_{i=0}^{5} y_i = -2, \ \sum_{i=0}^{5} (x_i - a)^2 = 10,$$

$$\sum_{i=0}^{5} (x_i - a)\, y_i = 3$$

and the system to determine the coefficient A and B is:

$$0\,A + 5\,B = -2$$

$$10\,A + 0\,B = 3$$

and from here

$$B = \frac{-2}{5} \quad \text{and} \quad A = \frac{3}{10}\,.$$

Hence, the unique discrete least squares approximant to y of degree 1 is:

$$\frac{3}{10}\,(x - 3003) - \frac{2}{5}\,.$$

Analogously, looking for the unique discrete least square approximant to y by polynomials of degree 2 we use Taylor's representation about the point a:

$$C\,(x-a)^2 + B\,(x-a) + A\,.$$

We calculate

$$\sum_{i=0}^{5} (x_i - a)^3 = 0, \ \sum_{i=0}^{5} (x_i - a)^4 = 34, \ \sum_{i=0}^{5} (x_i - a)^2\, y_i = -9$$

and the linear system to determine the unique discrete least squares approximant of degree 2 is:

$$10\,C + 0\,B + 5\,A = -2$$

$$0\,C + 10\,B + 0\,A = 3$$

$$34\,C + 0\,B + 10\,A = -9\,.$$

Hence,

$$C = -\frac{5}{14},\ B = \frac{3}{10},\ A = \frac{11}{35}$$

and the unique least squares approximant has the form:

$$-\frac{5}{14}\,(x - 3003)^2 + \frac{3}{10}\,(x - 3003) + \frac{11}{35}\,.$$