Review 1

Final Exam

Mathematics 201-103
Calculus I Commerce

Name: ____________________________________

REMARKS:

- The exam has 14 pages and 9 questions.
- Calculators are permitted.
- A sheet of formulas is included at the end of the exam.
- Read every question carefully and show your work unless are told otherwise.
- If you need more space to write your solutions, you may use the back of each page. If you do this, indicate to the marker that your solution is continued on the other side of the page.

Reserved for markers

1	2	3	4	5	6	7	8	9	Total
—	—	—	—	—	—	—	—	—	—
24	24	6	6	6	6	9	9	10	**100**

Question 1. (24 marks)
Find the following limits if they exist. If they do not exist, briefly indicate why not and determine whether they go to $-\infty$, ∞, or neither.

(a) $\lim_{x\to 4}\frac{\sqrt{x}-2}{2x-8}$ $= \lim_{x\to 4}\frac{\sqrt{x}-2}{2(x-4)} = \lim_{x\to 4}\frac{\sqrt{x}-2}{2((\sqrt{x})^2-2^2)}$

$= \lim_{x\to 4}\frac{\cancel{\sqrt{x}-2}}{2\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)} = \lim_{x\to 4}\frac{1}{2(\sqrt{x}+2)}$

$= \frac{1}{2(\sqrt{4}+2)} = \frac{1}{2(2+2)} = \boxed{\frac{1}{8}}$

(b) $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{x(x-1)}\right)$ $= \lim_{x\to 1}\left(\frac{x}{(x-1)\,x} - \frac{1}{x(x-1)}\right)$

$= \lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{(x-1)}\,x} = \lim_{x\to 1}\frac{1}{x} = \boxed{1}$

(c) $\lim_{x\to\infty} \frac{\sqrt{9x-3}}{4x^2+5x}$ $= \lim_{x\to\infty} \frac{\sqrt{9x-3}\,/x^2}{(4x^2+5x)/x^2} = \lim_{x\to\infty} \frac{\sqrt{\frac{9x-3}{x^4}}}{4+\frac{5}{x}}$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{9}{x^3}-\frac{3}{x^4}}}{4+\frac{5}{x}}$$

$$= \cancel{\lim_{x\to\infty}} \frac{\sqrt{0-0}}{4+0} = \boxed{0}$$

(d) $\lim_{x\to 3} \frac{|x-3|}{x-3}$ $= \begin{cases} \lim_{x\to 3^+} \frac{x-3}{x-3} = 1 \\ \lim_{x\to 3^-} \frac{-(x-3)}{x-3} = -1 \end{cases}$

So, the limit doesn't exist, because the left limit is different from the right limit.

Question 2 (24 marks)
Find the derivative for each of the following functions.

(a) $f(x) = 3x^4 - 4x\sqrt{x} + \frac{2}{x}$

$$f'(x) = \left(3x^4 - 4x^{\frac{3}{2}} + 2x^{-1}\right)'$$
$$= 12x^3 - 6x^{\frac{1}{2}} - 2x^{-2}$$

(b) $f(x) = x^3 e^{-x}$

$$f'(x) = (x^3 e^{-x})' = 3x^2 e^{-x} + x^3 e^{-x}(-x)'$$
$$= 3x^2 e^{-x} - x^3 e^{-x}$$

(c) $f(x) = (x+1)^2(3x-5)^3$

$$f'(x) = 2(x+1)(3x-5)^3 + (x+1)^2 3(3x-5)^2 \cdot (3x-5)'$$
$$= 2(x+1)(3x-5)^3 + 9(x+1)^2(3x-5)^2$$

Question 3 (6 points)

Let $f(x)=\dfrac{x-1}{x^2-2x-3}$. Find the values of x where this function is discontinuous. State the type of discontinuity as removable or other.

$$f(x)=\frac{x-1}{x^2-2x-3}=\frac{x-1}{(x-3)(x+1)}$$

There is NO definition for $f(x)$ at 3 and -1.

$$\lim_{x\to 3^-}\frac{x-1}{(x-3)(x+1)}=-\infty$$

$$\lim_{x\to 3^+}\frac{x-1}{(x-3)(x+1)}=\infty$$

$$\lim_{x\to -1^-}\frac{x-1}{(x-3)(x+1)}=-\infty$$

$$\lim_{x\to -1^+}\frac{x-1}{(x-3)(x+1)}=\infty$$

So, $f(x)$ is dicon continuous at $x=3$ and $x=-1$.

The type of discontinuity is infinity.

Question 5 (6 points)
Find the equation of the tangent line to the graph $y^2 - xy - 6 = 0$ at the point $P(1,3)$.

Taking derivative to $y^2 - xy - 6 = 0$ with respect to x
we have:

$$2yy' - y - xy' = 0$$

So $y' = \dfrac{y}{2y - x}$

at point $P(1.3)$, $y'\big|_{(x,y)=(1.3)} = \dfrac{3}{2\cdot 3 - 1} = \dfrac{3}{5}$
(Slope)

Thus, the tangent line is:

$$\frac{y-3}{x-1} = \frac{3}{5}$$

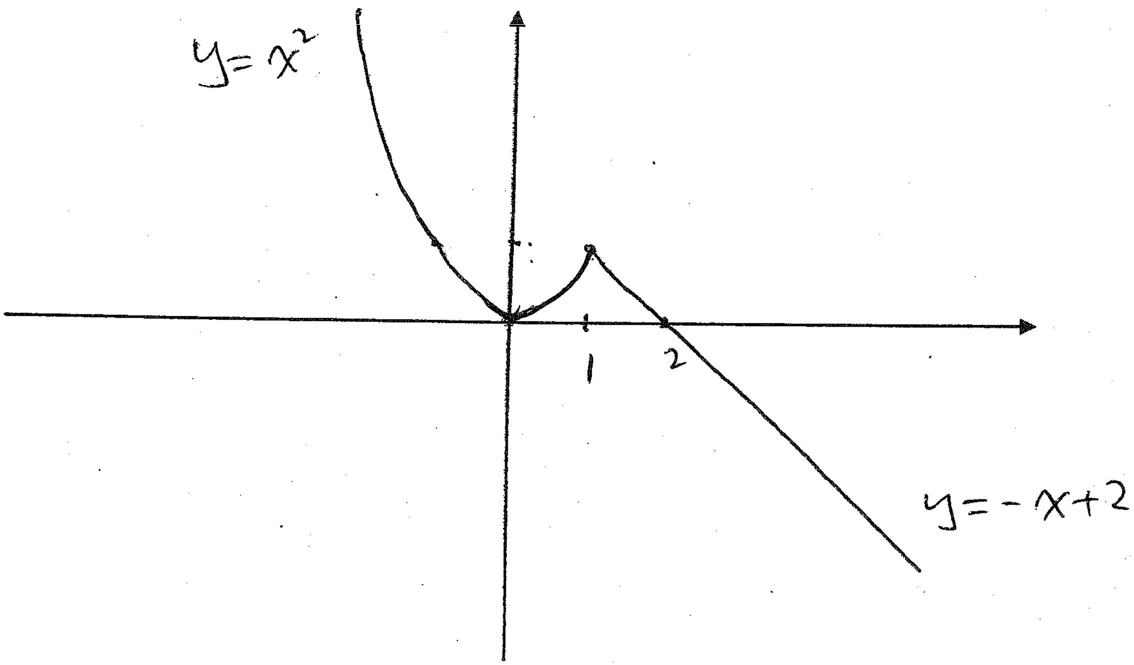
i.e. $y = \frac{3}{5}x + \frac{12}{5}$

Question 6 (6 points)

Let

$$f(x) = \begin{cases} x^2, & x \le 1 \\ -x+2, & x > 1 \end{cases}$$

(a) Sketch the graph of the function.



(b) For what value of x is this function NOT differentiable? Explain.

f(x) is not differentiable at $x = 1$,
because the point is a corner point (sharp point)

Question 7 (9 points)
In this exercise the goal is to sketch the graph of the function f. You are given the following information about the function.

- The domain of the function is $\mathbb{R}$
- $f(x) > 0$ when $x \neq 0$
- $f(0)=0,\quad f\left(\frac{1}{\sqrt{3}}\right)=f\left(-\frac{1}{\sqrt{3}}\right)=\frac{1}{4}$
- $f'(x)=\dfrac{2x}{(x^2+1)^2}$
- $f''(x)=\dfrac{2(1-3x^2)}{(x^2+1)^3}$
- $\lim_{x\to\infty} f(x)=1$
- $\lim_{x\to-\infty} f(x)=1$

In order to sketch the graph of the function, answer the following questions.
(a) Give the intervals where the function f is increasing and decreasing. Find the local maxima and minima (if any).

$f'(x)=\dfrac{2x}{(x^2+1)^2} \gtrless 0$ for $x \gtrless 0$.

So $f(x)$ is increasing in $(0,\infty)$ and decreasing in $(-\infty,0)$
$x=0$ is the critical number, and $f(0)$ is the local minimum, tested by the First Derivative Test

	f'	f
$(-\infty,0)$	$-$	$\searrow$
$x=0$		$\searrow,\nearrow$
$(0,\infty)$	$+$	$\nearrow$

(b) Give the intervals where the function is concave up and concave down, and find all points of inflection (if any).

$f''(x)=\dfrac{2(1-3x^2)}{(x^2+1)^3}=0 \Rightarrow x=\pm\frac{1}{\sqrt{3}}$ inflection Points

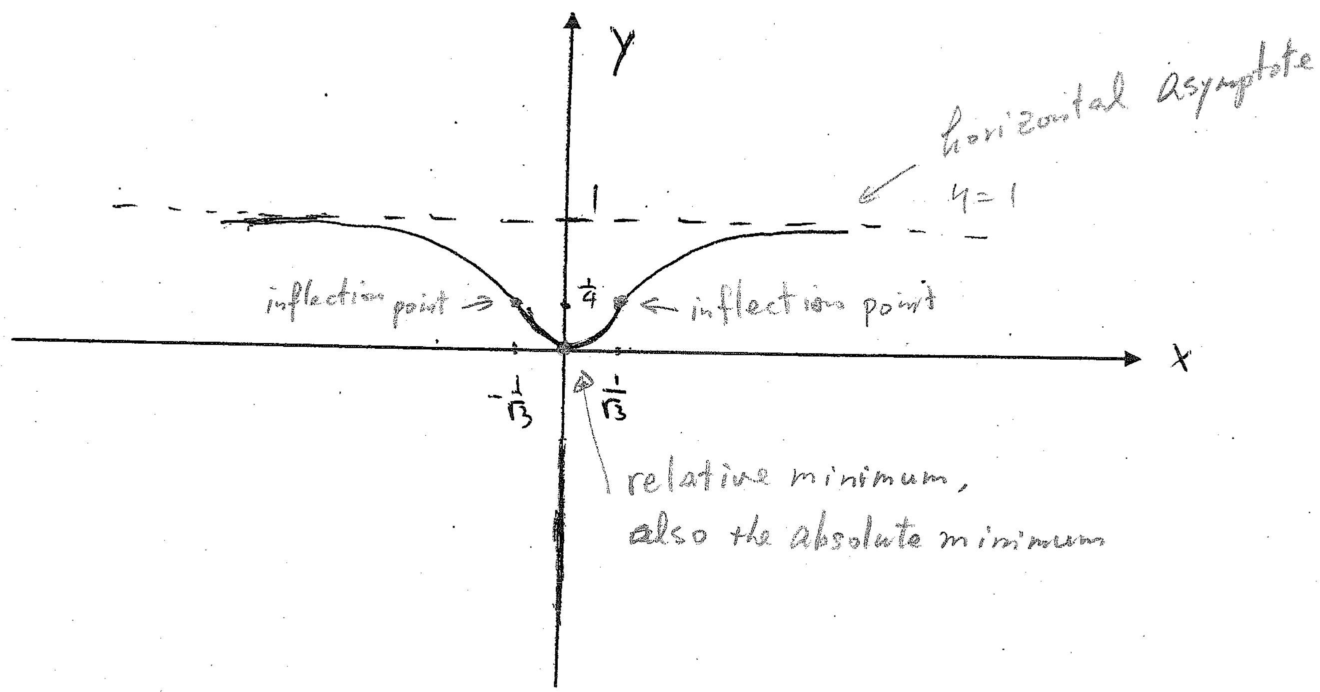
	f''	f
$(-\infty,-\frac{1}{\sqrt{3}})$	$-$	$\cap$
$x=-\frac{1}{\sqrt{3}}$		inflection Point
$(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$	$+$	$\cup$
$x=\frac{1}{\sqrt{3}}$		inflection Point
$(\frac{1}{\sqrt{3}},\infty)$	$-$	$\cap$

f: Concave upward in $(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$
Concave downward in $(-\infty,-\frac{1}{\sqrt{3}})\cup(\frac{1}{\sqrt{3}},\infty)$

(c) Give the equation of the vertical and horizontal asymptotes (if any).

No vertical asymptote. but one horizontal asymptote: $y=1$

(d) Gather all above information on the function f and sketch the graph of this function. On your graph indicate the equations of the asymptote(s), the coordinates of the local maxima and/or minima and the coordinates of the inflection points.



Question 8 (9 points)

The quantity of Sicard wristwatches demanded each month is related to the unit ptice by the demand equation: $p = \dfrac{50}{0.01x^2+1}$, $\quad (0 \le x \le 20)$.

(a) Find the equation of the revenue function.

$$R(x) = x\,p(x) = \frac{50x}{0.01x^2+1}$$

(b) Find the marginal revenue function.

$$R'(x) = \frac{(50x)'(0.01x^2+1) - 50x(0.01x^2+1)'}{(0.01x^2+1)^2}$$

$$= \frac{50(0.01x^2+1) - 50x \cdot 0.02x}{(0.01x^2+1)^2}$$

$$= \frac{50 - 0.5x^2}{(0.01x^2+1)^2}$$

(c) Compute $R'(2)$ and interpret your result.

$$R'(2) = \frac{50 - 0.5 \cdot 2^2}{(0.01 \cdot 2^2 + 1)^2} = \frac{48}{1.04^2} > 0$$

Question 9 (10 points)

A company manufactures and sells x videos video phones per week. The weekly price-demand and cost equations are $p = 500 - 0.5x$ and $C(x) = 20000 + 135x$.

(a) Find the equation of the profit function.

$$P(x) = xp(x) - C(x) = x(500 - 0.5x) - (20000 + 135x)$$
$$= 365x - 0.5x^2 - 20000$$

(b) How many phones should be produced to realize the maximum weekly profit?

$$P'(x) = 365 - 0.5 \cdot 2x = 365 - x = 0$$

$\boxed{x = 365}$ Critical number.

Since $P''(x) = (365 - x)' = -1 < 0$. $P(365)$ is a local maximum, also an absolute maximum. Therefore, to maximize the weekly profit, they need to produce 365 phones.

(c) How much should the company charge for the phones to realize the maximum profit?

$$p(365) = \cancel{20000 + 135 \cdot 365}$$
$$\cancel{= 69275}$$
$$= 500 - 0.5 \cdot 365 = \$317.5$$

(d) What is the maximum weekly profit?

$$P(365) = 365 \cdot 365 - 0.5 \cdot 365^2 - 20000$$
$$= 66512.5$$