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Traveling waves for time-delayed reaction diffusion equations with degenerate diffusion

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Abstract

This paper is concerned with time-delayed reaction–diffusion equations with degenerate diffusion. When the term for birth rate is a nonlocal integral with a heat kernel, the family of minimum wave speeds corresponding to all the degenerate diffusion coefficients is proved to admit a uniform positive infimum. However, when the term for birth rate is local, there is no positive infimum of all the minimum wave speeds. This difference indicates that the nonlocal effect plays a role as Laplacian such that a positive lower bound independent of the degenerate diffusion exists for the minimum wave speeds. The approach adopted for the proof is the monotone technique with the viscosity vanishing method. The degeneracy of diffusion for the equation causes us essential difficulty in the proof. A number of numerical simulations are also carried out at the end of the paper, which further numerically confirm our theoretical results.

Keywords: Traveling waves; Time-delay; Degenerate diffusion; Reaction–diffusion equations

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1. Introduction and preliminaries

In 1986, Metz and Diekmann [39] proposed the dynamical model of population with age-structure and diffusion:

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} = \widetilde{D}(a)\frac{\partial^2 v}{\partial^2 x} - \widetilde{d}(a)v, \qquad 0 < a < \tau, \tag{1.1}$$

where $v(t, x, a)$ is the population density of age a at location $x \in \Omega$ and time $t > 0$, τ is the mature time for the species, $\widetilde{D}(a)$ and $\widetilde{d}(a)$ are the diffusion rate and death rate of the population at age a. Let $u(t, x)$ be the population density of the mature at time t and point x

$$u(t, x) = \int_{\tau}^{\infty} v(t, x, a)da.$$

When the death rate d_m and diffusion rate D_m of mature population are constants, So et al. [47] derived the following reaction–diffusion equation (1.2) with nonlocal birth rate term from (1.1):

$$\frac{\partial u}{\partial t} = D_m \Delta u - d_m u + \int_{-\infty}^{+\infty} b(u(t-r, y)) f_\alpha(x-y)dy, \tag{1.2}$$

where $\alpha := \int_0^\tau D_{im}(a)da > 0$ represents the effect of the dispersal rate of immature population on the matured population and D_{im} is the diffusion rate of the immature population. f_α is the heat kernel in the form of

$$f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-y^2/4\alpha}, \qquad \int_{-\infty}^{\infty} f_\alpha(y)dy = 1. \tag{1.3}$$

Ecologically, since the diffusion phenomenon for the mature population at different time and different location are totally different, namely, D_m is variable and may be dependent on the population $u(t, x)$, so it is more practical and reasonable for us to consider the following time-delayed nonlinear diffusion equation:

$$\frac{\partial u}{\partial t} = \nabla(\varphi(u)\nabla u) - d(u) + \int_{-\infty}^{+\infty} b(u(t-r, y)) f_\alpha(x-y)dy. \tag{1.4}$$

Here, the diffusion of mature species is considered to be degenerate in the form of $-\nabla(\varphi(u)\nabla u)$ with $\varphi(u) = Dmu^{m-1}$ and $m > 1$, which is dependent on the population density due to the population pressure. See also the derivation and background stated later. Such a degenerate diffusion means that the smaller density, the slower spatial-diffusion, particularly, zero density implies non-diffusion. D represents the diffusivity of the mature population, and $d(u)$ is the death rate function in a general form.

When $\alpha \to 0^+$, namely, the immature population is almost non-mobile, by the property

$$\lim_{\alpha\to 0}\int_{-\infty}^{\infty} b(u(t-r,x-y))f_\alpha(y)dy = b(u(t-r,x)),$$

the nonlocal equation (1.4) is reduced to the local equation

$$\frac{\partial u}{\partial t} = D\Delta u^m - d(u) + b(u(t-r,x)). \tag{1.5}$$

For the time-delayed degenerate diffusion equations with nonlocality (1.4) and without nonlocality (1.5), their characters are essentially different. The local equation (1.5) is really degenerate in some sense for its space-diffusion, while the nonlocal equation (1.4) basically still behaves like a regular diffusion, because there is a good diffusion effect coming out from the nonlocality. In fact, for some "good" functions u, we formally have the following expansion for $\alpha \ll 1$ (see for example [31]),

$$\begin{aligned}
&\int_{\mathbb{R}^n} u(t,x-y)f_\alpha(y)dy - u(t,x)\\
&= \int_{\mathbb{R}^n} f_\alpha(y)[u(t,x-y)-u(t,x)]dy\\
&= \frac{1}{2}\int_{\mathbb{R}^n} f_\alpha(y)y^2dy\Delta u(t,x) + o(1)\Delta u(t,x)\int_{\mathbb{R}^n} f_\alpha(y)y^2dy\\
&\approx \frac{\alpha^2}{2}C_1\Delta u(t,x) + O(\alpha^3),
\end{aligned} \tag{1.6}$$

with $C_1 = \int_{\mathbb{R}^N} k(y)y_1^2dz$ and $k(z) = \frac{1}{\sqrt{4\pi}}e^{-y^2/4}$.

Since α depends on the diffusion of the immature population, an ecological explanation of the dispersal model with nonlocal term is that the diffusion rate of the immature population contributes to the effective dispersal rate of the mature population with the rate of $\frac{\alpha^2}{2}\int_{\mathbb{R}^N} k(y)y_1^2dz$ for small α. The mobility of the immature for many species indicates that a dynamical model with nonlocal birth rate function may be more appropriate.

Different from the previous study of the reaction–diffusion model for a single species with age structure, we treat the mature and immature dispersal separately for the heterogeneous populations. For the immature, we assume that the individuals randomly diffuse according to Fick's law. For the mature, we consider the nonlinear dispersal responding to overcrowding. When the individuals mature into adults, they may be mutually repulsive and become more active when they encounter more individuals. There is a nonlinear effect of population pressure upon dispersal [43]. The effect of the population pressure, which acts to enhance the dispersal of individuals as their density becomes high, is modeled expressing diffusion coefficient as functions of population density [18,41]. Incorporating the density-dependent diffusion coefficient $\varphi(u) = Dmu^{m-1}$ with $m > 1$ of the mature population, we obtain the model equation (1.4). In this case, large dispersal takes place in highly populated regions, but low mobility occurs in the regions of low density. This indicates that mature individuals can sense and react to the local population density, a type of positive feedback that increases with m [9].

In many realistic cases, the diffusion coefficient is not constant, which may be a consequence of the interaction between individuals [9,18,42,41,45]. Gurney et al. [18] and [46] proposed a model for animal dispersal in which they assumed that the diffusion coefficient depends linearly on population density $\varphi(u) \propto u$. Later, Gurtin and MacCamy [20] extended their model to a more general case in which $\varphi(u)$ is a function satisfies $\varphi'(0) = 0$, $\varphi'(u) > 0$ for $u > 0$. A special case of $\varphi(u)$ is $\varphi(u) = Dmu^{m-1}$ with $m > 1$. The density-dependent diffusion coefficient can be derived from a microscopic model in which individuals perform a biased random walk. Make a grid of mesh size h, and set $x = nh$. The derivation of the model begins with a master equation for a continuous-time and discrete-space random walk [49]

$$\frac{\partial u_i}{\partial t} = \mathcal{T}^+_{i-1} u_{i-1} + \mathcal{T}^-_{i+1} u_{i+1} - (\mathcal{T}^+_i + \mathcal{T}^-_i) u_i, \tag{1.7}$$

where $\mathcal{T}^\pm_i(\cdot)$ denote the transitional rates per unit time of a one-step jump to $x_{i\pm1}$ and u_i denotes the population density at x_i. Assume the transition rates depend on the population density at the point of departure alone with the following form

$$\mathcal{T}^\pm_i = \beta q(u_i), \tag{1.8}$$

where $q(u_i)$ is the jump probability which measures the tendency of species u leaving the site x_i and constant β is the intrinsic dispersal coefficient. Assume the jump probability increases with population density and satisfy the following property

$$q(0) = 0, \quad q(U_{max}) = 1,$$

where U_{max} is the carrying capacity at any site, namely, the jump probability is 1 when the population density exceeds maximum and it is zero when the species are absent. A natural choice for q is

$$q(u) = (u/U_{max})^{m-1}, \quad m > 1. \tag{1.9}$$

Expanding the right-hand side as a function of x to second order with respect to h, we obtain

$$\frac{\partial u}{\partial t} = \beta h^2 \frac{\partial^2}{\partial^2 x}\Big(q(u)u\Big) + O(h^4).$$

We assume that $q_h = \frac{k}{h^2} q$ for some scaling constant k. By taking the limit of $h \to 0$, we arrive at the following model

$$\frac{\partial u}{\partial t} = \beta_0 \frac{\partial}{\partial x}\Big(q(u)\frac{\partial u}{\partial x}\Big),$$

where $\beta_0 = k\beta$. Taking $q(u)$ as (1.9), we get

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(\Phi(u)\frac{\partial u}{\partial x}\Big),$$

where the density-dependent diffusion coefficient $\Phi(u) = D_0 u^{m-1}$, $D_0 = \frac{\alpha_0}{U_{max}^{m-1}}$ is a reference diffusivity. Thus, the effect of population pressure incorporate to the diffusivity [43]. When $m > 1$, the diffusion coefficient increases with population density. The relationship between biological dispersal and population density has been studied extensively [43,46]. Individuals tend to migrate from the nutrient-poor habitats of high density into regions of low density which have larger amounts of unconsumed food [41]. Several experiments have typically found an increase in diffusion coefficients as density is increased, such as population regulation of ant-lions [40], and the swarming of locusts [5,6]. In particular, Carl [7] found that arctic ground squirrels migrate from overcrowding regions into sparsely populated regions, even when the latter provide a less favorable habitat.

Our main purpose of this paper is to study the existence and non-existence of traveling wave solutions for (1.4). A traveling wave solution is a solution in the form of $u(t,x) = \phi(x + ct)$, where c is the wave speed.

From biological experiments, the functions $d(s)$ and $b(s)$ in (1.4) usually satisfy the following hypotheses [35]:

(H1) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $d(u_+) = b(u_+)$, and $d(s)$, $b(s) \in C^2[0, u_+]$;

(H2) $b'(0) > d'(0) \geq 0$, $0 \leq b'(u_+) < d'(u_+)$, and $d'(u_+)^2 > b'(0)b'(u_+)$;

(H3) For $0 \leq s \leq u_+$, $d'(s) \geq 0$, $b'(s) \geq 0$, $d''(s) \geq 0$, $b''(s) \leq 0$, but either $d''(s) > 0$ or $|b''(s)| > 0$.

Under the above hypothesis, both $u_- = 0$ and $u_+ > 0$ are constant equilibria of (1.1), and $u_- = 0$ is unstable and u_+ is stable for the spatially homogeneous equation associated with (1.1). Furthermore, both the birth rate function $b(u)$ and the death rate function $d(u)$ are nondecreasing, and $b(u)$ is concave downward and $d(u)$ is concave upward. These characters are summarized from the classical Fisher-KPP equation [1,14,56,62], see also a lots of evolution equation in ecology, for example, the well-studied nonlocal Nicholson's blowflies equation [12,13,28,34,48] with the death function $d(u) = \delta u$, the birth function

$$b_1(u) = pue^{-au^q}, \text{ or } b_2(u) = \frac{pu}{1 + au^q}, \quad p > 0, \quad q > 0, \quad a > 0;$$

and the age-structured population model [3,16,26,27] with

$$d(u) = \delta u^2, \text{ and } b(u) = pe^{-\gamma\tau}u, \quad p > 0, \quad \delta > 0, \quad \gamma > 0.$$

In the past decades, traveling waves for many systems of time-delayed reaction–diffusion equations arising from biological and physical applications have been studied intensively [8,10,11,15,17,23,32,50,52,53]. It was Schaaf [44] who first studied traveling wave solutions for the time delayed reaction–diffusion equation by maximum principles, the method of lower and upper solutions, and the phase plane techniques. The existence and non-existence of traveling waves were proved according to different size of c, namely, there is a critical wave speed $c^* > 0$, such that if $c > c^*$, there are only trivial waves; if $c < c^*$, there are nontrivial wave solutions and c^* is asymptotic speed of propagation. In [47], So et al. proved the existence of traveling waves for (1.2), which is a non-degenerate case of (1.4) describing the population distribution of single species with age-structure and spatial diffusion. Later on, Liang and Wu [29] studied theoretically the existence of the travelling waves for (1.2) three birth functions which have been widely used in the Nicholson's blowflies equation and showed the wave approximations numerically. Gourley and Kuang [26] studied the existence and the global stability of the age-structure single species diffusive delay population model

$$\frac{\partial u}{\partial t} = D\Delta u - \delta u^2 + pe^{-\gamma\tau} \int_{-\infty}^{+\infty} u(t-r, y) f_\alpha(x-y)dy. \tag{1.10}$$

The existence of traveling waves and the critical wave speed for this model were also established in [26,3,16,11,54]. In particular, when $\alpha \to 0^+$, the nonlocal equation (1.10) is reduced to the local time-delayed reaction–diffusion equation [19,36,33] for $\alpha = 0$. The first work related to the existence of traveling waves for this model was given by Al-Omari and Gourley [2] by the method of upper and lower solutions method. For the stability of critical and noncritical traveling waves, we refer to [22,23,27,33–38,54,56] and the references therein.

In realistic situation, a high population density results an increasing probability of individual leaving departure site [43]. So the degenerate diffusion term Δu^m with $m > 1$ is more ecological. Moreover, there are attractive phenomena due to the degeneracy of the diffusivity at $u = 0$. For example, a population which is initially confined to a bounded region spreads out at a finite speed, and may even remain confined for all time [4,20,58,59]. However, the case of time-delayed reaction diffusion equation with degenerate diffusion is more complex and the study of this type of equation is quite limited and incomplete. The bad effect of the degeneracy of spatial diffusion and time-delay causes us the essential difficulty for the existence of the traveling waves. Generally speaking, a traveling wave $\phi(x + ct)$ exists if $c \ge c^*$, while no traveling wave $\phi(x + ct)$ exists if $c < c^*$, the number c^* is called the minimum wave speed (or the critical wave speed). The critical wave speed c^* also coincides with the asymptotic speed of propagation [30,50,55], and it is very important in the study of biological invasions.

Our main observation here is that for the degenerate diffusion with nonlocal effect ($\alpha > 0$), the minimum wave speeds, which may depend on D, α and r, denoted by $c^*(D, \alpha, r)$, have a uniform positive infimum with respect to all $D > 0$; while this infimum is zero for the degenerate diffusion without nonlocal effect ($\alpha = 0$). This results may imply that the mobility of the immature contributes to the propagation of the whole population. When $\alpha > 0$, the limit of the minimum wave speed is a positive constant as D tends to zero, that is, the invasion speed of the population is not zero due to the diffusion of the immature population. When the effect of the immature dispersal is neglected ($\alpha = 0$), the limit of the minimum wave speed is zero as D tends to zero. In this case, there is no spreading when the mature population is immotile. We notice that, for such a degenerate diffusion case with a local birth rate (i.e., $m > 1$ and $\alpha = 0$), Huang et al. [21] first studied the existence of traveling waves by the method combining the portrait analysis with $r = 0$ and the perturbation analysis on small time-delay $0 < r \ll 1$, they also proved the L^1-stability of the wavefronts by the weighted energy method. But the traveling waves are proved to exist only for the case with the large wave speed $c \ge c_0 > 0$ and with the small time-delay $r \ll 1$ due to the restriction of the proof approach. Therefore, our results indicate that in the degenerate diffusion case the nonlocal convolution involving the heat kernel $f_\alpha(y)$ plays the smoothing effect. Since the nonlocal term can be approximately expanded as (1.6) for small α, we can expect that the nonlocal term plays the role of Laplacian, even though the original diffusion $-\Delta(u^m)$ is degenerate. As numerical reported in the last part of the paper, the numerical results by the direct iteration scheme for the nonlocal equation with degenerate diffusion yield a smooth traveling wave, while the direct iteration scheme for the local equation with degenerate diffusion gives some irregular oscillations, and we cannot get the traveling wave numerically.

Technically, we modify the iteration scheme by adding artificial viscosities to both sides of the local equation, then we obtain the expected traveling wave numerically. This also shows that the nonlocal effect is essential for the numerical scheme and the regularity of the traveling waves.

The rest of this paper is organized as follows. In section 2, we present the main results on the existence and nonexistence of traveling waves. In section 3, we show the existence of traveling wave solutions for both nonlocal and local cases. Section 4 is devoted to the proof of the nonexistence theorem of traveling wave solutions for the nonlocal time-delayed reaction diffusion equation with degenerate diffusion. Finally the numerical simulations of traveling waves are carried out in Section 5.

2. Main results

We consider the following initial-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u^m - d(u) + \displaystyle\int_{-\infty}^{+\infty} b(u(t-r,y)) f_\alpha(x-y)dy, & x \in \mathbb{R}, t > 0, \\ u(s,x) = u_0(s,x), & x \in \mathbb{R}, s \in [-r,0], \end{cases} \tag{2.1}$$

where $\alpha \geq 0$, $r \geq 0$, $m > 1$, $D > 0$, $u_0 \in L^2((-r,0)\times\Omega)$ for any compact set $\Omega \subset \mathbb{R}$. Since (2.1) is degenerate for $u = 0$, we employ the following definition of weak solutions.

Definition 2.1. A function $u \in L^2_{\text{loc}}((0,+\infty)\times\mathbb{R})$ is called a weak solution of (2.1) if $0 \leq u \leq u_+$, $\nabla u^m \in L^2_{\text{loc}}((0,+\infty)\times\mathbb{R})$, and for any $T > 0$ and $\psi \in C_0^\infty((-r,T)\times\mathbb{R})$

$$\begin{aligned} &-\int_0^T\int_{\mathbb{R}} u(t,x)\frac{\partial\psi}{\partial t}dxdt + D\int_0^T\int_{\mathbb{R}} \nabla u^m\cdot\nabla\psi dxdt + \int_0^T\int_{\mathbb{R}} d(u(t,x))\psi dxdt \\ &= \int_{\mathbb{R}} u_0(0,x)\psi(0,x)dx + \int_r^{\max\{T,r\}}\int_{\mathbb{R}}\int_{-\infty}^{+\infty} b(u(t-r,y))f_\alpha(x-y)\psi(x,t)dydxdt \\ &\quad + \int_0^{\min\{T,r\}}\int_{\mathbb{R}}\int_{-\infty}^{+\infty} b(u_0(t-r,y))f_\alpha(x-y)\psi(x,t)dydxdt. \end{aligned}$$

We are looking for the monotone increasing traveling wave solutions that connect the two equilibria $u_- = 0$ and $u_+ =: K$. Under the hypotheses (H1)–(H3), the birth function $b(u)$ is monotone increasing on $[u_1,u_2] = [0,K]$. Let $\phi(\xi)$, where $\xi = x + ct$ and $c > 0$, be the traveling wave solution of (2.1), we get (we write ξ as t for the sake of simplicity)

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)) + \displaystyle\int_{-\infty}^{+\infty} b(\phi(t-cr-y))f_\alpha(y)dy, \\ \phi(-\infty) = 0, \quad \phi(+\infty) = K. \end{cases} \tag{2.2}$$

We define upper, lower and traveling wave solutions for (2.2) as follows.

Definition 2.2. A function $\phi \in C(\mathbb{R};\mathbb{R})$ is called an upper (respectively lower) solution of (2.2) if $0 \le \phi \le K$, $\phi^m \in W^{1,2}_{\rm loc}$, $\lim_{t\to+\infty}\phi(t) = K$ and $\lim_{t\to-\infty}\phi(t) = 0$, and ϕ satisfies the following inequality

$$c\phi'(t) \ge (\le) D(\phi^m(t))'' - d(\phi(t)) + \int_{-\infty}^{+\infty} b(\phi(t-cr-y)) f_\alpha(y)dy,$$

$$\left(c\phi'(t) \le D(\phi^m(t))'' - d(\phi(t)) + \int_{-\infty}^{+\infty} b(\phi(t-cr-y)) f_\alpha(y)dy,\right)$$

in the sense of distributions. A function $\phi \in C(\mathbb{R};\mathbb{R})$ is called a monotone increasing traveling wave solution of (2.2) if ϕ is monotone increasing on $\mathbb{R}$, and ϕ is an upper solution as well as a lower solution of (2.2).

Since the diffusion in (2.2) is degenerate for $\phi = 0$, we define the following characteristic function for $c > 0$

$$\Delta_c(\lambda) = b'(0)e^{\alpha\lambda^2 - \lambda cr} - c\lambda - d'(0), \qquad \lambda > 0. \tag{2.3}$$

We will show in Lemma 3.2 that if $\alpha > 0$ then there exists a critical value $\hat{c} = \hat{c}(\alpha, r) > 0$ such that $\Delta_c(\lambda) = 0$ has real root if and only if $c \ge \hat{c}$. For $\alpha = 0$, we can see that $\Delta_c(\lambda) = 0$ has real root for all $c > 0$ and we define $\hat{c}(0, r) = 0$.

In order to show the important role of $\hat{c}(\alpha, r)$ playing in the analysis of traveling waves of (2.2), we need to compare it with the critical wave speed. For any given $m > 1$, $D > 0$, $\alpha \ge 0$, and $r \ge 0$, we define the critical wave speed $c^*(D, \alpha, r)$ for the degenerate diffusion equation (2.2) as follows

$$\begin{aligned} c^*(D,\alpha,r) := \inf\{c > 0;\ &(2.2) \text{ admits monotone increasing traveling wave solution} \\ &\text{satisfying the property (2.6) in Theorem 2.1}\}. \end{aligned} \tag{2.4}$$

For the classical Fisher-KPP equation $u_t - D\Delta u = u(1 - u/K)$, it is well known that $c^* = 2\sqrt{D}$, which vanishes as D tends to zero, and $\inf_{D>0} c^* = 0$.

The main purpose of this paper is to reveal that

$$\inf_{D>0} c^*(D,\alpha,r) = \hat{c}(\alpha,r) \begin{cases} > 0, & \alpha > 0, \\ = 0, & \alpha = 0. \end{cases} \tag{2.5}$$

That is, there exists a positive infimum of all the minimum wave speeds for all $D > 0$ if the degenerate diffusion involves nonlocal effect while this infimum is zero for the equation without nonlocal effect. This indicates the nonlocal effect plays a role as Laplace operator concerned with the traveling waves, which is coincident with the observation of formal expansion (1.6) and the phenomena in our simulation results.

Our main results are as follows. For the degenerate diffusion with nonlocal effect ($m > 1$ and $\alpha > 0$), there is a positive infimum for its minimum wave speed.

Theorem 2.1 *(Existence of non-critical traveling waves). Assume that $d(u)$ and $b(u)$ satisfy $(H1)$–$(H3)$, $\alpha > 0$, $m > 1$, $r \geq 0$. For any given $c > \hat{c}(\alpha, r)$, if D is sufficiently small or c is sufficiently large, then (2.2) admits at least one monotone increasing traveling wave solution $\phi(t)$ with speed c such that $0 < \phi(t) \leq K$ for all $t \in \mathbb{R}$. Moreover, there exist $\lambda > \lambda_1 > 0$ and a constant $C > 0$ such that*

$$|\phi(t) - Ke^{\lambda_1 t}| \leq Ce^{\lambda t}, \qquad t < 0, \tag{2.6}$$

where λ_1 is the least root of $\Delta_c(\lambda) = 0$ as shown in Lemma 3.2.

Theorem 2.2 *(Existence of critical traveling waves). Assume that $d(u)$ and $b(u)$ satisfy $(H1)$–$(H3)$, $\alpha > 0$, $m \geq 2$, $r \geq 0$. Then there exists a constant $C(m, \alpha, r) > 0$ such that for any $c \geq \hat{c}(\alpha, r)$, (2.2) admits at least one monotone increasing traveling wave solution $\phi(t)$ with speed c provided that $DK^{m-2} \leq C(m, \alpha, r)$. Moreover, $\phi(t)$ satisfies the property (2.6) in Theorem 2.1.*

Theorem 2.3 *(Non-existence of traveling waves). Assume that $d(u)$ and $b(u)$ satisfy $(H1) - (H3)$, $\alpha > 0$, $m > 1$, $r \geq 0$. For $c < \hat{c}(\alpha, r)$, (2.2) admits no monotone increasing traveling wave solution $\phi(t)$ with speed c that satisfies the property (2.6) in Theorem 2.1.*

For the non-degenerate diffusion case ($m = 1$), we have

Corollary 2.1 *(Minimum wave speed). Assume that $d(u)$ and $b(u)$ satisfy $(H1) - (H3)$, $\alpha \geq 0$, $m = 1$, $r \geq 0$. Then there exists $c^{**}(D, \alpha, r) > 0$ such that for $c > c^{**}(D, \alpha, r)$, (2.2) admits at least one monotone increasing traveling wave solution that satisfying*

$$|\phi(t) - Ke^{\lambda_1 t}| \leq Ce^{\lambda t}, \qquad t < 0,$$

for some constants $\lambda > \lambda_1 > 0$ and $C > 0$; while for $0 < c < c^{**}(D, \alpha, r)$, (2.2) admits no monotone increasing traveling wave solution of such kind.

For the degenerate diffusion without nonlocal effect ($m > 1$ and $\alpha = 0$), there is no positive infimum of minimum wave speed:

Theorem 2.4 *(No positive infimum of minimum wave speed for local but degenerate equation). Assume that $d(u)$ and $b(u)$ satisfy $(H1) - (H3)$, $\alpha = 0$, $m > 1$, $r \geq 0$. Then for any given $c > 0$, (2.2) admits at least one monotone traveling wave solution provided that D is sufficiently small; while for any $c \leq 0$, (2.2) admits no monotone increasing traveling wave solution that satisfying the property (2.6) in Theorem 2.1.*

Conversely, for any given $D > 0$, there exists a $r_0 > 0$ such that the critical wave speed $c^*(D, 0, r)$ defined in (2.4) is positive provided that $0 \leq r \leq r_0$.

Remark. According to Theorem 2.1 and Theorem 2.3, we see that for $m > 1$ and $\alpha > 0$,

$$\inf_{D>0} c^*(D, \alpha, r) = \hat{c}(\alpha, r) > 0.$$

Theorem 2.4 tells us that for $m > 1$ and $\alpha = 0$,

$$\inf_{D>0} c^*(D, 0, r) = \hat{c}(0, r) = 0.$$

Moreover, Theorem 2.2 implies that for $m \geq 2$ and $\alpha > 0$, there exists a constant $C(m, \alpha, r) > 0$ such that if $DK^{m-2} \leq C(m, \alpha, r)$ then

$$c^*(D, \alpha, r) = \hat{c}(\alpha, r).$$

For the degenerate diffusion without nonlocal effect, if the time delay is small, we find by Theorem 2.4 that

$$c^*(D, 0, r) > 0.$$

Further, we note that according to our proof, the critical traveling wave and non-critical traveling wave for the degenerate diffusion with nonlocal effect (in Theorem 2.1 and Theorem 2.2) and the non-degenerate diffusion equation (in Corollary 2.1) are all positive and smooth; the non-critical traveling wave for the degenerate diffusion without nonlocal effect (in Theorem 2.4) is positive and smooth while its critical traveling wave may be of semi-finite "sharp"-type (see [21]).

3. Existence of monotone traveling wave solutions

In this section, we employ the monotone iteration method to show the existence of monotone traveling wave solutions.

Compared with the linear diffusion case ($m = 1$), both the comparison principle and the solvability of degenerate elliptic problem ($m > 1$) are not obvious. The solvability of linear diffusion case can be easily showed by writing the explicit expression. We can not expect such kind of expressions due to the degenerate diffusion.

Let $H : C(\mathbb{R}; \mathbb{R}) \to C(\mathbb{R}; \mathbb{R})$ be defined as

$$H(\phi)(t) = \int_{-\infty}^{+\infty} b(\phi(t - cr - y)) f_\alpha(y) dy.$$

Under the hypotheses $(H1) - (H3)$, the birth function $b(u)$ is monotonically increasing on $[u_1, u_2] = [0, K]$. Then H is a monotone operator as showed by the following lemma (see Lemma 4.1 of [47]).

Lemma 3.1. *Assume that $b(u)$ satisfies $(H1) - (H3)$.*

(i) If $\phi(t) \geq 0$ for all $t \in \mathbb{R}$, then $H(\phi)(t) \geq 0$ for all $t \in \mathbb{R}$;

(ii) If $0 \leq \phi(t) \leq K$ and $\phi(t)$ is monotone increasing, then $H(\phi)(t)$ is monotone increasing;

(iii) If $0 \leq \phi(t) \leq \psi(t) \leq K$, then $H(\phi)(t) \leq H(\psi)(t)$.

For given $c > 0$, we define the following function

$$\Delta_c(\lambda) = b'(0)e^{\alpha\lambda^2 - \lambda cr} - c\lambda - d'(0), \qquad \lambda > 0.$$

We can verify that

Lemma 3.2. *Let $\alpha > 0$. There exists $\hat{c} = \hat{c}(\alpha, r) > 0$, such that*

(i) When $0 < c < \hat{c}$, then $\Delta_c(\lambda) > 0$ for all $\lambda \geq 0$;

(ii) When $c = \hat{c}$, then there exists $\lambda^* > 0$ such that $\Delta_{\hat{c}}(\lambda^*) = 0$ and $\Delta_{\hat{c}}(\lambda) > 0$ for all $\lambda \geq 0$, $\lambda \neq \lambda^*$;

(iii) When $c > \hat{c}$, then there exist $0 < \lambda_1 < \lambda_2$ such that $\Delta_c(\lambda_1) = \Delta_c(\lambda_2) = 0$, $\Delta_c(\lambda) > 0$ for $\lambda \in [0, \lambda_1) \cup (\lambda_2, +\infty)$ and $\Delta_c(\lambda) < 0$ for $\lambda \in (\lambda_1, \lambda_2)$.

Moreover, for any $\hat{m} > 1$ and $C > 0$, there exists a $c > \hat{c}$ such that $\Delta_c(\hat{m}\lambda_1(c)) < -C\lambda_1^2(c)$, where $\lambda_1(c)$ is the left root determined in (iii) above.

Proof. For fixed $c > 0$, let $f(\lambda) = \Delta_c(\lambda)$, $p = b'(0)$ and $\delta = d'(0)$ for simplicity in this proof. Then $f(0) = p - \delta > 0$ and

$$f'(\lambda) = pe^{\alpha\lambda^2 - \lambda cr}(2\alpha\lambda - cr) - c.$$

We note that $f'(\lambda) < 0$ for all $\lambda \in [0, cr/(2\alpha)]$. Both of the functions $2\alpha\lambda - cr$ and $e^{\alpha\lambda^2 - \lambda cr}$ are positive and strictly increasing with respect to λ for $\lambda \in (cr/(2\alpha), +\infty)$. Therefore, $f'(\lambda)$ is strictly increasing for $\lambda \in (cr/(2\alpha), +\infty)$, which means there exists $\lambda^* > 0$ such that $f'(\lambda^*) = 0$, $f'(\lambda) < 0$ for $\lambda \in (0, \lambda^*)$ and $f'(\lambda) > 0$ for $\lambda \in (\lambda^*, +\infty)$. We only need to show that $\min_{\lambda \geq 0} \Delta_c(\lambda)$ is strictly decreasing with respect to c. For fixed $\lambda \geq 0$,

$$\frac{\partial}{\partial c}\Delta_c(\lambda) = -pe^{\alpha\lambda^2 - \lambda cr}\lambda r - \lambda < 0.$$

It follows that

$$\Delta_{c_1}(\lambda) < \Delta_{c_2}(\lambda), \quad \lambda \geq 0,$$

for $c_1 > c_2$. Since both $\Delta_{c_1}(\lambda)$ and $\Delta_{c_2}(\lambda)$ attain there minimums, we can take minimum in the above inequality and get $\min_{\lambda \geq 0} \Delta_{c_1}(\lambda) < \min_{\lambda \geq 0} \Delta_{c_2}(\lambda)$ for $c_1 > c_2$.

Note that

$$\Delta_c(1) \leq pe^{\alpha} - c - \delta \leq 0,$$

if $c \geq c_0 = pe^{\alpha} - \delta$. That is, $\lambda_1(c) \leq 1$ if $c \geq c_0$. Now we only consider $0 \leq \lambda \leq \hat{m}$ and $c \geq c_0$. Then

$$\Delta_c'(\lambda) = pe^{\alpha\lambda^2 - \lambda cr}(2\alpha\lambda - cr) - c \leq 2\alpha\hat{m}pe^{\alpha\hat{m}^2} - c < 0,$$

if we further let $c > 2\alpha\hat{m}pe^{\alpha\hat{m}^2}$. Therefore,

$$
\begin{aligned}
\Delta_c(\hat{m}\lambda_1(c)) \leq & \Delta_c(\lambda_1(c)) + (2\alpha\hat{m}pe^{\alpha\hat{m}^2} - c)(\hat{m}\lambda_1(c) - \lambda_1(c)) \\
\leq & -(c - 2\alpha\hat{m}pe^{\alpha\hat{m}^2})(\hat{m}-1)\lambda_1(c) \\
< & -C\lambda_1(c) \leq -C\lambda_1^2(c),
\end{aligned}
$$

provided that

$$
c > \max\{2\alpha\hat{m}pe^{\alpha\hat{m}^2} + C/(\hat{m}-1), c_0\},
$$

since $\lambda_1(c) \leq 1$ for $c \geq c_0$. The proof is completed. □

First, we use the approximate Hohmgren's approach (see Theorem 6.5 in [51], Chapter 1.3 and 3.2 in [57]) to derive the comparison principle for degenerate elliptic problem on unbounded domain. To do so, we need to show the properties of solutions to the dual elliptic problem.

Lemma 3.3. *For any given $g(t) \geq 0$, $g(t) \in C_0^2(\mathbb{R})$, $\varepsilon \in (0,1)$, $\eta \in (0,1)$, and $0 \leq \beta(t) \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, $\liminf_{t\to+\infty} \beta(t) > 0$, $0 < \gamma(t) \in C^1(\overline{\mathbb{R}})$, $\gamma(t)$ is increasing for $t \leq t_0$ with some fixed $t_0 \in \mathbb{R}$, and $\liminf_{t\to+\infty} \gamma(t) > 0$, let $A > 1$ such that $\operatorname{supp} g \subset (-A, A)$ and $\psi_{\varepsilon,\eta}(t)$ be the solution of the following elliptic problem*

$$
\begin{cases}
-c\psi'(t) - D(\beta_\varepsilon(t) + \eta)\psi''(t) + \gamma_\varepsilon(t)\psi(t) = g(t), & t \in \mathbb{R}, \\
\psi(\pm\infty) = 0,
\end{cases}
\tag{3.1}
$$

where $\beta_\varepsilon(t)$ and $\gamma_\varepsilon(t)$ are the smooth approximations of $\beta(t)$ and $\gamma(t)$ such that $\beta(t) \leq \beta_\varepsilon(t) \leq \beta(t) + \varepsilon$ and $\gamma(t) \leq \gamma_\varepsilon(t) \leq \gamma(t) + \varepsilon$. Then there exist constants $k_1, k_2, C_1, C_2, C_3 > 0$ independent of ε, η and A such that $\psi_{\varepsilon,\eta}(t)$ has the following properties:

(i) $0 < \psi_{\varepsilon,\eta}(t) \leq \sup_{t\in(-A,A)} |g(t)|/\gamma(t)$;

(ii) $|\psi_{\varepsilon,\eta}(t)|, |\psi'_{\varepsilon,\eta}(t)|, (\beta_\varepsilon(t) + \eta)|\psi''_{\varepsilon,\eta}(t)| \leq C_1 e^{k_1 t}$ for all $t < -A-1$;

(iii) $|\psi_{\varepsilon,\eta}(t)|, |\psi'_{\varepsilon,\eta}(t)|, |\psi''_{\varepsilon,\eta}(t)| \leq C_2 e^{-k_2 t}$ for all $t > A+1$;

(iv)

$$
\int_{\mathbb{R}} (\beta_\varepsilon(t) + \eta)|\psi''_{\varepsilon,\eta}(t)|^2 dt \leq C_3.
$$

Proof. Since (3.1) is uniformly elliptic and $g \in C_0^2(\mathbb{R})$, the existence of $\psi_{\varepsilon,\eta}(t)$ is trivial and the maximum principle shows $0 < \psi_{\varepsilon,\eta}(t) \leq \sup|g|/\gamma$. We now prove the exponential decay of $\psi_{\varepsilon,\eta}(t)$ for $t < -A-1$ and $t > A+1$ respectively. We write $\psi_{\varepsilon,\eta}(t)$ as $\psi(t)$ for short. We may assume that $\beta(t) > 0$, $\gamma(t) > 0$ for $t > A$, and $\gamma(t)$ is increasing for $t \leq -A$. Otherwise, we can choose a larger A.

Consider (3.1) on $(-\infty, -A-1)$. Let $\psi(-A-1) = C_1$. Then

$$
\begin{cases}
-c\psi'(t) - D(\beta_\varepsilon(t) + \eta)\psi''(t) + \gamma_\varepsilon(t)\psi(t) = 0, & t \in (-\infty, -A-1), \\
\psi(-\infty) = 0, \quad \psi(-A-1) = C_1.
\end{cases}
\tag{3.2}
$$

We assert that $\psi'(t) \geq 0$ and $\psi''(t) \geq 0$ for all $t \in (-\infty, -A-1)$. We argue by contradiction. If there exists some $t_0 \in (-\infty, -A-1)$ such that $\psi'(t_0) < 0$, then according to (3.2) $\psi''(t_0) > 0$, which means there exists a left neighborhood of t_0 such that $\psi'(t) < \psi'(t_0) < 0$ on that interval. Continue this argument and we can show that $\psi'(t) < 0$ for all $t < t_0$ and contradicts to $\psi(-\infty) = 0$. If there exists some $t_0 \in (-\infty, -A-1)$ such that $\psi''(t_0) < 0$, then according to (3.2), $\psi'(t_0) > 0$, and there exists a left neighborhood (t_1, t_0) of t_0 such that $\psi'(t) > \psi'(t_0) > 0$ on that interval. We note that $\psi(t)$ is increasing and $\psi'(t)$ is decreasing on (t_1, t_0). It follows from (3.2) that

$$D(\beta_\varepsilon(t) + \eta)\psi''(t) = \gamma_\varepsilon(t)\psi(t) - c\psi'(t)$$

is increasing on (t_1, t_0). Therefore $D(\beta_\varepsilon(t) + \eta)\psi''(t) \leq D(\beta_\varepsilon(t_0) + \eta)\psi''(t_0) < 0$ and $\psi''(t) < -|(\beta_\varepsilon(t_0) + \eta)\psi''(t_0)|/(\sup|\beta| + 2)$ on (t_1, t_0). We can continue this argument to find a constant $C_0 > 0$ such that $\psi''(t) < -C_0$ for all $t < t_0$. This shows that there exists a $t_2 < t_0$ such that $\psi(t_2) < -1$, which contradicts to $\psi \geq 0$. Now that we have proved $\psi'' \geq 0$, we consider the following elliptic problem

$$\begin{cases} -c\widetilde{\psi}'(t) - D(\sup|\beta| + 2)\widetilde{\psi}''(t) = 0, \quad t \in (-\infty, -A-1), \\ \widetilde{\psi}(-\infty) = 0, \quad \widetilde{\psi}(-A-1) = \psi(-A-1). \end{cases} \tag{3.3}$$

Clearly,

$$-c\psi'(t) - D(\sup|\beta| + 2)\psi''(t) = D\big((\beta_\varepsilon(t) + \eta) - (\sup|\beta| + 2)\big)\psi''(t) - \gamma_\varepsilon(t)\psi(t) \leq 0.$$

Comparison principle of constant coefficients elliptic problems shows that

$$\psi(t) \leq \widetilde{\psi}(t), \quad t \in (-\infty, -A-1).$$

Since the coefficients of (3.3) are constants, we can employ the phase plane analysis to find $k_1, C_1 > 0$ independent of ε, η and A, such that $|\widetilde{\psi}(t)| \leq C_1 e^{k_1 t}$ for all $t < -A-1$. Note that $\psi' \geq 0$ and $\psi'' \geq 0$, (3.2) shows that $|\psi_{\varepsilon,\eta}(t)|, |\psi'_{\varepsilon,\eta}(t)|, (\beta_\varepsilon(t) + \eta)|\psi''_{\varepsilon,\eta}(t)| \leq C_1 e^{k_1 t}$ for all $t < -A-1$.

Consider (3.1) in $(A+1, +\infty)$. Let $\psi(A+1) = C_2$. Then

$$\begin{cases} -c\psi'(t) - D(\beta_\varepsilon(t) + \eta)\psi''(t) + \gamma_\varepsilon(t)\psi(t) = 0, \quad t \in (A+1, +\infty), \\ \psi(A+1) = C_2, \quad \psi(+\infty) = 0. \end{cases} \tag{3.4}$$

We assert that $\psi'(t) \leq 0$ and $\psi''(t) \geq 0$ for all $t \in (A+1, +\infty)$. We can argue by contradiction in a way similar to that on $(-\infty, -A-1)$. Here we omit it. Since $\liminf_{t\to+\infty} \beta(t) > 0$, we may assume that $\inf_{t>A+1} \beta(t) > 0$ and (3.4) is uniformly elliptic with its diffusion coefficient

$$0 < D \inf_{t>A+1} \beta(t) \leq D(\beta_\varepsilon(t) + \eta) \leq D(\sup_{t>A+1} \beta(t) + 2).$$

Comparison with the constant coefficients elliptic problems shows the exponential decay of ψ together with its derivatives.

Multiplying (3.1) by $-\psi''(t)$ and integrating over $\mathbb{R}$, we find

$$D\int_{-\infty}^{+\infty}(\beta_\varepsilon(t)+\eta)|\psi''(t)|^2dt+\int_{-\infty}^{+\infty}\gamma_\varepsilon(t)|\psi'(t)|^2dt$$

$$\leq\int_{-\infty}^{+\infty}|g''(t)||\psi(t)|dt+\int_{-\infty}^{+\infty}|\gamma_\varepsilon'(t)||\psi'(t)|\psi(t)dt\leq C_3,$$

since $g\in C_0^2(\mathbb{R})$, $\gamma\in C^1(\overline{\mathbb{R}})$, $|\psi'(t)|$ is exponentially decaying and

$$\int_{-\infty}^{+\infty}c\psi'(t)\psi''(t)dt=\frac{c}{2}|\psi'(t)|^2\Big|_{-\infty}^{+\infty}=0.$$

This completes the proof. □

Now, we can prove the following comparison principle of degenerate diffusion equation on unbounded domain.

Lemma 3.4 *(Comparison Principle). Let $\phi_1,\phi_2\in C(\mathbb{R};\mathbb{R})$ such that for $i=1,2$, $0\leq\phi_i\leq K$, $\phi_i^m\in W^{1,2}_{loc}$, $\phi_1(t)>0$ for all $t\in\mathbb{R}$, $\phi_i(t)$ is increasing for $t\leq t_0$ with some fixed $t_0\in\mathbb{R}$, $\liminf_{t\to\pm\infty}(\phi_1(t)-\phi_2(t))\geq 0$, $\liminf_{t\to+\infty}\phi_1(t)>0$ and ϕ_i satisfies the following inequality*

$$c\phi_1'(t)-D(\phi_1^m(t))''+d(\phi_1(t))\geq c\phi_2'(t)-D(\phi_2^m(t))''+d(\phi_2(t))$$

in the sense of distributions. Then $\phi_1(t)\geq\phi_2(t)$ for all $t\in\mathbb{R}$.

Proof. Let

$$\beta(t)=\begin{cases}\dfrac{\phi_1^m(t)-\phi_2^m(t)}{\phi_1(t)-\phi_2(t)}, & \phi_1(t)\neq\phi_2(t),\\ m\phi_1^{m-1}(t), & \phi_1(t)=\phi_2(t).\end{cases}$$

Then $0\leq\beta(t)\leq mK^{m-1}$ and $\beta(t)$ is continuous. We can rewrite $\beta(t)$ as

$$\beta(t)=m\int_0^1(s\phi_1(t)+(1-s)\phi_2(t))^{m-1}ds.$$

Therefore, $\beta(t)\in C(\mathbb{R})$ and $\liminf_{t\to+\infty}\beta(t)>0$. For any $\psi(t)\geq 0$, $\psi\in C_0^2(\mathbb{R})$ we have

$$\int_{-\infty}^{+\infty}\Big(-c(\phi_1(t)-\phi_2(t))\psi'(t)-D(\phi_1^m(t)-\phi_2^m(t))\psi''(t)+(d(\phi_1(t))-d(\phi_2(t)))\psi(t)\Big)dt\geq 0.$$

That is

$$\int_{-\infty}^{+\infty} (\phi_1(t) - \phi_2(t))\Big(- c\psi'(t) - D\beta(t)\psi''(t) + \gamma(t)\psi(t)\Big)dt \geq 0, \tag{3.5}$$

where

$$\gamma(t) = \int_0^1 d'(s\phi_1(t) + (1-s)\phi_2(t))ds = \begin{cases} \dfrac{d(\phi_1) - d(\phi_2)}{\phi_1(t) - \phi_2(t)}, & \phi_1(t) \neq \phi_2(t), \\ d'(\phi_1(t)), & \phi_1(t) = \phi_2(t). \end{cases}$$

We note that $0 < \gamma(t) \leq d'(K)$, $\gamma(t)$ is increasing for $t \leq t_0$ with some fixed $t_0 \in \mathbb{R}$, and $\gamma(t)$ is C^1 continuous since $\phi_1(t) > 0$ for all $t \in \mathbb{R}$ and $d \in C^2([0, K])$. Now for any given $g(t) \geq 0$, $g(t) \in C_0^2(\mathbb{R})$, $\varepsilon > 0$ and $\eta > 0$, let $\psi_{\varepsilon,\eta}(t)$ be the solution of the following elliptic problem

$$\begin{cases} -c\psi'(t) - D(\beta_\varepsilon(t) + \eta)\psi''(t) + \gamma_\varepsilon(t)\psi(t) = g(t), & t \in \mathbb{R}, \\ \psi(\pm\infty) = 0, \end{cases} \tag{3.6}$$

where $\beta_\varepsilon(t)$ and $\gamma_\varepsilon(t)$ are the smooth approximations of $\beta(t)$ and $\gamma(t)$ such that $\beta(t) \leq \beta_\varepsilon(t) \leq \beta(t) + \varepsilon$ and $\gamma(t) \leq \gamma_\varepsilon(t) \leq \gamma(t) + \varepsilon$. Lemma 3.3 shows that $0 \leq \psi_{\varepsilon,\eta}(t) \leq \max_{t\in\mathbb{R}} g(t)/\gamma(t)$, and both $\psi_{\varepsilon,\eta}(t)$, $\psi'_{\varepsilon,\eta}(t)$, $(\beta_\varepsilon(t) + \eta)\psi''_{\varepsilon,\eta}(t)$ decay to zero exponentially as $t \to \pm\infty$. Taking $\psi_{\varepsilon,\eta}(t)$ as the test function in (3.5), by (3.6) we find that

$$\begin{aligned} \int_{-\infty}^{+\infty} (\phi_1(t) - \phi_2(t))g(t)dt &\geq - \int_{-\infty}^{+\infty} D(\phi_1(t) - \phi_2(t))(\beta_\varepsilon(t) + \eta - \beta(t))\psi''_{\varepsilon,\eta}(t)dt \\ &\geq -2KD \int_{-\infty}^{+\infty} (|\beta_\varepsilon(t) - \beta(t)| + \eta)|\psi''_{\varepsilon,\eta}(t)|dt. \end{aligned} \tag{3.7}$$

Now for any $\mu > 0$, we can estimate the last term in (3.7) as

$$\begin{aligned} &\int_{-\infty}^{+\infty} (|\beta_\varepsilon(t) - \beta(t)| + \eta)|\psi''_{\varepsilon,\eta}(t)|dt \\ &\leq \int_{-\infty}^{-B} + \int_{-B}^{B} + \int_{B}^{+\infty} (|\beta_\varepsilon(t) - \beta(t)| + \eta)|\psi''_{\varepsilon,\eta}(t)|dt \\ &\leq 2\mu + \int_{-B}^{B} (|\beta_\varepsilon(t) - \beta(t)| + \eta)|\psi''_{\varepsilon,\eta}(t)|dt \end{aligned}$$

$$
\begin{aligned}
&\le 2\mu + \Big(\int_{-B}^{B} (|\beta_\varepsilon(t)-\beta(t)|+\eta)|\psi''_{\varepsilon,\eta}(t)|^2 dt\Big)^{\frac{1}{2}} \cdot \Big(\int_{-B}^{B} (|\beta_\varepsilon(t)-\beta(t)|+\eta)dt\Big)^{\frac{1}{2}} \\
&\le 2\mu + \Big(\int_{-\infty}^{\infty} (\beta_\varepsilon(t)+\eta)|\psi''_{\varepsilon,\eta}(t)|^2 dt\Big)^{\frac{1}{2}} \cdot (2B(|\beta_\varepsilon(t)-\beta(t)|+\eta))^{\frac{1}{2}} \\
&\le 3\mu,
\end{aligned}
$$

where $B > A+1$ such that $\int_{-\infty}^{-B}(\beta_\varepsilon(t)+\eta)|\psi''_{\varepsilon,\eta}(t)|dt \le \mu$ and $\int_{B}^{+\infty}(\beta_\varepsilon(t)+\eta)|\psi''_{\varepsilon,\eta}(t)|dt \le \mu$ since the integral is exponential decay independent of ε and η, and the last inequality is valid for sufficiently small ε and η as $\int_{-\infty}^{\infty}(\beta_\varepsilon(t)+\eta)|\psi''_{\varepsilon,\eta}(t)|^2 dt$ is uniformly bounded according to Lemma 3.3. Therefore, (3.7) reads

$$
\int_{-\infty}^{+\infty} (\phi_1(t)-\phi_2(t))g(t)dt \ge 0.
$$

Since $g(t) \ge 0$ is arbitrary, we conclude $\phi_1(t) \ge \phi_2(t)$ for all $t \in \mathbb{R}$. $\square$

Next, we show the solvability of the degenerate elliptic problem.

Lemma 3.5. *Assume that $\overline{\phi}(t)$ is an upper solution of (2.2) and $\overline{\phi}(t)$ is monotone increasing, $0 < \overline{\phi}(t) \le K$, $\lim_{t\to-\infty}\overline{\phi}(t) = 0$, $\lim_{t\to+\infty}\overline{\phi}(t) = K$. Then the following degenerate elliptic equation*

$$
\begin{cases}
c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) = H(\overline{\phi})(t), & t \in \mathbb{R}, \\
\lim\limits_{t\to-\infty}\phi(t) = 0, \quad \lim\limits_{t\to+\infty}\phi(t) = K,
\end{cases}
\tag{3.8}
$$

admits a monotone increasing solution $\phi(t)$ such that $0 < \phi(t) \le \overline{\phi}(t)$ for all $t \in \mathbb{R}$, and $\phi(t)$ is an upper solution of (2.2). Furthermore, there exists a constant $C > 0$ depending only on c, d, m, D, K, such that $\|\phi^m\|_{W^{1,2}(-A,A)} \le CA^{1/2}$ for all $A > 1$.

Proof. Let $f(t) = H(\overline{\phi})(t)$ for simplicity. Consider the following regularized problem for any $A > 1$

$$
\begin{cases}
c\phi'(t) = D\big(m(|\phi(t)|^2 + 1/A)^{(m-1)/2}\phi'(t)\big)' - d(\phi(t)) + f(t), & t \in (-A, A), \\
\phi(-A) = d^{-1}(f(-A)), \quad \phi(A) = d^{-1}(f(A)).
\end{cases}
\tag{3.9}
$$

Since $\overline{\phi}(t)$ is monotone increasing and $\lim_{t\to-\infty}\overline{\phi}(t) = 0$, $\lim_{t\to+\infty}\overline{\phi}(t) = K$, Lemma 3.1 shows that $f(t)$ is monotonically increasing. We can verify that $0 < f(t) < d(K) = b(K)$ and $\lim_{t\to-\infty} f(t) = 0$, $\lim_{t\to+\infty} f(t) = d(K)$. The unique existence of solution to (3.9) is trivial. The solution is denoted by ϕ_A. Comparison principle of elliptic equation shows that

$$
0 < d^{-1}(f(-A)) \le \phi_A(t) \le d^{-1}(f(A)) < K, \quad t \in (-A, A).
$$

In fact, if this is not true, we argue by contradiction. If there exists $t_0 \in (-A, A)$ such that $\phi_A(t_0) < d^{-1}(f(-A))$, then the minimum of $\phi_A(t)$ on $[-A, A]$ is less than $d^{-1}(f(-A))$ and is attained at some point $t^* \in (-A, A)$ since $\phi_A(\pm A) \geq d^{-1}(f(-A))$. At this point t^*, $\phi_A'(t^*) = 0$, $\phi_A''(t^*) \geq 0$, and by (3.9)

$$f(t^*) = c\phi_A'(t^*) - D\big(m(|\phi_A(t^*)|^2 + 1/A)^{(m-1)/2}\phi_A'(t^*)\big)' + d(\phi_A(t^*)) < f(-A),$$

which contradicts to the fact $f(t) \geq f(-A)$ for all $t \in [-A, A]$. The proof of $\phi_A(t) \leq d^{-1}(f(A))$ is similar to that of $d^{-1}(f(-A)) \leq \phi_A(t)$.

We assert that $\phi_A'(t) \geq 0$. Otherwise, there exists a $t_0 \in (-A, A)$ such that $\phi_A'(t_0) < 0$. Let (t_1, t_2) be the maximum interval such that $t_0 \in (t_1, t_2)$ and $\phi_A'(t) < 0$ for $t \in (t_1, t_2)$. We note that $\phi_A(t)$ attains its minimum at $-A$ and its maximum at A, which implies $\phi_A'(-A) \geq 0$ and $\phi_A'(A) \geq 0$. Thus, $\phi_A'(t_1) = \phi_A'(t_2) = 0$, $\phi_A(t_1) > \phi_A(t_2)$,

$$(m(|\phi_A(t)|^2 + 1/A)^{(m-1)/2}\phi_A'(t))'\big|_{t=t_1} \leq 0, \quad (m(|\phi_A(t)|^2 + 1/A)^{(m-1)/2}\phi_A'(t))'\big|_{t=t_2} \geq 0,$$

and

$$\begin{aligned} f(t_1) &= c\phi_A'(t_1) - D\big(m(|\phi_A(t_1)|^2 + 1/A)^{(m-1)/2}\phi_A'(t_1)\big)' + d(\phi_A(t_1)) \\ &> c\phi_A'(t_2) - D\big(m(|\phi_A(t_2)|^2 + 1/A)^{(m-1)/2}\phi_A'(t_2)\big)' + d(\phi_A(t_2)) \\ &= f(t_2), \quad t_1 < t_2, \end{aligned}$$

which contradicts to the monotone increasing of f. For $1 < B < A$, let $\eta(t)$ be the cut-off function such that $0 \leq \eta(t) \leq 1$, $\eta \in C_0^2((-B, B))$, $|\eta'(t)| \leq 2$ for $t \in (-B, B)$, $\eta(t) = 1$ for $t \in (-B+1, B-1)$. Multiply (3.9) by $\eta^2(t)\phi_A(t)$ and integrate over $(-A, A)$, we have

$$\begin{aligned} &\int_{-A}^{A} c\eta^2\phi_A(t)\phi_A'(t)dt + \int_{-A}^{A} Dm\eta^2(|\phi_A(t)|^2 + 1/A)^{(m-1)/2}|\phi_A'(t)|^2 dt \\ &\quad + \int_{-A}^{A} \eta^2 d(\phi_A(t))\phi_A(t)dt \\ &\leq \int_{-A}^{A} 2Dm\eta(|\phi_A(t)|^2 + 1/A)^{(m-1)/2}\phi_A(t)\phi_A'(t)|\eta'(t)|dt + \int_{-A}^{A} \eta^2\phi_A(t)f(t)dt \\ &\leq \frac{1}{2}\int_{-A}^{A} Dm\eta^2(|\phi_A(t)|^2 + 1/A)^{(m-1)/2}|\phi_A'(t)|^2 dt \\ &\quad + \int_{-A}^{A} 2Dm(|\phi_A(t)|^2 + 1/A)^{(m-1)/2}|\phi_A(t)|^2|\eta'(t)|^2 dt + 2d(K)KB. \end{aligned}$$

Therefore,

$$\frac{1}{2}\int_{-B+1}^{B-1} Dm(|\phi_A(t)|^2+1/A)^{(m-1)/2}|\phi_A'(t)|^2dt+\int_{-B+1}^{B-1} d(\phi_A(t))\phi_A(t)dt$$
$$\le \int_{-B}^{-B+1}+\int_{B-1}^{B} 2Dm(|\phi_A(t)|^2+1/A)^{(m-1)/2}|\phi_A(t)|^2|\eta'(t)|^2dt+2d(K)KB$$
$$\le 16Dm(K^2+1)^{(m-1)/2}K^2+2d(K)KB.$$

It follows that $\|\phi_A^m\|_{W^{1,2}(-B+1,B-1)}$ is bounded independent of A. We note that the embedding $W^{1,2}(-B+1,B-1)$ to $C^\gamma([-B+1,B-1])$ with $\gamma\in(0,\frac{1}{2})$ is compact, and $\phi_A^m\in C^\gamma([-B+1,B-1])$ implies $\phi_A\in C^{\gamma/m}([-B+1,B-1])$. There exist a subsequence of $\{\phi_A(t)\}_{A>1}$ denoted by $\{\phi_{A_n}(t)\}_{n\in\mathbb{N}}$ and a function $\phi(t)\in C^{\gamma/m}(\mathbb{R})$ such that $\phi^m\in W^{1,2}_{\text{loc}}(\mathbb{R})$, $0\le\phi\le K$, and $\phi_{A_n}(t)$ uniformly converges to $\phi(t)$ on any compact interval, $\phi^m_{A_n}(t)$ weakly converges to $\phi^m(t)$ in $W^{1,2}_{\text{loc}}(\mathbb{R})$. Since each $\phi_{A_n}(t)$ is monotonically increasing, we see that $\phi(t)$ is also increasing. We can verify that $\phi(t)$ is a solution of (3.8). Moreover, $\phi(t)>0$ for all $t\in\mathbb{R}$. Otherwise, there exists a t_0 such that $\phi(t_0)=0$, which is the minimum of $\phi(t)$ and $\phi^m(t)$. It follows $\phi'(t_0)=0$, $(\phi^m(t_0))''\ge 0$, and $f(t_0)=H(\overline{\phi})(t_0)=0$ according to (3.8). By the definition of $H(\overline{\phi})$, we see that $\overline{\phi}\equiv 0$ and it contradicts to the assumption $\overline{\phi}>0$.

Now we show that $\phi(t)\le\overline{\phi}(t)$ and $\phi(t)$ is an upper solution of (2.2). According to the definition of upper solution, we have

$$\begin{cases} c\overline{\phi}'(t)-D(\overline{\phi}^m(t))''+d(\overline{\phi}(t))\ge H(\overline{\phi})(t)=c\phi'(t)-D(\phi^m(t))''+d(\phi(t)), & t\in\mathbb{R},\\ \lim_{t\to-\infty}\phi(t)=0,\quad \lim_{t\to+\infty}\phi(t)=K,\quad \overline{\phi}(t)\ge 0,\quad \lim_{t\to+\infty}\overline{\phi}(t)=K. \end{cases}$$

The comparison principle Lemma 3.4 of degenerate elliptic problem implies that $\phi(t)\le\overline{\phi}(t)$ for all $t\in\mathbb{R}$. Lemma 3.1 shows that

$$c\phi'(t)-D(\phi^m(t))''+d(\phi(t))=H(\overline{\phi})(t)\ge H(\phi)(t).$$

It follows that $\phi(t)$ is an upper solution of (2.2). □

We also need to compare the solution of (3.8) with some lower solution $\underline{\phi}(t)$.

Lemma 3.6. *Assume that $\overline{\phi}(t)$ is a monotone increasing upper solution of (2.2) such that $0<\overline{\phi}(t)\le K$, $\lim_{t\to-\infty}\overline{\phi}(t)=0$, $\lim_{t\to+\infty}\overline{\phi}(t)=K$, $\underline{\phi}(t)$ is a lower solution of (2.2), $\underline{\phi}(t)$ is increasing for $t\le t_0$ with some fixed $t_0\in\mathbb{R}$, and $\underline{\phi}(t)\le\overline{\phi}(t)$ for all $t\in\mathbb{R}$. Let $\phi(t)$ be the solution of (3.8) in Lemma 3.5. Then $\phi(t)\ge\underline{\phi}(t)$ for all $t\in\mathbb{R}$.*

Proof. Since $\underline{\phi}(t)\le\overline{\phi}(t)$, Lemma 3.1 shows that $H(\overline{\phi})(t)\ge H(\underline{\phi})(t)$ for all $t\in\mathbb{R}$. Therefore,

$$\begin{aligned} c\phi'(t)-D(\phi^m(t))''+d(\phi(t)) =& H(\overline{\phi})(t)\\ &\ge H(\underline{\phi})(t)\ge c\underline{\phi}'(t)-D(\underline{\phi}^m(t))''+d(\underline{\phi}(t)),\quad t\in\mathbb{R}, \end{aligned}$$
$$\lim_{t\to-\infty}\phi(t)=0,\quad \lim_{t\to+\infty}\phi(t)=K,\quad \lim_{t\to-\infty}\underline{\phi}(t)=0,\quad \underline{\phi}(t)\le K.$$

The comparison principle Lemma 3.4 of degenerate elliptic problem implies that $\phi(t) \geq \underline{\phi}(t)$ for all $t \in \mathbb{R}$. $\square$

We construct a pair of upper and lower solutions that satisfy the assumption in Lemma 3.6. For $c > \hat{c}(\alpha, r)$ with $\hat{c}(\alpha, r) > 0$ being the constant in Lemma 3.2, define the functions $\overline{\phi}$ and $\underline{\phi}$ by

$$\begin{cases} \overline{\phi}(t) = \min\{K, Ke^{\lambda_1 t} + Ke^{\hat{m}\lambda_1 t}\}, \\ \underline{\phi}(t) = \max\{0, Ke^{\lambda_1 t} - MKe^{\hat{m}\lambda_1 t}\}, \end{cases} \tag{3.10}$$

where $1 < \hat{m} < \min\{m, 2\}$ such that $\hat{m}\lambda_1 \in (\lambda_1, \lambda_2)$, λ_1, λ_2 are the two roots of $\Delta_c(\lambda_1) = \Delta_c(\lambda_2) = 0$, $M > 1$ is a positive constant to be determined.

Lemma 3.7. *Assume that (i) $c > \hat{c}(\alpha, r)$ is given, D is sufficiently small; or (ii) c is sufficiently large. Then the function $\overline{\phi}(t)$ defined by (3.10) is a monotone increasing upper solution of (2.2) such that $0 < \overline{\phi}(t) \leq K$, $\lim_{t\to-\infty} \overline{\phi}(t) = 0$, $\lim_{t\to+\infty} \overline{\phi}(t) = K$.*

Proof. We note that $Ke^{\lambda_1 t} + Ke^{\hat{m}\lambda_1 t}$ are strictly monotone increasing in $\mathbb{R}$. Let t_0 be the unique solution of $e^{\lambda_1 t} + e^{\hat{m}\lambda_1 t} = 1$. We point out that $\overline{\phi}^m(t) \in W^{1,\infty}(\mathbb{R})$, $(\overline{\phi}^m(t))' > 0$ in $(-\infty, t_0)$, and $(\overline{\phi}^m(t))' = 0$ in $(t_0, +\infty)$. Therefore, $\overline{\phi}^m(t) \notin W^{2,1}(\mathbb{R})$. Actually, $(\overline{\phi}^m(t))''$ is the sum of a negative measure and a L^∞ function at some neighborhood of t_0. We only need to check (2.1) in $(-\infty, t_0)$ and (t_0, ∞) separately.

Case (i) For $t \in (t_0, \infty)$. We have $\overline{\phi}(t) = K$ and $\overline{\phi}'(t) = (\overline{\phi}^m(t))'' = 0$ for $t > t_0$. Since $0 \leq \overline{\phi}(t) \leq K$ for all $t \in \mathbb{R}$ and the function $b(s)$ is increasing on $[0, K]$, we have

$$\begin{aligned} c\overline{\phi}'(t) - D(\overline{\phi}^m(t))'' + d(\overline{\phi}(t)) - \int_{-\infty}^{+\infty} b(\overline{\phi}(t - cr - y)) f_\alpha(y) dy \\ = d(K) - \int_{-\infty}^{+\infty} b(\overline{\phi}(t - cr - y)) f_\alpha(y) dy \\ \geq d(K) - \int_{-\infty}^{+\infty} b(K) f_\alpha(y) dy = d(K) - b(K) = 0. \end{aligned}$$

Case (ii) For $t \in (-\infty, t_0)$. Then $\overline{\phi}(t) = Ke^{\lambda_1 t} + Ke^{\hat{m}\lambda_1 t}$,

$$\overline{\phi}'(t) = K\lambda_1 e^{\lambda_1 t} + K\hat{m}\lambda_1 e^{\hat{m}\lambda_1 t},$$

and

$$\overline{\phi}''(t) = K\lambda_1^2 e^{\lambda_1 t} + K\hat{m}^2\lambda_1^2 e^{\hat{m}\lambda_1 t},$$

for all $t < t_0$. We have

$$
\begin{aligned}
(\overline{\phi}^m(t))'' =& m\overline{\phi}^{m-1}\cdot\overline{\phi}''(t)+m(m-1)\overline{\phi}^{m-2}\cdot(\overline{\phi}'(t))^2\\
=& m(Ke^{\lambda_1 t}+Ke^{\hat{m}\lambda_1 t})^{m-1}(K\lambda_1^2e^{\lambda_1 t}+K\hat{m}^2\lambda_1^2e^{\hat{m}\lambda_1 t})\\
&+m(m-1)(Ke^{\lambda_1 t}+Ke^{\hat{m}\lambda_1 t})^{m-2}(K\lambda_1e^{\lambda_1 t}+K\hat{m}\lambda_1e^{\hat{m}\lambda_1 t})^2, \quad t<t_0.
\end{aligned}
$$

We note that $\overline{\phi}(t)\le Ke^{\lambda_1 t}+Ke^{\hat{m}\lambda_1 t}$ for all $t\in\mathbb{R}$, and $b(s)\le b'(0)s$, $d(s)\ge d'(0)s$ for $s\in[0,K]$ since $d''(s)\ge 0$ and $b''(s)\le 0$. Therefore, for $t<t_0$

$$
\begin{aligned}
& c\overline{\phi}'(t)-D(\overline{\phi}^m(t))''+d(\overline{\phi}(t))-\int_{-\infty}^{+\infty}b(\overline{\phi}(t-cr-y))f_\alpha(y)dy\\
&\ge c(K\lambda_1e^{\lambda_1 t}+K\hat{m}\lambda_1e^{\hat{m}\lambda_1 t})-D(\overline{\phi}^m(t))''+d'(0)(Ke^{\lambda_1 t}+Ke^{\hat{m}\lambda_1 t})\\
&\quad -b'(0)\int_{-\infty}^{+\infty}(Ke^{\lambda_1(t-cr-y)}+Ke^{\hat{m}\lambda_1(t-cr-y)})f_\alpha(y)dy\\
&\ge Ke^{\lambda_1 t}(c\lambda_1+d'(0))+Ke^{\hat{m}\lambda_1 t}(c\hat{m}\lambda_1+d'(0))-D(\overline{\phi}^m(t))''\\
&\quad -b'(0)\int_{-\infty}^{+\infty}(Ke^{\lambda_1(t-cr-y)}+Ke^{\hat{m}\lambda_1(t-cr-y)})f_\alpha(y)dy\\
&\ge Ke^{\lambda_1 t}(c\lambda_1+d'(0)-b'(0)e^{\alpha\lambda_1^2-\lambda_1 cr})\\
&\quad +Ke^{\hat{m}\lambda_1 t}(c\hat{m}\lambda_1+d'(0)-b'(0)e^{\alpha\hat{m}^2\lambda_1^2-\hat{m}\lambda_1 cr})-D(\overline{\phi}^m(t))''\\
&=-Ke^{\lambda_1 t}\Delta_c(\lambda_1)-Ke^{\hat{m}\lambda_1 t}\Delta_c(\hat{m}\lambda_1)-D(\overline{\phi}^m(t))''\\
&=-Ke^{\hat{m}\lambda_1 t}\Delta_c(\hat{m}\lambda_1)-D(\overline{\phi}^m(t))''\\
&\ge 0,
\end{aligned}
$$

provided that

$$
-Ke^{\hat{m}\lambda_1 t}\Delta_c(\hat{m}\lambda_1)-D(\overline{\phi}^m(t))''\ge 0, \quad t<t_0, \tag{3.11}
$$

since $\hat{m}\lambda_1\in(\lambda_1,\lambda_2)$ and $\Delta_c(\lambda_1)=0$. Note that $t_0<0$ and $e^{\lambda_1 t}\ge e^{\hat{m}\lambda_1 t}$ for $t<t_0$. Then we have

$$
\begin{aligned}
D|(\overline{\phi}^m(t))''| \le& Dm(2Ke^{\lambda_1 t})^{m-1}(1+\hat{m}^2)K\lambda_1^2e^{\lambda_1 t}\\
&+Dm(m-1)(2Ke^{\lambda_1 t})^{m-2}((1+\hat{m})K\lambda_1e^{\lambda_1 t})^2\\
\le& DmK^m\lambda_1^2(2^{m-1}(1+\hat{m}^2)+(m-1)2^{m-2}(1+\hat{m})^2)e^{m\lambda_1 t}\\
=& ADK^m\lambda_1^2e^{m\lambda_1 t}, \quad t<t_0,
\end{aligned}
$$

where $A=m(2^{m-1}(1+\hat{m}^2)+(m-1)2^{m-2}(1+\hat{m})^2)$. A sufficient condition for (3.11) is

$$
ADK^{m-1}\lambda_1^2\le-\Delta_c(\hat{m}\lambda_1) \tag{3.12}
$$

since $t < t_0 < 0$, $\hat{m} < m$ and $\Delta_c(\hat{m}\lambda_1) < 0$. For given $c > \hat{c}(\alpha, r)$, (3.12) is valid if D or K is sufficiently small. Otherwise, (3.12) is valid if c is sufficiently large according to Lemma 3.2 for any given $1 < \hat{m} < \min\{m, 2\}$, and $\hat{m}\lambda_1 \in (\lambda_1, \lambda_2)$ is valid automatically since $\Delta_c(\hat{m}\lambda_1) < 0$. $\square$

Lemma 3.8. *For any $c > \hat{c}(\alpha, r)$, the function $\underline{\phi}(t)$ defined by (3.10) is a lower solution of (2.2) if the constant $M > 1$ is sufficiently large. Moreover, $\underline{\phi}(t)$ is increasing for $t \le t_0$ with some fixed $t_0 \in \mathbb{R}$.*

Proof. Let t_1 be the unique solution of $e^{\lambda_1 t} = Me^{\hat{m}\lambda_1 t}$. Similar to the proof of Lemma 3.7, $\underline{\phi}^m(t) \in W^{1,\infty}(\mathbb{R})$, $(\underline{\phi}^m(t))' < 0$ in some left neighborhood of t_1, and $(\underline{\phi}^m(t))' = 0$ in $(t_1, +\infty)$. Therefore, $\underline{\phi}^m(t) \notin W^{2,1}(\mathbb{R})$ and $(\underline{\phi}^m(t))''$ is the sum of a positive measure and a L^∞ function at some neighborhood of t_1. We only need to check (2.1) in $(-\infty, t_1)$ and (t_1, ∞) separately.

Case (i) For $t \in (t_1, \infty)$. We have $\underline{\phi}(t) \ge 0$ for all $y \in \mathbb{R}$. Then

$$\begin{aligned}
&c\underline{\phi}'(t) - D(\underline{\phi}^m(t))'' + d(\underline{\phi}(t)) - \int_{-\infty}^{+\infty} b(\underline{\phi}(t - cr - y)) f_\alpha(y)dy \\
&\le -\int_{-\infty}^{+\infty} b(\overline{\phi}(t - cr - y)) f_\alpha(y)dy \\
&\le 0, \qquad t > t_1.
\end{aligned}$$

Case (ii) For $t \in (-\infty, t_1)$. Then $\underline{\phi}(t) = Ke^{\lambda_1 t} - MKe^{\hat{m}\lambda_1 t} > 0$,

$$\underline{\phi}'(t) = K\lambda_1 e^{\lambda_1 t} - MK\hat{m}\lambda_1 e^{\hat{m}\lambda_1 t},$$

and

$$\underline{\phi}''(t) = K\lambda_1^2 e^{\lambda_1 t} - MK\hat{m}^2\lambda_1^2 e^{\hat{m}\lambda_1 t},$$

for all $t < t_1$. We have

$$\begin{aligned}
(\underline{\phi}^m(t))'' =& m\underline{\phi}^{m-1} \cdot \underline{\phi}''(t) + m(m-1)\underline{\phi}^{m-2} \cdot (\underline{\phi}'(t))^2 \\
\ge& m(Ke^{\lambda_1 t} - MKe^{\hat{m}\lambda_1 t})^{m-1}(K\lambda_1^2 e^{\lambda_1 t} - MK\hat{m}^2\lambda_1^2 e^{\hat{m}\lambda_1 t}) \\
\ge& -m(Ke^{\lambda_1 t} - MKe^{\hat{m}\lambda_1 t})^{m-1} MK\hat{m}^2\lambda_1^2 e^{\hat{m}\lambda_1 t} \\
\ge& -mMK^m\hat{m}^2\lambda_1^2 e^{\hat{m}\lambda_1 t + (m-1)\lambda_1 t}, \qquad t < t_1.
\end{aligned}$$

Let $h(t) = Ke^{\lambda_1 t} - MKe^{\hat{m}\lambda_1 t}$ for $t \in \mathbb{R}$. Then $\underline{\phi}(t) \ge h(t)$. Since $b(s)$ is monotone increasing on $[0, K]$, $b'(0) > 0$ and $b \in C^2([0, K])$, there exists a constant $A > 0$ such that $b(s) \ge b'(0)s(1 - As)$ for $s \in (-\infty, K]$ (we can extend $b(s)$ to $(-\infty, 0)$), similar to the proof of Lemma 4.6 in [47], we have

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} b(\underline{\phi}(t-cr-y)) f_\alpha(y)dy \\
&\geq \int_{-\infty}^{+\infty} b(h(t-cr-y)) f_\alpha(y)dy \\
&\geq b'(0) \int_{-\infty}^{+\infty} h(t-cr-y)(1-Ah(t-cr-y)) f_\alpha(y)dy \\
&\geq b'(0)Ke^{\lambda_1 t} e^{\alpha\lambda_1^2-\lambda_1 cr} - b'(0)MKe^{\hat{m}\lambda_1 t} e^{\alpha\hat{m}^2\lambda_1^2-\hat{m}\lambda_1 cr} \\
&\quad - b'(0)K^2Ae^{2\lambda_1 t} \int_{-\infty}^{+\infty} e^{-2\lambda_1(y+cr)}(1-Me^{(\hat{m}-1)\lambda_1(t-y-cr)})^2 f_\alpha(y)dy \\
&\geq b'(0)Ke^{\lambda_1 t} e^{\alpha\lambda_1^2-\lambda_1 cr} - b'(0)MKe^{\hat{m}\lambda_1 t} e^{\alpha\hat{m}^2\lambda_1^2-\hat{m}\lambda_1 cr} - B(t)e^{2\lambda_1 t},
\end{aligned} \tag{3.13}
$$

where

$$
\begin{aligned}
B(t) &= b'(0)K^2A \int_{-\infty}^{+\infty} e^{-2\lambda_1(y+cr)}(1-Me^{(\hat{m}-1)\lambda_1(t-y-cr)})^2 f_\alpha(y)dy \\
&\leq b'(0)K^2A \int_{-\infty}^{+\infty} e^{-2\lambda_1(y+cr)}(1+Me^{(\hat{m}-1)\lambda_1(t-y-cr)})^2 f_\alpha(y)dy \\
&\leq b'(0)K^2A \int_{-\infty}^{+\infty} e^{-2\lambda_1(y+cr)}(1+e^{-(\hat{m}-1)\lambda_1(y+cr)})^2 f_\alpha(y)dy \\
&=: B_0, \quad t < t_1,
\end{aligned}
$$

since $Me^{(\hat{m}-1)\lambda_1 t} < 1$ for $t < t_1$. We note that $d(s) \leq d'(0)s + Es^2$ for $s \in (-\infty, K]$ with some constant $E > 0$ as $d \in C^2([0, K])$. Now we have

$$
\begin{aligned}
&c\underline{\phi}'(t) - D(\underline{\phi}^m(t))'' + d(\underline{\phi}(t)) - \int_{-\infty}^{\infty} b(\underline{\phi}(t-cr-y)) f_\alpha(y)dy \\
&\leq -Ke^{\lambda_1 t}\Delta_c(\lambda_1) + MKe^{\hat{m}\lambda_1 t}\Delta_c(\hat{m}\lambda_1) - D(\underline{\phi}^m(t))'' + B(t)e^{2\lambda_1 t} + E(\underline{\phi}(t))^2 \\
&\leq MKe^{\hat{m}\lambda_1 t}\Delta_c(\hat{m}\lambda_1) + DmMK^m\hat{m}^2\lambda_1^2 e^{\hat{m}\lambda_1 t+(m-1)\lambda_1 t} + (B_0+EK^2)e^{2\lambda_1 t} \\
&\leq MKe^{\hat{m}\lambda_1 t}\Delta_c(\hat{m}\lambda_1) + DmMK^m\hat{m}^2\lambda_1^2 e^{\hat{m}\lambda_1 t}/M^{(m-1)/(\hat{m}-1)} + (B_0+EK^2)e^{2\lambda_1 t}
\end{aligned}
$$

$$=\Delta_c(\hat{m}\lambda_1)e^{\hat{m}\lambda_1 t}M\Big(K+\frac{DmK^m\hat{m}^2\lambda_1^2}{\Delta_c(\hat{m}\lambda_1)M^{(m-1)/(\hat{m}-1)}}+\frac{(B_0+EK^2)e^{(2-\hat{m})\lambda_1 t}}{\Delta_c(\hat{m}\lambda_1)M}\Big)$$
$$\le 0,$$

provided that $M>1$ sufficiently large such that

$$K+\frac{DmK^m\hat{m}^2\lambda_1^2}{\Delta_c(\hat{m}\lambda_1)M^{(m-1)/(\hat{m}-1)}}+\frac{(B_0+EK^2)e^{(2-\hat{m})\lambda_1 t}}{\Delta_c(\hat{m}\lambda_1)M}>0,$$

since $\Delta_c(\hat{m}\lambda_1)<0$. □

We employ the monotone iteration to find the traveling wave solution.

Lemma 3.9. *For $c>\hat{c}(\alpha,r)$, let $\overline{\phi}(t)$ and $\underline{\phi}(t)$ be defined in (3.10). Let $\phi_0(t)=\overline{\phi}(t)$, and $\phi_i(t)$, $i=1,2,\dots$, be the solution of the following iteration problem*

$$\begin{cases} c\phi_i'(t)-D(\phi_i^m(t))''+d(\phi_i(t))=H(\phi_{i-1})(t), & t\in\mathbb{R},\\ \lim\limits_{t\to-\infty}\phi_i(t)=0, \quad \lim\limits_{t\to+\infty}\phi_i(t)=K. \end{cases}$$

Then $\underline{\phi}(t)\le\cdots\le\phi_i(t)\le\phi_{i-1}\le\cdots\le\overline{\phi}(t)$, $\phi_i(t)>0$ for all $t\in\mathbb{R}$, and there exists a function $\phi\in C(\mathbb{R};\mathbb{R})$ such that $\lim_{i\to\infty}\phi_i(t)=\phi(t)$, $0<\phi(t)\le K$ and $\phi(t)$ is the solution of (2.2).

Proof. According to Lemma 3.5 and Lemma 3.6, we see that $0<\phi_1(t)\le\phi_0(t)\le K$, $\phi_1(t)$ is a monotone increasing upper solution of (2.2), and $\phi_1(t)\ge\underline{\phi}(t)$ for all $t\in\mathbb{R}$. Using Lemma 3.5 and Lemma 3.6 again, we find that $0<\phi_2(t)\le\phi_1(t)\le K$, $\phi_2(t)$ is a monotone increasing upper solution of (2.2), and $\phi_2(t)\ge\underline{\phi}(t)$ for all $t\in\mathbb{R}$. We can deduce by induction that the above assertions are valid for $\phi_i(t)$. Since $\phi_i(t)$ are monotone decreasing with respect to i, and Lemma 3.5 implies that $\|\phi_i^m(t)\|_{W^{1,2}}$ are uniformly bounded on any compact interval, there exists a function $\phi\in C(\mathbb{R};\mathbb{R})$ such that $\lim_{i\to\infty}\phi_i(t)=\phi(t)$, and $\phi(t)$ is the solution of (2.2) The proof of $\phi(t)>0$ is similar to that of Lemma 3.5. □

Therefore, we have proved the existence Theorem 2.1 of degenerate diffusion equation with time-delay and nonlocal effect.

Proof of Theorem 2.1. This is proved in Lemma 3.9. □

Theorem 2.1 shows the existence of traveling waves for any given $c>\hat{c}(\alpha,r)$ provided that D is sufficiently small which may depend on c and vanish as c tends to $\hat{c}$. In order to investigate the traveling wave with critical wave speed $c^*(D,\alpha,r)$, we construct another upper solution. For $c\ge\hat{c}$, we define the function $\overline{\phi}$ by

$$\overline{\phi}(t)=\min\{K,Ke^{\lambda_1 t}\}, \tag{3.14}$$

where λ_1 is the least root of $\Delta_c(\lambda_1)=0$.

Lemma 3.10. *For $m \geq 2$ there exists a constant $C(m,\alpha,r) > 0$ such that if $DK^{m-2} \leq C(m,\alpha,r)$, then for any $c \geq \hat{c}(\alpha,r)$, the function $\overline{\phi}(t)$ defined by (3.14) is a monotone increasing upper solution of (2.2) such that $0 < \overline{\phi}(t) \leq K$, $\lim_{t\to-\infty}\overline{\phi}(t) = 0$, $\lim_{t\to+\infty}\overline{\phi}(t) = K$.*

Proof. The proof of this upper solution is similar to that in Lemma 3.7. We only need to check the case (ii) therein. For $t \in (-\infty, 0)$, $\overline{\phi}(t) = Ke^{\lambda_1 t}$, $\overline{\phi}'(t) = K\lambda_1 e^{\lambda_1 t}$ for all $t < 0$. We have

$$(\overline{\phi}^m(t))'' = m^2\lambda_1^2 K^m e^{m\lambda_1 t}, \quad t < 0.$$

We note that $\overline{\phi}(t) \leq Ke^{\lambda_1 t}$ for all $t \in \mathbb{R}$, $b(s)$ is increasing on $[0, K]$, and $b(s) \leq b'(0)s - C_b s^2$, $d(s) \geq d'(0)s + C_d s^2$ for all $s \in [0, K]$ with $C_b \geq 0$, $C_d \geq 0$, $C_b + C_d > 0$, since either $d''(s) > 0$ or $b''(s) < 0$ according to the assumption (H3). (a) If $C_d > 0$, then for $t < 0$

$$\begin{aligned}
c\overline{\phi}'(t) &- D(\overline{\phi}^m(t))'' + d(\overline{\phi}(t)) - \int_{-\infty}^{+\infty} b(\overline{\phi}(t-cr-y)) f_\alpha(y)dy \\
&\geq cK\lambda_1 e^{\lambda_1 t} - D(\overline{\phi}^m(t))'' + d'(0)Ke^{\lambda_1 t} + C_d K^2 e^{2\lambda_1 t} \\
&\quad - b'(0)\int_{-\infty}^{+\infty} Ke^{\lambda_1(t-cr-y)} f_\alpha(y)dy \\
&\geq Ke^{\lambda_1 t}(c\lambda_1 + d'(0)) - D(\overline{\phi}^m(t))'' + C_d K^2 e^{2\lambda_1 t} - b'(0)e^{\alpha\lambda_1^2 - \lambda_1 cr}Ke^{\lambda_1 t} \\
&\geq Ke^{\lambda_1 t}(c\lambda_1 + d'(0) - b'(0)e^{\alpha\lambda_1^2 - \lambda_1 cr}) + C_d K^2 e^{2\lambda_1 t} - D(\overline{\phi}^m(t))'' \\
&= -Ke^{\lambda_1 t}\Delta_c(\lambda_1) + C_d K^2 e^{2\lambda_1 t} - Dm^2\lambda_1^2 K^m e^{m\lambda_1 t} \\
&\geq 0, \qquad t < 0,
\end{aligned}$$

provided that

$$Dm^2\lambda_1^2 K^{m-2} \leq C_d, \tag{3.15}$$

since $\Delta_c(\lambda_1) = 0$ and $m \geq 2$. (b) If $C_b > 0$, then for $t < 0$

$$\begin{aligned}
c\overline{\phi}'(t) &- D(\overline{\phi}^m(t))'' + d(\overline{\phi}(t)) - \int_{-\infty}^{+\infty} b(\overline{\phi}(t-cr-y)) f_\alpha(y)dy \\
&\geq cK\lambda_1 e^{\lambda_1 t} - D(\overline{\phi}^m(t))'' + d'(0)Ke^{\lambda_1 t} - \int_{-\infty}^{+\infty} b(Ke^{\lambda_1(t-cr-y)}) f_\alpha(y)dy \\
&\geq Ke^{\lambda_1 t}(c\lambda_1 + d'(0)) - D(\overline{\phi}^m(t))'' \\
&\quad - b'(0)\int_{-\infty}^{+\infty} Ke^{\lambda_1(t-cr-y)} f_\alpha(y)dy + C_b\int_{-\infty}^{+\infty} K^2 e^{2\lambda_1(t-cr-y)} f_\alpha(y)dy
\end{aligned}$$

$$\begin{aligned}
&\geq Ke^{\lambda_1 t}(c\lambda_1 + d'(0)) - D(\overline{\phi}^m(t))'' - b'(0)e^{\alpha\lambda_1^2 - \lambda_1 cr} Ke^{\lambda_1 t} + C_b K^2 e^{4\alpha\lambda_1^2 - 2\lambda_1 cr} e^{2\lambda_1 t} \\
&\geq Ke^{\lambda_1 t}(c\lambda_1 + d'(0) - b'(0)e^{\alpha\lambda_1^2 - \lambda_1 cr}) + C_b K^2 e^{4\alpha\lambda_1^2 - 2\lambda_1 cr} e^{2\lambda_1 t} - D(\overline{\phi}^m(t))'' \\
&= -Ke^{\lambda_1 t}\Delta_c(\lambda_1) + C_b K^2 e^{4\alpha\lambda_1^2 - 2\lambda_1 cr} e^{2\lambda_1 t} - Dm^2\lambda_1^2 K^m e^{m\lambda_1 t} \\
&\geq 0, \qquad t < 0,
\end{aligned}$$

provided that

$$Dm^2\lambda_1^2 K^{m-2} \leq C_b e^{4\alpha\lambda_1^2 - 2\lambda_1 cr}, \qquad t < 0, \tag{3.16}$$

since $\Delta_c(\lambda_1) = 0$ and $m \geq 2$. For $c \geq \hat{c}$, let

$$C_m(c) = \begin{cases} C_d/(m^2\lambda_1^2), & C_d > 0, \\ C_b e^{4\alpha\lambda_1^2 - 2\lambda_1 cr}/(m^2\lambda_1^2), & C_d = 0,\ C_b > 0. \end{cases}$$

Since either $C_d > 0$ or $C_b > 0$, we see that $C_m(c) > 0$. Note that λ_1 is decreasing with respect to $c \geq \hat{c}$, $\lambda_1(\hat{c}) > 0$, and

$$\Delta_c(\lambda_1) = b'(0)e^{\alpha\lambda_1^2 - \lambda_1 cr} - \lambda_1 c - d'(0) = 0.$$

We have

$$e^{4\alpha\lambda_1^2 - 2\lambda_1 cr} = e^{2\alpha\lambda_1^2} e^{2\alpha\lambda_1^2 - 2\lambda_1 cr} = e^{2\alpha\lambda_1^2}\left(\frac{\lambda_1 c + d'(0)}{b'(0)}\right)^2 \geq \left(\frac{d'(0)}{b'(0)}\right)^2.$$

Therefore

$$C_m(c) \geq \begin{cases} C_d/(m^2\lambda_1(\hat{c})^2), & C_d > 0, \\ C_b(d'(0)/b'(0))^2/(m^2\lambda_1(\hat{c})^2), & C_d = 0,\ C_b > 0, \end{cases}$$

for all $c \geq \hat{c}$. Hereafter we denote the right-sided constant in the above inequality by $C(m, \alpha, r)$ as $\hat{c} = \hat{c}(\alpha, r)$ depends on α and r. Then either (3.15) or (3.16) is valid if $DK^{m-2} \leq C(m, \alpha, r)$ and thus $\overline{\phi}$ is an upper solution. □

As we have constructed upper solutions for all $c \geq \hat{c}$, we present the following existence of critical wave for degenerate diffusion ($m \geq 2$) with nonlocal effect and time delay.

Proof of Theorem 2.2. For any $c > \hat{c}$, the function $\overline{\phi}$ defined by (3.14) is an upper solution according to Lemma 3.10, and the function $\underline{\phi}$ defined by (3.10) is a lower solution according to Lemma 3.8 if $M > 1$ is sufficiently large. Similar to the proof of Lemma 3.9, we see that (2.2) admits at least one monotone increasing traveling wave solution $\phi_c(t)$ with speed c such that $0 < \phi_c(t) \leq K$ for all $t \in \mathbb{R}$. We may assume that $\phi_c(0) = 1/2$ by shifting. The estimates in Lemma 3.5 is also valid uniformly for $\phi_c(t)$. There exist a subsequence of $\{\phi_c\}_{c>\hat{c}}$ denoted by $\{\phi_{c_k}\}_{k\in\mathbb{N}}$ and a function $0 \leq \phi_{\hat{c}} \leq K$ such that $\phi_{\hat{c}}^m \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ and ϕ_{c_k} uniformly converges to $\phi_{\hat{c}}$ on any compact interval, $\phi_{c_k}^m$ weakly converges to $\phi_{\hat{c}}^m$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R})$. We can prove that $\phi_{\hat{c}}$ is a monotone increasing traveling wave of (2.2) with critical speed $\hat{c}$ by testing the equation (2.2) of ϕ_{c_k} with any $\psi \in C_0^\infty(\mathbb{R})$ and letting k tends to infinity. The property $\phi_{\hat{c}} > 0$ can be proved similar to that in Lemma 3.5. □

For the linear diffusion case ($m = 1$), we now prove the existence of critical traveling wave speed Corollary 2.1.

Proof of Corollary 2.1. The existence is proved in [47]. And the nonexistence result can be proved in a way similar to the proof of Theorem 2.3 in next section. □

Similar to the proof of existence of monotone traveling wave solution to the degenerate diffusion equation with time-delay and nonlocal effect. We can prove the existence of monotone traveling wave solution to the degenerate diffusion equation without nonlocal effect.

Lemma 3.11. *Assume that $\alpha = 0$ and $m > 1$. (i) If $r = 0$, then for any given $D > 0$, the critical wave speed $c^*(D, 0, 0)$ defined in (2.4) is positive. (ii) For any given $D > 0$ there exists a $r_0 > 0$ such that the critical wave speed $c^*(D, 0, r)$ defined in (2.4) is positive if $0 < r \le r_0$.*

Proof. For the degenerate diffusion equation without time delay and nonlocal effect, Huang et al. [21] proved the existence of positive critical wave speed $c^*(D, 0, 0) > 0$ for a typical type of $b(u)$ and $d(u)$. They also proved the existence of traveling wave with some speed $c > c^*(D, 0, 0)$ and small time delay r for the time delayed degenerate diffusion equation. We can verify that these results are valid for the general type of $b(u)$ and $d(u)$ satisfying $(H1)$–$(H3)$. The main approach is the generalized phase plane analysis, see for example [21,24,25,60,61]. If $\alpha = 0$, (2.2) reads

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)) + b(\phi(t - cr)), \\ \phi(-\infty) = 0, \qquad \phi(+\infty) = K. \end{cases} \tag{3.17}$$

Let $\psi(t) = D(\phi^m(t))'$. We are looking for the non-critical traveling waves that are positive and smooth and then the infimum of these wave speeds is the critical wave speed. Now (3.17) is transformed into

$$\begin{cases} \phi'(t) = \dfrac{\psi(t)}{Dm\phi^{m-t}(t)} =: \Phi, \\ \psi'(t) = \dfrac{c\psi(t)}{Dm\phi^{m-t}(t)} - \big(b(\phi_{cr}(t)) - d(\phi(t))\big) =: \Psi, \end{cases} \tag{3.18}$$

where $\phi_{cr}(t) = \phi(t - cr)$.

(i) If $r = 0$, we note that for $\phi(t) > 0$ and $\phi'(t) > 0$, we can use ϕ as the variable of ψ and then (3.18) is equivalent to

$$\frac{d\psi}{d\phi} = c - \frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\psi} = \frac{\Psi}{\Phi}. \tag{3.19}$$

The monotone increasing traveling wave is a trajectory in the phase plane (ϕ, ψ) of (3.19) such that $\psi(0) = 0$, $\psi(K) = 0$, $\psi(\phi) > 0$ for $\phi \in (0, K)$. We solve (3.19) with the condition

$$\psi(K)=0, \qquad \psi(\phi)>0, \quad \phi\in(\beta,K), \tag{3.20}$$

where $\beta\in[0,K)$ and (β,K) is the maximum interval such that $(\phi,\psi)\in(0,K)\times(0,+\infty)$. The existence of the solution denoted by $\psi_c(\phi)$ to such kind singular ODE (3.19) follows from the following observation: (a) Let Γ_c be the curve of

$$\tilde{\psi}(\phi):=\frac{Dm\phi^{m-1}(b(\phi)-d(\phi))}{c}.$$

Then it divides $(0,K)\times(0,+\infty)$ into two parts, $E_1:=\{(\phi,\psi);\phi\in(0,K),0<\psi<\tilde{\psi}(\phi)\}$ and $E_2:=((0,K)\times(0,+\infty))\setminus\overline{E}_1$. (b) For any $(\phi,\psi)\in E_1$, a direct calculation shows that $\Psi/\Phi<0$; while for any $(\phi,\psi)\in E_2$, $\Psi/\Phi>0$. Since $\phi(t)$ is increasing with respect to t, we know that the trajectory passing through a point $(\phi,\psi)\in E_1$ at $t=t_0$ must cross Γ_c from E_2 into E_1 before t_0 and it cannot cross $\{(\phi,\psi);\phi\in(0,K),\psi=0\}$ before t_0. (c) We can solve (3.19) with the condition $\psi(K)=\varepsilon>0$ and let ε tends to zero to approximate (3.19)–(3.20). We see that the maximum interval is $(0,K)$ and $\beta=0$. Moreover, the solution of (3.19)–(3.20) satisfies $\psi_c(\phi)\le\sup_{\phi\in(0,K)}\tilde{\psi}(\phi)$.

Next, we show that for sufficiently large c, the trajectory corresponding to the solution of (3.19)–(3.20) will arrive at the point $(0,0)$ and then it will correspond to a monotone increasing traveling wave as we want. Let $\bar{\psi}(\phi)=B\phi^r$ with $1\le r\le m$. Then,

$$\frac{d\bar{\psi}}{d\phi}<c-\frac{Dm\phi^{m-1}(b(\phi)-d(\phi))}{\bar{\psi}}, \quad \phi\in(0,K)$$

is equivalent to

$$Br\phi^{r-1}+\frac{Dm\phi^{m-1-r}(b(\phi)-d(\phi))}{B}<c, \quad \phi\in(0,K),$$

which holds for c greater than some constant $c_0>0$ since $b(0)-d(0)=0$ and $b(\phi)-d(\phi)\in C^2([0,K])$. For $c>c_0$, we note that $\bar{\psi}(K)=BK^r>0=\psi_c(K)$, then by comparison, we have $\psi_c(\phi)\le\bar{\psi}(\phi)$ for $\phi\in(0,K)$ and $\psi_c(0)=0$.

The last step is to show that $\psi_c(0)>0$ if c is sufficiently small and positive. The integral over $(0,K)$ of (3.19) with condition (3.20) shows that

$$-\frac{(\psi_c(0))^2}{2}=c\int_0^K\psi_c(\phi)d\phi-Dm\int_0^K\phi^{m-1}(b(\phi)-d(\phi))d\phi. \tag{3.21}$$

If $\psi_c(0)=0$, then (3.19) tells us $\frac{d\psi_c}{d\phi}\le c$ and $\psi_c(\phi)\le c\phi$, which yields $\int_0^K\psi_c(\phi)d\phi\le cK^2/2$. Further, according to (3.21)

$$Dm\int_0^K\phi^{m-1}(b(\phi)-d(\phi))d\phi=c\int_0^K\psi_c(\phi)d\phi\le c^2K^2/2.$$

Therefore, if c is small enough such that the above inequality is not true, then $\psi_c(0)$ cannot be zero and there does not exist monotone increasing traveling wave.

(ii) If $r > 0$, inspired by [25,21], we define ϕ_{cr} as a function of ϕ and $\psi(\phi)$ (here we can also regard ψ as a function of ϕ) as follows

$$\begin{cases} cr = \displaystyle\int_{\phi_{cr}}^{\phi} \frac{Dms^{m-1}}{\psi(s)} ds, \text{ if } \displaystyle\int_0^{\phi} \frac{Dms^{m-1}}{\psi(s)} ds \geq cr, \\ \phi_{cr} = 0, \text{ if } \displaystyle\int_0^{\phi} \frac{Dms^{m-1}}{\psi(s)} ds < cr. \end{cases} \tag{3.22}$$

Consider the following problem

$$\begin{cases} \dfrac{d\psi}{d\phi} = c - \dfrac{Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))}{\psi} = \dfrac{\Psi}{\Phi}, \\ \psi(0) = \psi(K) = 0, \qquad \psi(\phi) > 0, \quad \phi \in (0, K). \end{cases} \tag{3.23}$$

The existence of monotone increasing traveling waves for small time delay r and some wave speed $c > c^*(D, 0, 0)$ is proved in [21].

Here in order to show that $c^*(D, 0, r) > 0$ for small time delay, we only need to prove that there exist $r_0 > 0$ and $c_0 > 0$ such that if $c < c_0$ and $0 < r \leq r_0$ then (3.23) has no solution. Let $\psi_{c,r}$ be such a solution and Γ_c, $\tilde{\psi}$ be defined as in (i). We note that $\phi_{cr} < \phi$ and $b(s)$ is strictly increasing. Then we have for any $(\phi, \psi) \in \Gamma_c$ in the phase plane of (3.23),

$$\frac{\Psi}{\Phi} = c - \frac{Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))}{\psi} > c - \frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\psi} = 0.$$

We can also check that $\Psi/\Phi > 0$ for $(\phi, \psi) \in E_2$. Let (γ, K) with $\gamma \in (0, K)$ be the maximum interval such that $\tilde{\psi}$ is decreasing. We see that γ is well defined since $\tilde{\psi}'(K) < 0$ and $\tilde{\psi} \in C^2([0, K])$. The above analysis shows that

$$\psi_{c,r}(\phi) \leq \sup_{\phi \in (0,K)} \tilde{\psi}(\phi), \quad \phi \in (0, K), \tag{3.24}$$

and

$$\psi_{c,r}(\phi) < \tilde{\psi}(\phi), \quad \phi \in (\gamma, K). \tag{3.25}$$

Otherwise, $\psi_{c,r}(K)$ will be positive, which contradicts to our assumption.

We also need to study the behavior of $\psi_{c,r}$ near $\phi = 0$. For this purpose, let $\psi_1(\phi)$ be the solution of

$$\begin{cases} \dfrac{d\psi}{d\phi} = c + \dfrac{Dm\phi^{m-1}d(\phi)}{\psi}, \\ \psi(0) = 0, \qquad \psi(\phi) > 0, \quad \phi \in (0, K), \end{cases}$$

whose existence can be proved in a similar way as Lemma 2.2 in [25], together with the property that there exists $C_1 > 0$ such that

$$\psi_1(\phi) \le C_1\phi, \quad \phi \in (0, K).$$

Here $C_1 = C_1(c_0)$ is a constant depending on the upper bound of c, which is c_0, independent of r. The comparison principle tells us that

$$\psi_{c,r}(\phi) \le \psi_1(\phi) \le C_1\phi, \quad \phi \in (0, K). \tag{3.26}$$

Let $\varepsilon \in (0, \gamma)$ be a constant such that

$$\int_0^\varepsilon \phi^{m-1} d(\phi) d\phi < \frac{1}{4}\int_\varepsilon^\gamma \phi^{m-1}(b(\phi) - d(\phi)) d\phi. \tag{3.27}$$

It is easy to check that ε (and γ) only depends on the structure of $b(s)$ and $d(s)$. Using the estimates (3.24) and (3.25), we assert that there exists a constant $r_0 > 0$ such that if $\psi_{c,r}$ is the solution of (3.23) and $r \le r_0$, $cr \le r_0$, then $b(\phi_{cr}) - d(\phi) > 0$ for all $\phi \in (\varepsilon, K)$. (a) For $\phi \in (\gamma, K)$, we have

$$\psi_{c,r} = Dm\phi^{m-1}\phi'(t) < \tilde{\psi} = \frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{c},$$

that is, $\phi'(t) < (b(\phi) - d(\phi))/c$, and

$$cr = \int_{\phi_{cr}}^{\phi} \frac{Dms^{m-1}}{\psi_{c,r}(s)} ds > \int_{\phi_{cr}}^{\phi} \frac{c}{b(s) - d(s)} ds \ge \int_{\phi_{cr}}^{\phi} \frac{c}{(d'(K) - b'(K))(K - s)} ds$$

since $(b - d)$ is concave and $b(K) - d(K) = 0$. It follows that

$$K - \phi_{cr} \le (K - \phi)e^{(d'(K)-b'(K))r},$$

which together with the fact that $b(K) - d(K) = 0$ and $b'(K) < d'(K)$, yields $b(\phi_{cr}) - d(\phi) > 0$ for $r \le r_0$ with some constant $r_0 > 0$. (b) For $\phi \in (\varepsilon, \gamma)$, we see that

$$\inf_{\phi \in (\varepsilon,\gamma)} (b(\phi) - d(\phi)) =: \delta > 0$$

and

$$cr = \int_{\phi_{cr}}^{\phi} \frac{Dms^{m-1}}{\psi_{c,r}(s)} ds > \int_{\phi_{cr}}^{\phi} \frac{Dms^{m-1}}{\sup_{\phi \in (0,K)} \tilde{\psi}(\phi)} ds.$$

A direct calculation shows that $b(\phi) - b(\phi_{cr}) \le \delta/2$ for $cr \le r_0$ if we choose r_0 sufficiently small. Then

$$b(\phi_{cr}) - d(\phi) = (b(\phi) - d(\phi)) - (b(\phi) - b(\phi_{cr})) \geq \frac{b(\phi) - d(\phi)}{2}, \quad \phi \in (\varepsilon, \gamma). \tag{3.28}$$

The first integral of (3.23) over $(0, K)$ implies that

$$\begin{aligned} c\int_0^K \psi_{c,r}(\phi)d\phi &= \int_0^K Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))d\phi \\ &= \int_0^\varepsilon + \int_\varepsilon^\gamma + \int_\gamma^K Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))d\phi \\ &\geq -\int_0^\varepsilon Dm\phi^{m-1}d(\phi)d\phi + \int_\varepsilon^\gamma Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))d\phi \\ &\geq \left(-\frac{1}{4} + \frac{1}{2}\right)\int_\varepsilon^\gamma Dm\phi^{m-1}(b(\phi) - d(\phi))d\phi, \end{aligned}$$

where we have used (3.27) and (3.28). Additionally, (3.26) tells us

$$\int_0^K \psi_{c,r}(\phi)d\phi \leq C_1\int_0^K \phi d\phi = \frac{C_1K^2}{2}.$$

Now, if we choose $c_0 > 0$ such that

$$\frac{C_1K^2}{2}c_0 < \frac{1}{4}\int_\varepsilon^\gamma Dm\phi^{m-1}(b(\phi) - d(\phi))d\phi,$$

then there cannot be any monotone increasing traveling wave with speed $c < c_0$ and $r \leq r_0$. The proof is completed. □

Proof of Theorem 2.4. Let

$$\Delta_c(\lambda) = b'(0)e^{-\lambda cr} - c\lambda - d'(0), \quad \lambda > 0.$$

For any $c > 0$, $\Delta_c(0) = b'(0) - d'(0) > 0$, $\Delta_c(\lambda)$ is strictly decreasing with respect to λ, and

$$\Delta_c((b'(0) - d'(0))/c) \leq b'(0) - (b'(0) - d'(0)) - d'(0) = 0.$$

Hence, there exists an unique $\lambda_1 > 0$ such that $\Delta_c(\lambda_1) = 0$, $\Delta_c(\lambda) > 0$ for all $\lambda \in (0, \lambda_1)$, and $\Delta_c(\lambda) < 0$ for all $\lambda \in (\lambda_1, +\infty)$. The comparison principle Lemma 3.4, the construction of a pair of upper and lower solutions by (3.10), the iteration procedure Lemma 3.9 are still valid for nonlocal case for any $c > 0$. The nonexistence result can be proved in a way similar to the proof of Theorem 2.3 in next section. The last assertion of this theorem is proved in Lemma 3.11. □

Remark. In order to prove the existence of critical wave with speed $\hat{c}$ for degenerate diffusion ($m \geq 2$) with nonlocal effect ($\alpha > 0$) and time delay, we have constructed upper solutions for all $c \geq \hat{c}$ of the form (3.14). We note that for the degenerate diffusion equation without nonlocal effect ($\alpha = 0$), the construction of upper solutions by (3.14) is invalid, so as the proof of existence of critical waves, since the function $C_m(c)$ defined in the proof of Lemma 3.10 vanishes as c tends to 0. This suggests the nonexistence of positive infimum of critical traveling wave speeds in this case.

4. Nonexistence of traveling waves

We present the following proof of nonexistence theorem of traveling wave solutions with speed $c < \hat{c}$ for the degenerate diffusion equation with time-delay and nonlocal effect.

Proof of Theorem 2.3. Suppose $\phi(t)$ is a monotone increasing traveling wave solution of (2.2) and

$$|\phi(t) - Ke^{\lambda_1 t}| \leq Ce^{\lambda t}, \qquad t < 0, \tag{4.1}$$

for some constants $\lambda > \lambda_1 > 0$ and $C > 0$. Note that

$$\begin{cases} c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) = H(\phi)(t), & t \in \mathbb{R}, \\ \lim_{t\to-\infty} \phi(t) = 0, \quad \lim_{t\to+\infty} \phi(t) = K. \end{cases} \tag{4.2}$$

According to Lemma 3.2, for any given $c < \hat{c}(\alpha, r)$, the continuous function $\Delta_c(\lambda) > 0$ for all $\lambda \geq 0$, $\Delta_c(\lambda)$ is increasing for large λ. It follows that

$$\Delta_c(\lambda) = b'(0)e^{\alpha\lambda^2 - \lambda cr} - \lambda c - d'(0) \geq \min_{\lambda\geq 0} \Delta_c(\lambda) = \Delta_0 > 0, \qquad \lambda \geq 0.$$

Let $\varepsilon > 0$ be sufficiently small such that

$$(1-\varepsilon)b'(0)e^{\alpha\lambda^2 - \lambda cr} - \lambda c - d'(0) \geq \Delta_0 - \varepsilon b'(0)e^{\alpha\lambda^2 - \lambda cr} \geq \frac{1}{2}\Delta_0, \quad \lambda \in [0, 2\lambda_1]. \tag{4.3}$$

We may assume that $C \geq K$, then $Ke^{\lambda_1 t} - Ce^{\lambda t} \leq 0 \leq \phi(t)$ for $t \geq 0$, and $Ke^{\lambda_1 t} - Ce^{\lambda t} \leq \phi(t)$ for all $t \in \mathbb{R}$. Lemma 3.1 implies that

$$H(\phi)(t) \geq H(Ke^{\lambda_1 t} - Ce^{\lambda t})(t).$$

Similar to the proof of (3.13) in Lemma 3.8, we have

$$\begin{aligned} H(\phi)(t) &\geq H(Ke^{\lambda_1 t} - Ce^{\lambda t})(t) \\ &\geq b'(0)Ke^{\alpha\lambda_1^2 - \lambda_1 cr}e^{\lambda_1 t} - b'(0)Ce^{\alpha\lambda^2 - \lambda cr}e^{\lambda t} - B_0 e^{2\lambda_1 t} \\ &\geq b'(0)Ke^{\alpha\lambda_1^2 - \lambda_1 cr}e^{\lambda_1 t} - \hat{C}e^{\hat{\lambda} t}, \qquad t < 0, \end{aligned}$$

where $B_0 > 0$ is the constant in the proof of Lemma 3.8, $\hat{C} > 0$, $\hat{\lambda} = \min\{\lambda, 2\lambda_1\} > \lambda_1 > 0$. Let $t_0 < 0$, such that

$$\hat{C}e^{\hat{\lambda}t} \leq \varepsilon b'(0)Ke^{\alpha\lambda_1^2-\lambda_1 cr}e^{\lambda_1 t}, \qquad t < t_0.$$

Therefore, by (4.3),

$$\begin{aligned}
H(\phi)(t) \geq & b'(0)Ke^{\alpha\lambda_1^2-\lambda_1 cr}e^{\lambda_1 t} - \hat{C}e^{\hat{\lambda}t} \\
\geq & (1-\varepsilon)b'(0)Ke^{\alpha\lambda_1^2-\lambda_1 cr}e^{\lambda_1 t} \\
\geq & (\lambda_1 c + d'(0) + \frac{1}{2}\Delta_0)Ke^{\lambda_1 t} \\
\geq & (\lambda_1 c + d'(0) + \frac{1}{4}\Delta_0)(Ke^{\lambda_1 t} + Ce^{\lambda t}) \\
\geq & (\lambda_1 c + d'(0) + \frac{1}{4}\Delta_0)\phi(t), \qquad t < t_0,
\end{aligned}$$

provided that

$$(\lambda_1 c + d'(0) + \frac{1}{4}\Delta_0)Ce^{\lambda t} \leq \frac{1}{4}\Delta_0 Ke^{\lambda_1 t}, \qquad t < t_0.$$

We may take a smaller t_0 to let the above inequality hold since $\lambda > \lambda_1$. By (4.2), we see that

$$\begin{cases} c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) \geq (\lambda_1 c + d'(0) + \frac{1}{4}\Delta_0)\phi(t), & t < t_0, \\ \lim\limits_{t\to-\infty} \phi(t) = 0, \quad \phi(t_0) = \phi(t_0). \end{cases} \tag{4.4}$$

We note that there exists a constant $E > 0$ such that $d(s) \leq d'(0)s + Es^2$ for $s \in [0, K]$ since $d \in c^2([0, K])$. Therefore,

$$\begin{aligned}
d(\phi(t)) \leq & d'(0)\phi(t) + E(\phi(t))^2 \\
\leq & d'(0)\phi(t) + E(Ke^{\lambda_1 t} + Ce^{\lambda t})^2 \\
\leq & (d'(0) + \frac{1}{8}\Delta_0)\phi(t), \qquad t < t_0,
\end{aligned}$$

provided that

$$E(Ke^{\lambda_1 t} + Ce^{\lambda t})^2 \leq \frac{1}{8}\Delta_0(Ke^{\lambda_1 t} - Ce^{\lambda t}), \qquad t < t_0,$$

which is also valid if we choose a smaller t_0. We can rewrite (4.4) into

$$\begin{cases} c\phi'(t) - D(\phi^m(t))'' \geq (\lambda_1 c + \frac{1}{8}\Delta_0)\phi(t), & t < t_0, \\ \lim\limits_{t\to-\infty} \phi(t) = 0, \quad \phi(t_0) = \phi(t_0). \end{cases} \tag{4.5}$$

We assert that $\phi'(t) > 0$ for all $t < t_0$. Otherwise, if $\phi'(t_1) = 0$ for some $t_1 < t_0$, then $\phi^m(t)'|_{t=t_1} = 0$ and (4.4) implies

$$-D(\phi^m(t))''|_{t=t_1} \geq (\lambda_1 c + \frac{1}{8}\Delta_0)\phi(t_1) > 0.$$

It follows that $\phi^m(t)$ attains its strictly maximum at t_1, which contradicts to the monotone increasing of $\phi(t)$ and so as $\phi^m(t)$.

Now we employ the phase plane analysis to get more properties of ϕ. Since $\phi'(t) > 0$ for all $t < t_0$, let

$$\psi(t) = (\phi^m(t))' = m\phi^{m-1}\phi'(t) > 0, \qquad t < t_0. \tag{4.6}$$

Then the second order degenerate differential inequality (4.4) implies

$$\begin{cases} \dfrac{d\phi}{dt} = \dfrac{\psi(t)}{m\phi^{m-1}}, \\ D\dfrac{d\psi}{dt} \leq \dfrac{c\psi(t)}{m\phi^{m-1}} - (\lambda_1 c + \dfrac{1}{8}\Delta_0)\phi(t), \qquad t < t_0. \end{cases}$$

Note that $\phi'(t) > 0$ and set $\phi_0 = \phi(t_0)$. We can use $\phi(t)$ as the argument of $\psi(t)$ to get

$$\begin{cases} D\dfrac{d\psi(\phi)}{d\phi} \leq c - \dfrac{m(\lambda_1 c + \Delta_0/8)\phi^m}{\psi} =: G(\phi, \psi), \\ 0 < \phi < \phi_0, \quad \psi(\phi) > 0, \quad \psi(0) = 0. \end{cases} \tag{4.7}$$

Hereafter in this proof, we write $\psi(t)$ in the sense of (4.6) and we write $\psi(\phi)$ if we regard ψ as a function of ϕ, i.e. $\psi(\hat{\phi}) = \psi(\phi^{-1}(\hat{\phi}))$ for $\hat{\phi} \in (0, \phi_0)$ and $\phi^{-1}(\hat{\phi})$ is the inverse function of $\phi(t)$ since $\phi'(t) > 0$. Let Γ and Ω be the following curve and region in the (ϕ, ψ) phase plane

$$\Gamma = \{(\phi, \psi); 0 < \phi < \phi_0, \psi = \frac{m(\lambda_1 c + \Delta_0/8)\phi^m}{c}\},$$

$$\Omega = \{(\phi, \psi); 0 < \phi < \phi_0, 0 < \psi < \frac{m(\lambda_1 c + \Delta_0/8)\phi^m}{c}\}.$$

Clearly, if $(\phi, \psi) \in \Omega$, then $G(\phi, \psi) < 0$ and (4.7) shows

$$D\frac{d\psi(\phi)}{d\phi} \leq G(\phi, \psi) < 0.$$

The graph of $\psi(\phi)$ cannot run through Γ when ϕ grows, it must start from $(0, 0)$ and run into $(\mathbb{R}^+ \times \mathbb{R}^+)\backslash\Omega$ for $0 < \phi < \phi_1$ with some $\phi_1 \leq \phi_0$. That is,

$$\psi(\phi) \geq \frac{m(\lambda_1 c + \Delta_0/8)\phi^m}{c}, \quad 0 < \phi < \phi_1.$$

It follows that

$$\phi'(t) = \frac{\psi(t)}{m\phi^{m-1}} \geq \frac{(\lambda_1 c + \Delta_0/8)\phi}{c} = (\lambda_1 + \frac{\Delta_0}{8c})\phi(t), \quad t < t_1,$$

where $t_1 \leq t_0$ such that $\phi(t_1) = \phi_1$. That is,

$$(\log \phi(t))' \geq \lambda_1 + \frac{\Delta_0}{8c}, \quad t < t_1. \tag{4.8}$$

Integrating (4.8) over (t, t_1) yields

$$\log \phi(t_1) - \log \phi(t) \geq (\lambda_1 + \frac{\Delta_0}{8c})(t_1 - t),$$

which means

$$\phi(t) \leq \phi(t_1) e^{(\lambda_1 + \frac{\Delta_0}{8c})(t - t_1)}, \qquad t < t_1.$$

This contradicts to (4.1) since $\lambda_1 + \Delta_0/(8c) > \lambda_1$. $\square$

5. Numerical simulations of travelling waves

In this section, we numerically study travelling wavefronts of (1.4). We consider the Nicholson's blowflies equation with degenerate diffusion

$$\frac{\partial u}{\partial t} = D\Delta u^m - \delta u + p \int_{-\infty}^{+\infty} u(t - r, y) e^{-au(t-r,y)} f_\alpha(x - y) dy, \quad x \in \mathbb{R}, \ t > 0,$$

where $m > 1$, $\alpha \geq 0$. If $\alpha > 0$, the equation is nonlocal; while if $\alpha \to 0$, the equation reduces to a local one. It possesses two constant equilibria $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{p}{\delta} := K$. When $1 < \frac{p}{\delta} \leq e$, the birth rate function $b(u) = pue^{-au}$ is monotone increasing, and $b(u)$, $d(u)$ satisfy the hypotheses (H1)–(H3).

The travelling wavefront equation, derived in Section 2 is as follows

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - \delta\phi(t) + p \displaystyle\int_{-\infty}^{+\infty} \phi(t - cr - y) e^{-a\phi(t-cr-y)} f_\alpha(y) dy, \\ \phi(-\infty) = 0, \quad \phi(+\infty) = \frac{1}{a} \ln \frac{p}{\delta}. \end{cases} \tag{5.1}$$

Our aim is to examine the roles of varying biological parameters in the change of the existence, shape, and size of the respective waves. We solve the nonlinear travelling wavefront equations in Section 3 by using the finite element method and iteration technique.

The main difficulties of numerical simulation lie in: (i) degenerate diffusion $m > 1$; (ii) nonlinearity caused by both the nonlinear diffusion and the birth or death functions $b(u)$, $d(u)$; (iii) time delayed; (iv) nonlocal effect convolution for $\alpha > 0$; (v) unbounded domain.

Framework for nonlocal case $\alpha > 0$. In the proof of Theorem 2.1, we have theoretically proved that the iteration scheme presented in Lemma 3.9 admits a convergence sequence to the traveling wave solution if there are suitable upper and lower solutions. In numerical simulations, we only deal with bounded domain and the corresponding version is

$$\begin{cases} c\phi_i'(t) - D(\phi_i^m(t))'' + \delta\phi_i(t) = p\displaystyle\int_{-M}^{M} \phi_{i-1}(t-cr-y)e^{-a\phi_{i-1}(t-cr-y)} f_\alpha(y)dy, \\ \phi_i(-M)=0, \quad \phi_i(M)=K, \end{cases} \tag{5.2}$$

where $M>0$ is sufficiently large. We note that to deal with a larger speed c or time delay r, one must choose a larger M for the accuracy. The scheme (5.2) is nonlinear due to the degenerate diffusion. The existing method is not applicable, so we need to develop a modified iteration scheme. Consider the following second order differential problem

$$\begin{cases} c\phi'(t) - D(m\widetilde{\phi}^{m-1}\phi'(t))' + \delta\phi(t) = p\displaystyle\int_{-M}^{M} \widetilde{\phi}(t-cr-y)e^{-a\widetilde{\phi}(t-cr-y)} f_\alpha(y)dy, \\ \phi(-M)=0, \quad \phi(M)=K, \end{cases} \tag{5.3}$$

where $\widetilde{\phi}\in C^2(\overline{\mathbb{R}})$, $\widetilde{\phi}$ is monotone increasing and $\lim_{t\to-\infty}\widetilde{\phi}(t)=0$, $\lim_{t\to+\infty}\widetilde{\phi}(t)=K$. The solution is denoted by $\phi=F(\widetilde{\phi})$. If the operator F admits a fixed point ϕ such that $\phi=F(\phi)$ and ϕ is monotone increasing, we might regard it as an approximate solution of (5.1). For any given $\widetilde{\phi}$, we can use the finite element method to numerically solve the problem (5.3). Let $-M=t_0<t_1<t_2<\cdots<t_N=M$ be the partition with each step $\Delta x=2M/N$ and $\{\psi_j\}_{j=0}^N$ be the associated trigonometric basis function. For $\widetilde{\phi}(t)=\sum_{i=1}^{N-1}\widetilde{u}_i\psi_i(t)+K\psi_N(t)$ and $\phi(t)=\sum_{i=1}^{N-1}u_i\psi_i(t)+K\psi_N(t)$ with known $\{\widetilde{u}_i\}_{i=1}^{N-1}$ and unknown $\{u_i\}_{i=1}^{N-1}$, the FEM scheme of (5.3) is

$$c\langle\phi'(t),\psi_j(t)\rangle + mD\langle\widetilde{\phi}^{m-1}\phi'(t),\psi_j'(t)\rangle + \delta\langle\phi(t),\psi_j(t)\rangle = p\langle\widetilde{\phi}_{cr}e^{-a\widetilde{\phi}_{cr}} * f_\alpha, \psi_j(t)\rangle,$$

for $j=1,2,\ldots,N-1$, where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product and

$$\widetilde{\phi}_{cr}(t)=\widetilde{\phi}(t-cr)=\sum_{i=1+i_0}^{N-1+i_0}\widetilde{u}_i\psi_{i+i_0}(t)+K\psi_{N+i_0}(t)$$

with $i_0=cr/\Delta x$ (we may assume that i_0 is an integer). The search of fixed point can be carried on by iteration technique with shifting. We start with a test profile $\phi_0(t)$, for example, $Ke^{\lambda t}/(e^{\lambda t}+e^{-\lambda t})$ with some $\lambda>0$, and compute

$$\phi_i(t)=F(T(\phi_{i-1}(t))), \qquad i=1,2,\ldots,$$

where T is the shifting operator such that $\phi(t)=T(\widetilde{\phi}(t))=\widetilde{\phi}(t+k\Delta x)$ with the selection of k that makes $\phi(t^*)=1/2$ with some t^* in a fixed subinterval $[t_{i^*},t_{i^*+1}]$. We note that (i) if ϕ is a fixed point of $\phi=F(T(\phi))$, then $\phi=T(\phi)$ and ϕ is a fixed point of $\phi=F(\phi)$; (ii) without this shifting, the iteration might not be convergent and thus lead to inaccurate results. Illustrated example of the iteration process is shown in Fig. 1. Here and after, we only draw the graphic on a suitable subinterval of $[-M,M]$.

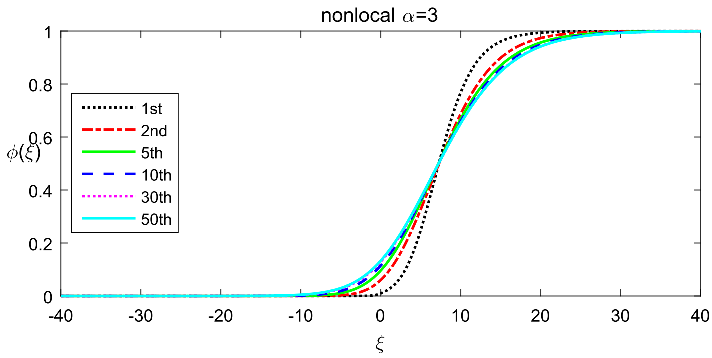

Fig. 1. The iteration process of the traveling wave for nonlocal case with parameters: $D = 1$, $m = 2$, $d = 1$, $p = 2$, $a = \log(p/d)$, $K = 1$, $\alpha = 3$, $r = 3$, $c = 2$, $M = 80$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Framework for local case $\alpha = 0$. Consider the following second order differential problem with artificial viscosities $-\mu\phi''$ on both sides of the equation

$$\begin{cases} c\phi'(t) - D(m\widetilde{\phi}^{m-1}\phi'(t))' - \mu\phi''(t) + \delta\phi(t) = p\widetilde{\phi}(t-cr)e^{-a\widetilde{\phi}(t-cr)} - \mu\widetilde{\phi}''(t), \\ \phi(-M) = 0, \quad \phi(M) = K, \end{cases} \tag{5.4}$$

where $\widetilde{\phi} \in C^2(\overline{\mathbb{R}})$, $\widetilde{\phi}$ is monotone increasing and $\lim_{t\to-\infty}\widetilde{\phi}(t) = 0$, $\lim_{t\to+\infty}\widetilde{\phi}(t) = K$, $\mu > 0$ is the regularization parameter. The solution of (5.4) is denoted by $\phi = G(\widetilde{\phi})$. Similar to the above nonlocal case, if the operator G admits a fixed point ϕ such that $\phi = G(\phi)$ and ϕ is monotone increasing, we may regard it as an approximate solution of (5.1) for $\alpha = 0$. For $\widetilde{\phi}(t) = \sum_{i=1}^{N-1}\widetilde{u}_i\psi_i(t) + K\psi_N(t)$ and $\phi(t) = \sum_{i=1}^{N-1}u_i\psi_i(t) + K\psi_N(t)$ with known $\{\widetilde{u}_i\}_{i=1}^{N-1}$ and unknown $\{u_i\}_{i=1}^{N-1}$, the FEM scheme of (5.4) is

$$\begin{aligned} &c\langle\phi'(t), \psi_j(t)\rangle + mD\langle\widetilde{\phi}^{m-1}\phi'(t), \psi_j'(t)\rangle + \mu\langle\phi'(t), \psi_j'(t)\rangle + \delta\langle\phi(t), \psi_j(t)\rangle \\ &\quad = p\langle\widetilde{\phi}_{cr}e^{-a\widetilde{\phi}_{cr}}, \psi_j(t)\rangle + \mu\langle\widetilde{\phi}'(t), \psi_j'(t)\rangle, \end{aligned}$$

for $j = 1, 2, \ldots, N-1$. The search of fixed point can also be carried on by iteration technique with shifting similar to the nonlocal case. Fig. 2 shows the comparison of the iteration process between the cases with and without regularization.

The difference between numerical simulations for nonlocal case and local case is that we add a regularization $-\mu\phi''(t)$ for local case. This is because the simulation without regularization in this case seems not to be convergent. This indicates that the nonlocal effect plays a role as non-degenerate diffusion, which consists with the observation of theoretical results we proved. See the illustrated example of the iteration process in Fig. 2. From the numerical simulation, we see that the iteration without regularization is ill conditioned when $\phi(\xi)$ tends to zero.

Existence and nonexistence. In Section 3 and Section 4, we have proved the existence and nonexistence of monotone traveling waves for degenerate diffusion with nonlocal effect corresponding to $c > \hat{c}(\alpha, r)$ and $c < \hat{c}(\alpha, r)$ and the existence of traveling waves for the local case. The numerical simulation for nonlocal case $\alpha > 0$ seems to admit both traveling waves for $c > \hat{c}$

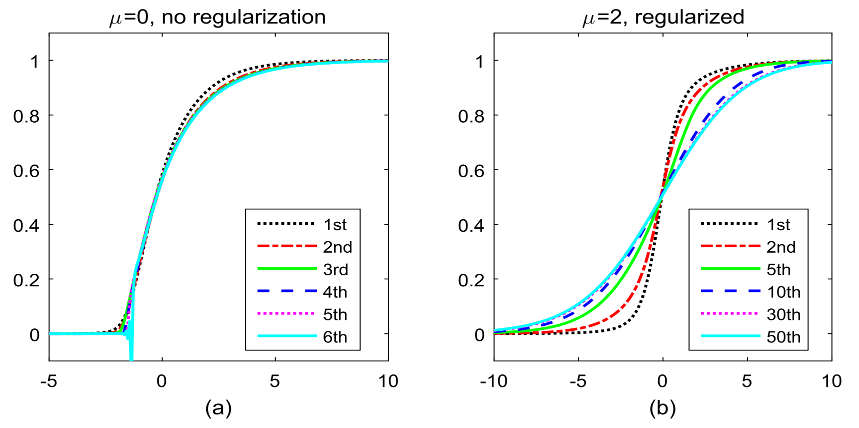

Fig. 2. The iterative process of the traveling wave for the local case: (a) the iteration without regularization does not converge; (b) the iteration with regularization converges. Here $D = 1$, $d = 1$, $m = 2$, $p = 2$, $a = \log(p/d)$, $K = 1$, $\alpha = 0$, $r = 4$, $c = 0.1$, $M = 80$.

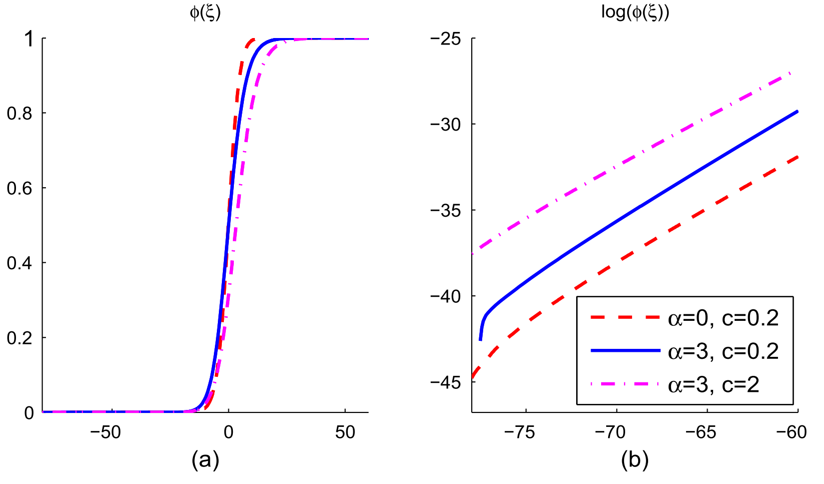

Fig. 3. (a) $\phi(\xi)$–ξ and (b) $\log(\phi(\xi))$–ξ graphics of (i) local case; (ii) nonlocal case with $c < \hat{c}$; (iii) nonlocal case with $c > \hat{c}$. We have theoretical proved the nonexistence in case (ii). Here $D = 1$, $d = 1$, $p = 2$, $m = 2$, $a = \log(p/d)$, $K = 1$, $r = 1$, $M = 80$.

and $c < \hat{c}$, as shown in Fig. 3(a). However, if we plot the graphics of $\log(\phi(\xi))$, there is an observable difference between them, see Fig. 3(b). There is a blank in the graphic of $\log(\phi(\xi))$ for the numerical solution in the nonlocal case with $c < \hat{c}$, which shows $\phi(\xi) \equiv 0$ for some interval (t_1, t_2). This is not the true solution as we have shown in Lemma 3.9 that all the monotone solution satisfies $\phi(t) > 0$ for all $t \in \mathbb{R}$. We note that for $t \to -\infty$, all of $\phi(t)$ and its derivatives are decaying to zero exponentially. This might be one of the reason that the numerical approach admits a false solution. Moreover, the numerical results in Fig. 3 illustrate that the degenerate diffusion equations of local and nonlocal cases with same parameters have an essential distinction in the existence of the monotone traveling waves.

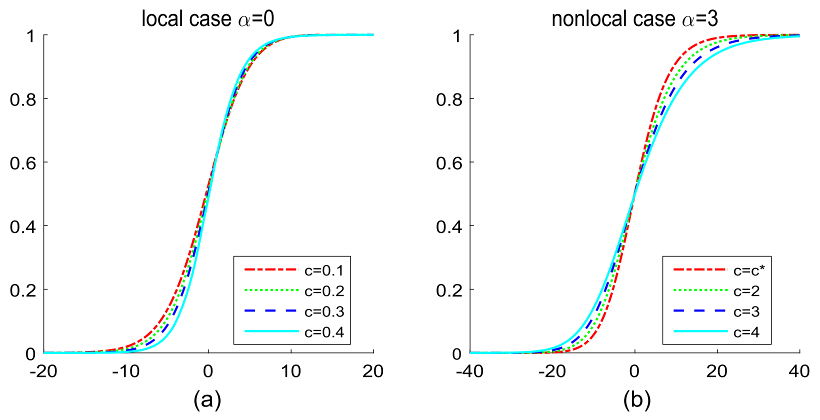

Fig. 4. The traveling waves of (a) local case with $c = 0.1, 0.2, 0.3, 0.4$ and (b) nonlocal case for $c = c^*, 2, 3, 4$ with the critical speed $c^* = 1.652987$.

Minimal traveling wave speed $c^*(D, \alpha, r)$ for nonlocal case. There is a positive infimum of critical traveling wave speeds $c^*(D, \alpha, r)$ for degenerate diffusion with nonlocal effect, see Theorem 2.1 and Theorem 2.3. The critical value (i.e. the infimum) $\hat{c}(\alpha, r)$ is determined by Lemma 3.2. That is, $\hat{c}$ is the unique positive number such that

$$\min_{\lambda \in (0,+\infty)} \Delta_{\hat{c}}(\lambda) = 0.$$

For any $c > 0$, let $\lambda^*(c)$ be the unique positive number such that $\frac{\partial}{\partial \lambda} \Delta_c(\lambda)|_{\lambda=\lambda^*(c)} = 0$, whose unique existence is proved in Lemma 3.2. Then $\hat{c}$ is the unique root of $\Delta_c(\lambda^*(c)) = 0$ for $c > 0$. Although we cannot give its explicit expression, we can solve it numerically.

Our numerical experiments show that the traveling wavefronts are monotone under the conditions of Theorem 2.1 and Theorem 2.4. This consists with the theoretical results obtained above. For simplicity, throughout this section we fix the parameters $D = \delta = 1$, $m = 2$, $a = \log(p/d)$, $K = 1$, and leave p, r, c and α free.

Numerical simulations – monotone case: (i) the effect of wave speed c. Consider the case where $1 < \frac{p}{\delta} \leq e$. We take the birth rate parameter $p = 2$, $M = 80$ and the time delay $r = 1$. Then we numerically observe travelling wave solutions for both nonlocal with wave speed c from the value $c = 0.1$ to $c = 0.4$ and local cases with wave speed c from the value $c = c^*$ and $c = 2$ to $c = 4$. Here the critical wave speed $c^* = 1.652987$ for $p = 2$, $\alpha = 3$, $r = 1$ as we numerically solved the characteristic equation. The numerical results are shown in Fig. 4. Fig. 4 illustrates the change of shapes of the traveling wavefronts for the local and nonlocal cases when the wave speed c varies. The front of the traveling wave becomes sharper and sharper in the local case as c increases, while it becomes more smooth when c is increasing in the nonlocal case within our simulations.

Numerical simulations – monotone case: (ii) the effect of time delay r. Consider the case where $1 < \frac{p}{\delta} \leq e$. We take $p = 2$ for the birth function, the wave speed $c = 0.2$ for local case, $c = 2$ for nonlocal case, and $M = 160$. Then we numerically observe travelling wave solutions for both nonlocal and local cases. We choose the time delay r from the value $r = 1$ to $r = 3$. The numerical results are shown in Fig. 5. Here we have chosen $\alpha = 0.1$ in the nonlocal case. There are two reasons concerned: (i) if we take α larger, the graphics are less distinguishable due to the

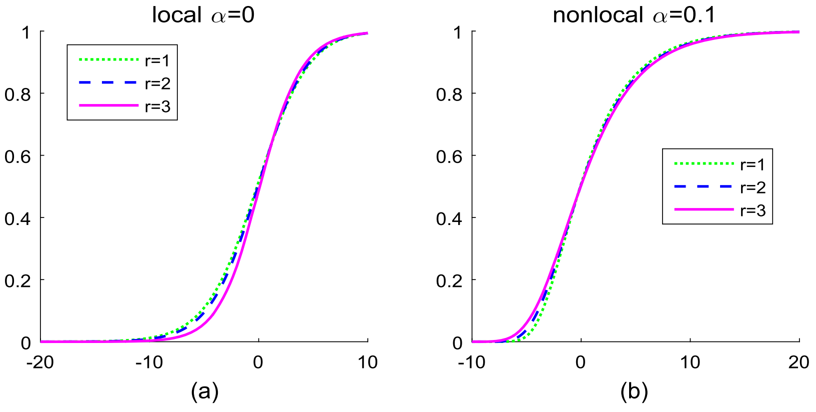

Fig. 5. The traveling waves of (a) local case with $c = 0.2$ and (b) nonlocal case with $c = 2$ for $r = 1, 2, 3$.

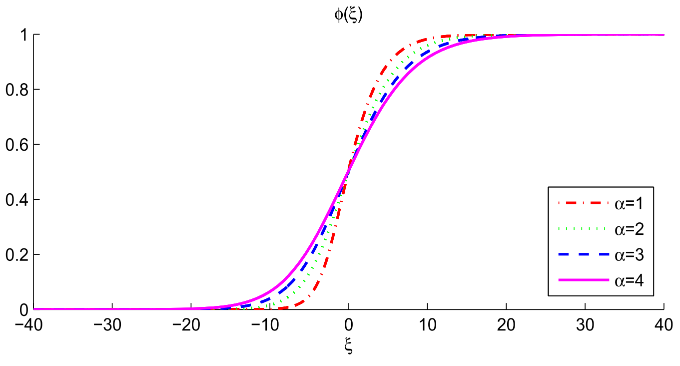

Fig. 6. The traveling waves of nonlocal case for $\alpha = 1, 2, 3, 4$.

nonlocal effect; (ii) this selection guarantees $c > \hat{c}$ for all $r = 1, 2, 3$, since both α and r affect the value of $\hat{c}$. In this case, the front of the travelling wave becomes sharper and sharper in the local case as the time delay r increases, while it becomes more smooth when the time delay r is increasing in the nonlocal case.

Numerical simulations – monotone case: (iii) the nonlocal effect α. Consider the case where $1 < \frac{p}{\delta} \leq e$. We take $p = 2$ for the birth function, time delay $r = 1$, $c = 2$ for nonlocal case, and $M = 160$. We choose the nonlocal parameter α from the value $\alpha = 1$ to $\alpha = 4$. The numerical results are shown in Fig. 6. The numerical results showed in Fig. 6 demonstrates that the larger nonlocal effect parameter α leads to the smoother wavefront.

Numerical simulations – non-monotone case. In this case, we consider the effect of the large birth rate parameter p on the existence of travelling wave solutions when the monotone condition is not satisfied. We take the wave speed $c = 0.2$ for local case and $c = 2$ for nonlocal case, the time delay $r = 1$, $M = 160$. We choose the birth rate parameter p from the value $p = 2$ to $p = 8$. The numerical results given in Fig. 7 shows that the solution $u(t, x)$ behaves like oscillatory traveling waves and the amplitude of the oscillatory traveling waves increases as p/δ increases.

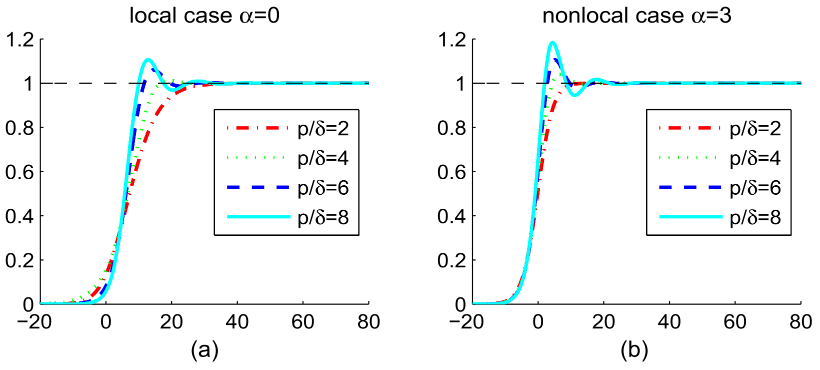

Fig. 7. The traveling waves of (a) local case with $c = 0.2$ and (b) nonlocal case with $c = 2$ and $\alpha = 1$ for $p/\delta = 2, 4, 6, 8$.

Numerical simulations – initial value problem. We consider the following initial value problem for local case $\alpha = 0$

$$\begin{cases} \dfrac{\partial v}{\partial t} = D\Delta v^m - \delta v + pv(t-r,x)e^{-av(t-r,x)}, & x \in \mathbb{R},\ t > 0, \\ v(s,x) = v_0(s,x), & x \in \mathbb{R},\ s \in [-r,0]. \end{cases} \tag{5.5}$$

Without loss of generality, we may always fix $\delta = 1$, $D = 1$, $m = 2$, $p = 2$, $a = \log(p/d)$, $K = 1$. The initial value is taken as

$$v_0(s,x) = \frac{K}{1+e^{-kx}} + 0.05(\sin x)e^{-0.01(x-50)^2}, \quad s \in [-r,0],$$

which implies $|v_0(s,x)| = O(1)e^{-k|x|}$ for $x \to -\infty$ and $|v_0(s,x) - K| = O(1)e^{-k|x|}$ for $x \to +\infty$ with $k > 0$.

We adopt the implicit finite difference approximation with a backward scheme for the time derivative and a central scheme for the spatial derivative to a finite computational domain $[-L, L]$. Here, we let $L = 400$ so that the computational domain is sufficiently large to avoid artificial numerical reflection. The sizes of the temporal step and spatial step are chosen as $\Delta t = 0.02$ and $\Delta x = 0.2$. We take the time delay $r = 0.1$, total time $T = 10$, and $k = 1$ in the initial data $v_0(s,x)$. The numerical simulations in Figs. 8 and 9 present the large time behavior of the solution of the degenerate reaction–diffusion equation with local birth rate (5.5). It converges to a stable monotone traveling wave.

Fig. 8(a) shows that, after a small initial oscillation, the solution $v(t,x)$ quickly behaves like a monotone traveling wave which moves in the negative x-direction as time increases (i.e., the wave speed $c > 0$). From the contour map shown in Fig. 8(b), we observe that the interface region of left and right states, v_- and v_+, moves in the negative x-direction as time increases. The contour lines are straight and the width of interface region at each time appears constant. We also can estimate the traveling speed for the solution $v(t,x)$ as $c \approx 1.0400$. In Fig. 9, the increasing shape of the solution $v(t,x)$, at different times $t = 2, 4, 6, 8, 10$ is the same and travels in the negative x-direction as time increases. These phenomena indicate that there is no change of the wave's shape for the sense of stability and the solution $v(t,x)$ converges time-asymptotically to the monotone traveling wave $\phi(x+ct)$ with $c \approx 1.0400$.

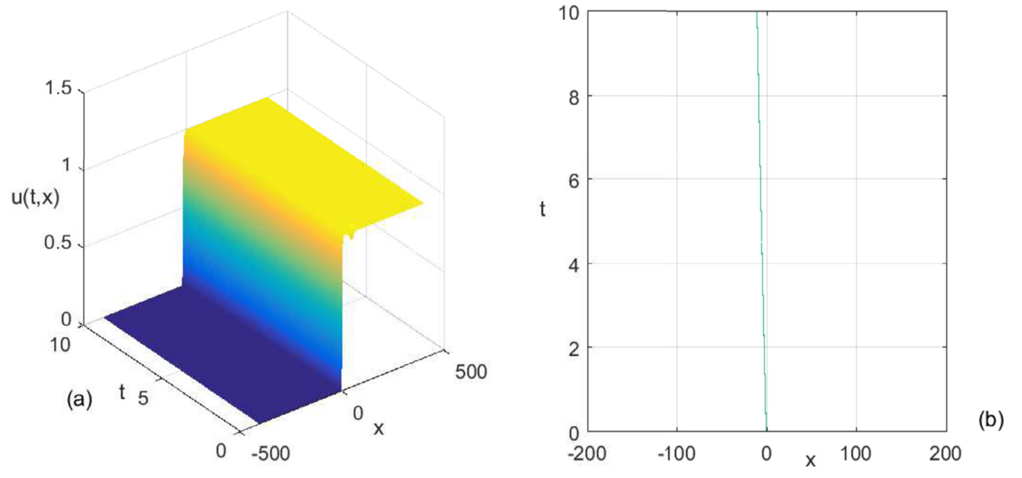

Fig. 8. The solution $v(t,x)$ behaves like a stable monotone traveling wave $\phi(x+ct)$ with small wave speed $c = 1.0400 > 0$. (a) Three-dimensional mesh of $v(t,x)$, and (b) the contour of $v(t,x)$.

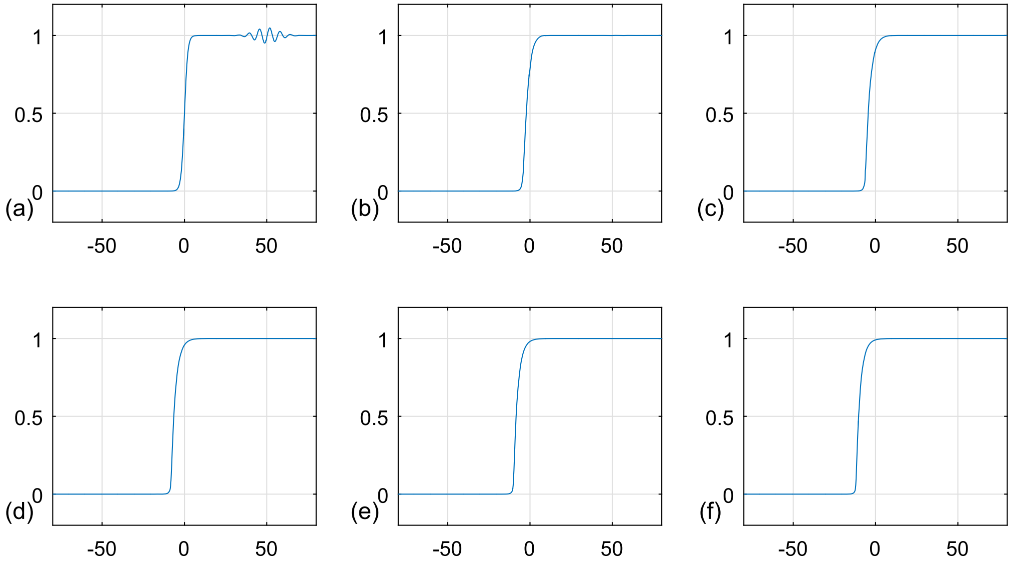

Fig. 9. From (a) to (f), the solution $v(t,x)$ plotted at times $t = 0, 2, 4, 6, 8, 10$, which behaves like a stable monotone traveling wave and moves in the negative x-direction as time increases.

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References

[1] S. Ai, Traveling wave fronts for generalized fisher equations with spatio-temporal delays, J. Differential Equations 232 (2007) 104–133.
[2] J. Al-Omari, S.A. Gourley, Monotone travelling fronts in an age-structured reaction–diffusion model of a single species, J. Math. Biol. 45 (2002) 294–312.
[3] J. Al-Omari, S.A. Gourley, Monotone wave-fronts in a structured population model with distributed maturation delay, IMA J. Appl. Math. 70 (2005) 858–879.
[4] D.G. Aronson, Density-dependent interaction–diffusion systems, in: W.E. Stewart, W.H. Ray, C.C. Conley (Eds.), Dynamics and Modelling of Reactive Systems, Academic Press, New York, New York, 1980, pp. 161–176.
[5] S. Bazazi, J. Buhl, J.J. Hale, M.L. Anstey, G.A. Sword, S.J. Simpson, I.D. Couzin, Collective motion and cannibalism in locust migratory bands, Curr. Biol. 18 (2008) 1–5.
[6] J. Buhl, D.J.T. Sumpter, I.D. Couzin, E.M. Despland, J.J. Hale, E. Miller, S.J. Simpson, From disorder to order in marching locusts, Science 312 (2006) 1402–1406.
[7] E.A. Carl, Population control in arctic ground squirrels, Ecology 52 (3) (1971) 395–413.
[8] I.L. Chern, M. Mei, X. Yang, Q. Zhang, Stability of non-monotone critical traveling waves for reaction–diffusion equations with time-delay, J. Dynam. Differential Equations 259 (2015) 1503–1541.
[9] E.H. Colombo, C. Anteneodo, Nonlinear population dynamics in a bounded habitat, J. Theoret. Biol. 446 (2018) 11–18.
[10] D. Duehring, W. Huang, Periodic traveling waves for diffusion equations with time delayed and non-local responding reaction, J. Dynam. Differential Equations 19 (2007) 457–477.
[11] J. Fang, J. Wei, X.Q. Zhao, Spatial dynamics of a nonlocal and time-delayed reaction–diffusion system, J. Differential Equations 245 (2008) 2749–2770.
[12] T. Faria, W. Huang, J. Wu, Traveling waves for delayed reaction–diffusion equations with non-local response, Proc. Roy. Soc. Sect. A 462 (2006) 229–261.
[13] T. Faria, S. Trofimchuk, Nonmonotone travelling waves in a single species reaction–diffusion equation with delay, J. Differential Equations 228 (2006) 357–376.
[14] S.A. Gourley, Travelling front solutions of a nonlocal fisher equation, J. Math. Biol. 41 (2000) 272–284.
[15] S.A. Gourley, Linear stability of traveling fronts in an age-structured reaction–diffusion population model, Quart. J. Mech. Appl. Math. 58 (2005) 257–268.
[16] S.A. Gourley, Y. Kuang, A stage structured predator-prey model and its dependence on maturation delay and death rate, J. Math. Biol. 49 (2004) 188–200.
[17] S.A. Gourley, J. Wu, Delayed nonlocal diffusive system in biological invasion and disease spread, Fields Inst. Commun. 48 (2006) 137–200.
[18] W.S.C. Gurney, R.M. Nisbet, The regulation of inhomogeneous population, J. Theoret. Biol. 52 (1975) 441–457.
[19] W.S.C. Gurney, S.P. Blythe, R.M. Nisbet, Nicholson's blowflies revisited, Nature 287 (1980) 17–21.
[20] M.E. Gurtin, R.C. MacCamy, On the diffusion of biological populations, Math. Biosci. 33 (1) (1977) 35–49.
[21] R. Huang, C. Jin, M. Mei, J. Yin, Existence and stability of traveling waves for degenerate reaction–diffusion equation with time delay, J. Nonlinear Sci. 28 (2018) 1011–1042, https://doi.org/10.1007/s00332-017-9439-5.
[22] R. Huang, M. Mei, Y. Wang, Planar traveling waves for nonlocal dispersion equation with monostable nonlinearity, Discrete Contin. Dyn. Syst. 32 (2012) 3621–3649.
[23] R. Huang, M. Mei, K. Zhang, Q. Zhang, Asymptotic stability of non-monotone traveling waves for time-delayed nonlocal dispersion equations, Discrete Contin. Dyn. Syst. 36 (2016) 1331–1353.
[24] S.M. Ji, J.X. Yin, R. Huang, Oscillatory traveling waves of polytropic filtration equation with generalized Fisher-KPP sources, J. Math. Anal. Appl. 419 (2014) 68–78.
[25] C.H. Jin, J.X. Yin, Traveling wavefronts for a time delayed non-Newtonian filtration equation, Phys. D 241 (2012) 1789–1803.
[26] Y. Kuang, S.A. Gourley, Wavefronts and global stability in a time-delayed population model with stage structure, Proc. R. Soc. Lond. Ser. A 459 (2003) 1563–1579.

[27] G. Li, M. Mei, Y.S. Wong, Nonlinear stability of travelling wavefronts in an agestructured reaction–diffusion population model, Math. Biosci. Eng. 5 (2008) 85–100.
[28] W.T. Li, S. Ruan, Z.C. Wang, On the diffusive Nicholson's blowflies equation with nonlocal delay, J. Nonlinear Sci. 17 (2007) 505–525.
[29] D. Liang, J. Wu, Travelling waves and numerical approximations in a reaction advection diffusion equation with nonlocal delayed effects, J. Nonlinear Sci. 13 (2003) 289–310.
[30] X. Liang, X.Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, Comm. Pure Appl. Math. 60 (2007) 1–40.
[31] C.Y. Kao, Y. Lou, Random dispersal vs. nonlocal dispersal, Discrete Contin. Dyn. Syst. 26 (2010) 551–596.
[32] S. Ma, Traveling waves for non-local delayed diffusion equations via auxiliary equations, J. Differential Equations 237 (2007) 259–277.
[33] M. Mei, C.K. Lin, C.T. Lin, Joseph W.-H. So, Traveling wavefronts for time-delayed reaction–diffusion equation: (i) local nonlinearity, J. Differential Equations 247 (2009) 495–510.
[34] M. Mei, C.K. Lin, C.T. Lin, Joseph W.-H. So, Traveling wavefronts for time-delayed reaction–diffusion equation:(ii) nonlocal nonlinearity, J. Differential Equations 247 (2009) 511–529.
[35] M. Mei, C. Ou, X.Q. Zhao, Global stability of monostable traveling waves for nonlocal time-delayed reaction–diffusion equations, SIAM J. Math. Anal. 42 (2010) 2762–2790.
[36] M. Mei, J.W.-H. So, M. Li, S. Shen, Asymptotic stability of traveling waves for the Nicholson's blowflies equation with diffusion, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 579–594.
[37] M. Mei, Y.S. Wong, Novel stability results for traveling wavefronts in an age-structured reaction–diffusion equations, Math. Biosci. Eng. 6 (2009) 743–752.
[38] M. Mei, J.W.-H. So, Stability of strong travelling waves for a non-local time-delayed reaction–diffusion equation, Proc. Roy. Soc. Edinburgh Sect. A 138 (2008) 551–568.
[39] J.A.J. Metz, O. Diekmann, The Dynamics of Physiologically Structured Populations, Springer, New York, 1986.
[40] M. Morisita, Measuring of habitat value by "environmental density" method, in: G.P. Patil, E.C. Pielou, W.E. Waters (Eds.), Statistical Ecology 1. Spatial Patterns and Statistical Distributions, vol. 1, Pennsylvania State University Press, University Park, 1971, p. 379.
[41] J.D. Murry, Mathematical Biology I: An Introduction, Springer, New York, USA, 2002.
[42] W.I. Newman, Some exact solutions to a nonlinear diffusion problem in population genetics and combustion, J. Theoret. Biol. 85 (1980) 325–334.
[43] A. Okubo, S.A. Levin, Diffusion and Ecological Problems: Modern Perspectives, Springer Science Business Media, 2013.
[44] K.W. Schaaf, Asymptotic behavior and traveling wave solutions for parabolic functional differential equations, Trans. Amer. Math. Soc. 302 (1987) 587–615.
[45] N. Shigesada, Spatial distribution of dispersing animals, J. Math. Biol. 9 (1980) 85–96.
[46] N. Shiguesada, K. Kawasaki, E. Teramoto, Spatial segregation of interacting species, J. Theoret. Biol. 79 (1979) 83–99.
[47] J.W-H. So, J. Wu, X. Zou, A reaction–diffusion model for a single species with age structure travelling wavefronts on unbounded domains, Proc. R. Soc. Lond. Ser. A 457 (2012) 1841–1853.
[48] J.W-H. So, Y. Yang, Dirichlet problem for the diffusive Nicholson's blowflies equation, J. Differential Equations 150 (1998) 317–348.
[49] A. Stevens, H.G. Othmer, Aggregation, blowup, and collapse: the abc's of taxis in reinforced random walks, SIAM J. Appl. Math. 57 (4) (2001) 1044–1081.
[50] H.R. Thieme, X.Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction–diffusion models, J. Differential Equations 195 (2003) 430–470.
[51] J.L. Vàzquez, The Porous Medium Equation: Mathematical Theory, Oxford Univ. Press, 2006.
[52] H. Wang, On the existence of traveling waves for delayed reaction–diffusion equations, J. Differential Equations 247 (2009) 887–905.
[53] W. Wang, X.Q. Zhao, A nonlocal and time-delayed reaction–diffusion model of dengue transmission, SIAM J. Appl. Math. 71 (2011) 147–168.
[54] Z.C. Wang, W.T. Li, S. Ruan, Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay, J. Differential Equations 238 (2008) 153–200.
[55] P. Weng, H. Huang, J. Wu, Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction, IMA J. Appl. Math. 68 (2003) 409–439.
[56] Y. Wu, X. Xing, Stability of traveling waves with critical speeds for p-degree fisher-type equations, Discrete Contin. Dyn. Syst. 20 (2008) 1123–1139.

[57] Z. Wu, J. Zhao, J. Yin, H. Li, Nonlinear Diffusion Equations, World Scientific Publishing Co. Put. Ltd., 2001.
[58] T. Xu, S. Ji, C. Jin, M. Mei, J. Yin, Early and late stage profile for a new chemotaxis model with density-dependent jump probability and quorum-sensing mechanisms, arXiv:1711.08114, 2017.
[59] T. Xu, S. Ji, M. Mei, J. Yin, Global existence of solutions to a chemotaxis-haptotaxis model with density-dependent jump probability and quorum-sensing mechanisms, Math. Methods Appl. Sci. (2018) 1–19.
[60] T. Xu, C. Jin, S. Ji, Discontinuous traveling wave entropy solutions of a modified Allen–Cahn model, Acta Math. Sci. 37B (2017) 1–21.
[61] T. Xu, J. Yin, S. Ji, Discontinuous traveling wave entropy solutions for a sedimentation–consolidation model, Nonlinear Anal. 159 (2017) 468–481.
[62] X. Zou, Delay induced traveling wave fronts in reaction diffusion equations of Kpp-Fisher type, J. Comput. Appl. Math. 146 (2002) 309–321.