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Global existence of solutions to a chemotaxis-haptotaxis model with density-dependent jump probability and quorum-sensing mechanisms

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In this paper, we first derive a new chemotaxis-haptotaxis model of cancer invasion of tissue with density-dependent jump probability and quorum-sensing mechanisms, which is with degeneracy in diffusion. In the presence of generic logistic damping, we then prove the global existence of weak solutions. The approach adopted is the compactness analysis with Moser-type iteration and artificial viscosity-vanishing technique.

KEYWORDS
cancer invasion model, chemotaxis, global existence, haptotaxis, logistic source, nonlinear diffusion

1 | INTRODUCTION

Cancer invasion consists of several important steps involving different biological mechanisms, and a variety of mathematical models have been developed for various aspects of cancer invasion. Tao and Winkler[1] first derived the chemotaxis-haptotaxis model:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (D(u)\nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1-u-w), \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, \\ \dfrac{\partial w}{\partial t} = -wv, \qquad x \in \Omega, \quad t > 0, \end{cases} \tag{1}$$

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in a bounded smooth domain $\Omega \subset \mathbb{R}^n$, where u, v, and w denote the relative density of cells, the concentration of matrix degrading enzymes (MDE), and the density of extracellular matrix (ECM), respectively, and the diffusivity is $D(u) = \delta(u+1)^{m-1}$. They showed the global existence of a unique classical solution to the above-mentioned model (1) by developing some L_p-estimate techniques. The conclusion is that large values of m seem to enhance the tendency towards the global solvability. When $m > 2 - 2/n$, Wang[2] further obtained the existence of global-in-time solutions for the system (1). It is worthy of mentioning that the diffusion coefficient in their studies was nonlinear but eventually assumed to be nondegenerate. However, biological experiments suggest that no cell migration (in particular no diffusivity) occurs in regions where the tissue is absent.[3] To account for this biological feature, various taxis models with degenerate diffusion have been paid more attention during the last decades. They describe the model for chemosensitive movement,[4,10] moving towards the gradient of nondiffusible signals (haptotaxis),[11,12] or incorporating both chemotaxis and haptotaxis effect.[13-16] Particularly, in Li and Lankeit,[14] it is proved that, for sufficiently regular initial data, the bounded solutions of (1) time globally exist for the cases of nondegenerate diffusion and degenerate diffusion whenever $m > 2-2/n$ with $n = 2, 3, 4$. Furthermore, the existence of a unique global classical solution for the nondegenerate diffusion of (1) and a global weak solution for degenerate case in the 2 space dimensions were investigated in Zheng et al,[16] recently.

On the other hand, Painter and Hillen[17] proposed the transition probability method to model the movement of cell population. They introduced volume filling approach combining neighbour- and gradient-based rules; that is, particles have a finite volume and that cells cannot move into regions that are already filled by other cells. In general, the jump probabilities depend on a variety of environmental factors (eg, other cell populations[18] or chemicals). Painter and Sherratt[19] further presented 4 different sensing strategies. Cell movement involves the processing of multiple signals, each of them may act on the cell in different ways. Inspired by the idea of Painter et al,[17,19] recently in Xu et al,[20] we derived the new chemotaxis model with density-dependent jump probability and quorum-sensing mechanisms combining the strictly local and gradient-based strategies.

Subsequent to Xu et al,[20] in this paper, we first derive the new chemotaxis-haptotaxis model of cancer invasion with density-dependent jump probability and quorum-sensing mechanisms (for details of how to derive the new model, we refer to the next section):

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_u \Delta(q(u)u) - \bar{\chi}_v \nabla\cdot(\phi_v(u)q(u)u\nabla v) - \bar{\chi}_w \nabla\cdot(\phi_w(u)q(u)u\nabla w) + ru^{\sigma}(1-\mu u - w), \\ \dfrac{\partial v}{\partial t} = D_v \Delta v - \gamma v + \xi u, \\ \dfrac{\partial w}{\partial t} = -\lambda w v, \qquad x \in \Omega, \quad t > 0, \end{cases} \tag{2}$$

where, as mentioned before, u, v, and w denote the relative density of cells, the concentration of MDE, and the density of ECM, respectively, and γ, ξ, and λ represent the decay rate of MDE, the production rate of MDE, the decay rate of ECM causing by MDE, respectively; $\bar{\chi}_v$ and $\bar{\chi}_w$ measure the chemotactic and haptotactic sensitivity, respectively. Parameters D_u, D_v, r, and μ are the cell diffusion coefficient, the chemical diffusion coefficient, the proliferation rate, and reciprocal of carrying capacity, respectively, and where $\Omega \subset \mathbb{R}^n$ with $n \geqslant 1$ denotes the physical domain under consideration. $q(u)$ denotes the jump probability of a cell depending on the population pressure at its present location, which is increasing with respect to u with the following properties:

$$q(0) = 0, \qquad q(1) = 1,$$

namely, the jump probability is 1 when the cell density exceeds maximum and it is zero when the cell density is zero. $\phi_v(u)$ and $\phi_w(u)$ are the density-dependent chemotactic and haptotactic functions responding to quorum-sensing mechanisms, respectively, while $\phi_v(u)$ can be sign-changing representing the phenomenon that some chemicals have been shown to elicit both attractive and repellent responses.[21,22] Moreover, some reasonable structure conditions on $\phi_v(s)$, $\phi_w(s)$, and $q(s)$ are also required in discussing the existence of solutions, which we leave in Section 2 after the formulation of this model. Without loss of generality, throughout the paper, we assume the following positive coefficients as

$$\gamma = \xi = \lambda = r = \bar{\chi}_v = \bar{\chi}_w = D_u = D_v = 1$$

for simplification. The second purpose of the paper is to establish the global existence of weak solutions to the system (2) by the energy estimates with artificial viscosity, Moser-type iteration, and the compactness analysis with viscosity-vanishing technique. The main difficulty is the degeneracy of diffusion for the system (2), which causes the solutions lack the basic regularities, and we have to treat it carefully by the viscosity-vanishing method.

The rest of the paper is organized as follows. In Section 2, we derive the new chemotaxis-haptotaxis model and state the main theorem. Section 3 is devoted to the proof of global existence of weak solutions to the corresponding chemotaxis system.

2 | FORMULATIONS AND MAIN RESULTS

In this section, we first derive a new chemotaxis-haptotaxis model with degenerate diffusion and density-dependent chemotactic and haptotaxis sensitivity; then we state our main results on the global existence of the weak solution to the new model.

Chaplain and Lolas[23] introduced a model for tumor invasion mechanism, which describes tumor invasion phenomenon in accounting for the role of a diffusive chemical substance, the so-called MDE, which decays nondiffusive static healthy tissue (ECM). In this model, both the enzyme and the healthy tissue can attract the cancer cells in the sense that the cancer cells bias their movement along the gradients of the concentrations of both ECM and MDE, where these processes, namely, taxis toward nondiffusible and diffusible quantity, are usually referred as haptotaxis and chemotaxis. Using the similar modeling approach mentioned in Xu et al,[20] we extend the Chaplain and Lolas model to a new one, incorporating haptotaxis and chemotaxis effect on the cell movement, ie, the transitional-probabilities

$$\mathcal{T}_i^{\pm} = q(u_i)(\alpha + \beta_v(z_i)(\tau_v(v_{i\pm1}) - \tau_v(v_i)) + \beta_w(z_i)(\tau_w(w_{i\pm1}) - \tau_w(w_i))),$$

where $\beta_v(z)$ and $\beta_w(z)$ are chemotactic and haptotactic functions responding to quorum-sensing mechanisms, respectively. By the similar process in Xu et al,[20] following the approach of Stevens and Othmer[24] (see also[25,26]), we get the following model:

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 (q(u)u)}{\partial x^2} - \frac{\partial}{\partial x}\left(\chi_v(v)\beta_v(z)q(u)u\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}\left(\chi_w(w)\beta_w(z)q(u)u\frac{\partial w}{\partial x}\right),$$

where $\chi_v(v) = 2k\dfrac{d\tau(v)}{dv}$ and $\chi_w(w) = 2k\dfrac{d\tau(w)}{dw}$ are the functions of chemotaxis and haptotaxis sensitivities, respectively. Furthermore, we assume that there is a linear dependence for τ on signal concentration, ie, $\chi_v(v) = \bar{\chi}_v$ and $\chi_w(w) = \bar{\chi}_w$, where $\bar{\chi}_v$ and $\bar{\chi}_w$ are constants. Apart from that, we consider a modification of the Verhulst logistic growth term to model organ size evolution introduced by Blumberg[27] and Turner et al,[28] which is called hyperlogistic function, accordingly

$$f(u) = ru^{\sigma}(1 - \mu u - w).$$

In the special case where the quorum sensing molecule z not diffusing and a monotone increasing function of the cell density, $z = z(u)$. Denote $\beta_v(z) = \beta_v(z(u)) := \phi_v(u)$, $\beta_w(z) = \beta_v(z(u)) := \phi_w(u)$. Assume that the attractive effect of haptotaxis concentration w is weaken with the increasing concentration of z; namely, β_w is a nonnegative and nonincreasing function. And z switches the response to chemotaxis concentration v from attractant at low concentrations of v to repellent at high concentrations; namely, β_v is a sign-changing and nonincreasing function, eg, $\beta_v(z) = 1 - z/z*$.[17,29] Including cell kinetics and signal dynamics, we derive the resulting model for the cell movement:

$$\frac{\partial u}{\partial t} = \underbrace{D_u\Delta(q(u)u)}_{\text{dispersion}} - \underbrace{\bar{\chi}_v\nabla\cdot(\phi_v(u)q(u)u\nabla v)}_{\text{chemotaxis}} - \underbrace{\bar{\chi}_w\nabla\cdot(\phi_w(u)q(u)u\nabla w)}_{\text{haptotaxis}} + \underbrace{ru^{\sigma}(1-\mu u - w)}_{\text{proliferation}}.$$

Incorporating the kinetic equation of ECM and MDE, we arrive at a modified Chaplain and Lolas' chemotaxis-haptotaxis model (2).

Since degenerate diffusion equation may not have classical solutions in general, we need to formulate the following definition of weak solutions.

Definition 2.1. Let $T \in (0, \infty)$. A triple (u, v, w) is said to be a weak solution to the problem (2) in $Q_T = \Omega \times (0, T)$ if

(i) $u \in L^\infty(Q_T)$, $\nabla(q(u)u) \in L^2((0,T); L^2(\Omega))$, and $q(u)u_t \in L^2((0,T); L^2(\Omega))$;
(ii) $v \in L^\infty(Q_T) \cap L^2((0,T);W^{2,2}(\Omega)) \cap W^{1,2}((0,T);L^2(\Omega))$;
(iii) $w \in L^\infty(Q_T)$, $w_t \in L^2((0,T); L^2(\Omega))$;
(iv) The identities

$$\begin{aligned}&\int_0^T\int_\Omega u\varphi_t dxdt + \int_\Omega u_0\varphi(x,0)dx\\&= \int_0^T\int_\Omega \nabla(q(u)u)\cdot\nabla\varphi dxdt - \int_0^T\int_\Omega \phi_v(u)q(u)u\nabla v\cdot\nabla\varphi dxdt\\&\quad - \int_0^T\int_\Omega \phi_w(u)q(u)u\nabla w\cdot\nabla\varphi dxdt - \int_0^T\int_\Omega \mu u^\sigma(1-u-w)\varphi dxdt,\end{aligned}$$

and

$$\int_0^T\int_\Omega v_t\psi dxdt + \int_0^T\int_\Omega \nabla v\cdot\nabla\psi dxdt = \int_0^T\int_\Omega (u-v)\psi dxdt,$$

and

$$\int_0^T\int_\Omega w_t\psi dxdt = -\int_0^T\int_\Omega wz\psi dxdt$$

hold for all $\varphi,\psi \in L^2((0,T);W^{1,2}(\Omega)) \cap W^{1,2}((0,T);L^2(\Omega))$ with $\varphi(x,T)=0$, $x\in\Omega$;
(v) (v,w) takes the value (v_0,w_0) in the sense of trace at $t=0$.
If (u, v, w) is a weak solution of (2) in Q_T for any $T \in (0,\infty)$, then we call it a global weak solution.

Throughout this paper, we assume that

(H1) $q(u) = u^{m-1}$, $m>1$, $\sigma>m$, $\mu>0$;
(H2) u_0,v_0, and w_0 are nonnegative functions, $u_0 \in C^0(\bar{\Omega})$, $v_0 \in W^{2,\infty}(\Omega)$, $w_0 \in C^{2+\theta}(\bar{\Omega})$ with $\theta \in (0,1)$, and $\frac{\partial w_0}{\partial \nu} = 0$ on $\partial\Omega$;
(H3) $\phi_v(s)$ and $\phi_w(s)$ are continuously differentiable with

$$|\phi_v(s)| \le 1, \quad |\phi_v'(s)| \le 1, \quad 0 \le \phi_w(s) \le 1, \quad |\phi_v'(s)| \le 1.$$

Theorem 2.1. *Under the above assumptions (H1)-(H3), the problem (2) admits a global weak solution (*u,v, w*), satisfying that there exists a constant* C *such that*

$$\sup_{t\in\mathbb{R}^+}\left\{\|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)} + \|w\|_{W^{1,\infty}(\Omega)}\right\} \le C,$$

and $v \in L^2((0,T); W^{2,2}(\Omega))$, $u^m \in L^2((0,T);W^{1,2}(\Omega))$, $u^{\frac{m+1}{2}} \in W^{1,2}((0,T);L^2(\Omega))$ *for any* $T \in (0,\infty)$.

Remark 2.1. If $\sigma = m$ and μ is sufficiently large, then the same result in the theorem is also valid.

3 | PROOF OF THE MAIN RESULTS

We prove the existence of a global weak solution in this section. We first use the artificial viscosity method to get smooth approximate solutions. Despite the absence of comparison principle, we can prove a special case compared with a lower solution, which is helpful for establishing the regularity estimates. By making use of the special structure of dispersion, we carry on the estimates on u^m in $W^{1,2}(Q_T)$, instead of u. These energy estimates ensure the global existence of weak solution.

Consider the following corresponding regularized problem:

$$\begin{cases} u_t = \nabla\cdot(m(a_\varepsilon(u))^{m-1}\nabla u) - \nabla\cdot(u^m\phi_v(u)\nabla v) - \nabla\cdot(u^m\phi_w(u)\nabla w) + \mu|u|^{\sigma-1}u(1-u-w) + \varepsilon,\\ v_t = \Delta v - v + u,\\ w_t = -wv, \qquad x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, \qquad x\in\partial\Omega,\ t>0,\\ u(x,0) = u_{0\varepsilon}(x), \quad v(x,0) = v_{0\varepsilon}(x), \quad w(x,0) = w_{0\varepsilon}(x), \quad x\in\Omega,\end{cases} \tag{3}$$

where $\varepsilon \in (0,1)$, $a_\varepsilon \in C^\infty(\mathbb{R})$, $a_\varepsilon(s) = s+\varepsilon$ for $s \geq 0$, $a_\varepsilon(s) = \varepsilon/2$ for $s < -\varepsilon$, a_ε is monotone increasing with $0 \leq a_\varepsilon' \leqslant 1$, and $u_{0\varepsilon}$,$v_{0\varepsilon}$, and $w_{0\varepsilon}$ are smooth approximation functions of u_0,v_0, and w_0, respectively, with

$$\varepsilon \leq u_{0\varepsilon} \leq u_0 + \varepsilon, \quad 0 \leq v_{0\varepsilon} \leq v_0 + \varepsilon, \quad 0 \leq w_{0\varepsilon} \leq w_0 + \varepsilon,$$
$$|\nabla u_{0\varepsilon}| \leq 2|\nabla u_0|, \quad |\nabla v_{0\varepsilon}| \leq 2|\nabla v_0|, \quad |\nabla w_{0\varepsilon}| \leq 2|\nabla w_0|, \quad |\Delta w_{0\varepsilon}| \leq 2|\Delta w_0|,$$

and $\frac{\partial w_{0\varepsilon}}{\partial \nu} = 0$ on $\partial\Omega$. Without loss of generality, we may assume that ϕ_v and ϕ_w are smooth enough. The local existence and uniqueness of the solution to the regularized problem (3) are trivial, and we denote the unique solution by $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$. Let $(0, T_{\max})$ be its maximal existence interval.

Generally, there is no comparison principle for the coupled parabolic system. However, we prove the following assertion compared with some special lower solutions.

Lemma 3.1. *There holds* $u_\varepsilon \geq 0$, $v_\varepsilon \geq 0$ *and* $w_\varepsilon \geq 0$ *for all* $x \in \Omega$ *and* $t \in (0, T_{\max})$.

Proof. We denote u_ε,v_ε,w_ε by u,v,w in this proof for the sake of simplicity. We argue by contradictions. Since $u_{0\varepsilon} \geq \varepsilon > 0$, there exists $t_0 \in (0, T_{\max})$ such that $u > 0$ for all $x \in \Omega$ and $t \in (0,t_0)$, $u(x_0,t_0) = 0$ for some $x_0 \in \bar{\Omega}$ and $u(x,t_0) \geq 0$ for all $x \in \Omega$.

Now, we divide this proof into 2 parts. If $x_0 \in \Omega$, then $\nabla u(x_0,t_0)=0$, and at this point, we have

$$\begin{aligned}
&\nabla \cdot (m(a_\varepsilon(u))^{m-1}\nabla u) = m(a_\varepsilon(u))^{m-1}\Delta u + m(m-1)a_\varepsilon'(u)|\nabla u|^2 \geqslant 0,\\
&\nabla \cdot (u^m\phi_v(u)\nabla v) = u^m\phi_v(u)\Delta v + (mu^{m-1}\phi_v(u) + u^m\phi_v'(u))\nabla u\cdot\nabla v = 0,\\
&\nabla \cdot (u^m\phi_w(u)\nabla w) = u^m\phi_w(u)\Delta w + (mu^{m-1}\phi_w(u) + u^m\phi_w'(u))\nabla u\cdot\nabla w = 0,\\
&\mu|u|^{\sigma-1}u(1-u-w) = 0,
\end{aligned}$$

which contradict to $\frac{\partial u}{\partial t}(x_0, t_0) \leqslant 0$.

If $x_0 \in \partial\Omega$, then $\frac{\partial u}{\partial \tau}(x_0, t_0) = 0$, $\frac{\partial^2 u}{\partial \tau^2}(x_0, t_0) \geq 0$ for any tangent vector τ, and the boundary condition shows that $\frac{\partial v}{\partial u}(x_0, t_0) = 0$. We assert that $\frac{\partial^2 u}{\partial \nu^2}(x_0, t_0) \geq 0$. In fact, if it were not true, Taylor expansion at (x_0,t_0) shows that there would exist a point $x' \in \Omega$ such that $u(x',t_0) < 0$. Therefore, we also have $\nabla u(x_0,t_0) = 0$ and the above equalities. Those contradictions imply that $u \geq 0$. The nonnegative property of v and w is trivial.

Since $u_\varepsilon \geq 0$, the first equation of (3) is equivalent to

$$\frac{\partial u}{\partial t} = \Delta(u+\varepsilon)^m - \nabla \cdot (u^m\phi_v(u)\nabla v) - \nabla \cdot (u^m\phi_w(u)\nabla w) + \mu u^\sigma(1-u-w) + \varepsilon, \quad u \geq 0.$$

Now we present some energy estimates independent of time t and the parameter ε.

Lemma 3.2. *It holds*

$$\int_\Omega u_\varepsilon(\cdot, t)dx \leq \max\left\{\int_\Omega u_0 dx + |\Omega|, \left(\frac{2(C_1 + |\Omega|)}{\mu C_2}\right)^{1/(\sigma+1)}\right\},$$

for all $t \in (0, T_{\max})$, *where* $C_1 = \mu 2^\sigma|\Omega|$ *and* $C_2 = 1/|\Omega|^\sigma$.

Proof. We denote u_ε, v_ε, w_ε by u, v, w in this proof for the sake of simplicity. From the third equation of (3), we see that

$$w(x,t) = w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau}.$$

Since u is nonnegative and $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w_0}{\partial \nu} = 0$ on $\partial\Omega$, integration of the first equation of (3) over Ω yields

$$\frac{d}{dt}\int_\Omega u dx \leq \mu\int_\Omega u^\sigma dx - \mu\int_\Omega u^{\sigma+1}dx + |\Omega|,$$

for all $t \in (0, T_{\max})$. We note that

$$\mu\int_\Omega u^\sigma dx \le \frac{1}{2}\mu\int_\Omega u^{\sigma+1}dx + C_1,$$

and

$$\int_\Omega u^{\sigma+1}dx \ge C_2\left(\int_\Omega u dx\right)^{\sigma+1},$$

where $C_1 = \mu 2^\sigma|\Omega|$ and $C_2 = 1/|\Omega|^\sigma$. Let $y(t) = \int_\Omega u(\cdot,t)dx$ for $t \in [0, T_{\max})$. We find

$$y'(t) \le C_1 + |\Omega| - \frac{\mu C_2}{2}(y(t))^{\sigma+1}.$$

By an ODE comparison, this shows that

$$y(t) \le \max\left\{y(0), \left(\frac{2(C_1+|\Omega|)}{\mu C_2}\right)^{1/(\sigma+1)}\right\}$$

for all $t \in (0, T_{\max})$.

Here, we recall some lemmas about the L^p-L^q type estimates for the components of the solution.

Lemma 3.3. *Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of (3) in $\Omega \times (0, T_{\max})$, $p \ge 1$,*

$$\begin{cases} q \in [1, \frac{np}{n-2p}), & p \le \frac{n}{2}, \\ q \in [1, \infty], & p > \frac{n}{2}, \end{cases}$$

and

$$\begin{cases} s \in [1, \frac{np}{n-p}), & p \le n, \\ s \in [1, \infty], & p > n. \end{cases}$$

Then, there exist C(p,q) > 0, C(p,s) > 0 and C(p) > 0, such that for any $T \in (0, T_{\max}]$, we have

$$\sup_{t\in(0,T)} \|v_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le C(p,q)\Big(\sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} + \|v_0\|_{L^q(\Omega)}\Big),$$

$$\sup_{t\in(0,T)} \|\nabla v_\varepsilon(\cdot,t)\|_{L^s(\Omega)} \le C(p,s)\Big(\sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} + \|\nabla v_0\|_{L^s(\Omega)}\Big),$$

and

$$\int_0^T\int_\Omega e^{\frac{1}{2}ps}\left(|v_\varepsilon(x,s)|^p + |\nabla v_\varepsilon(x,s)|^p + |\Delta v_\varepsilon(x,s)|^p + \left|\frac{\partial}{\partial s}v_\varepsilon(x,s)\right|^p\right)dxdt$$
$$\le C(p)\int_0^T\int_\Omega e^{\frac{1}{2}ps}|u_\varepsilon(x,s)|^p dxdt + C(p)\|v_0\|^p_{L^p(\Omega)} + C(p)\|\Delta v_0\|^p_{L^p(\Omega)}.$$

Proof. This follows from the standard L^p-L^q type estimates for the Neumann heat semigroup, and we refer the readers to Fujie et al[30] and Cao[31] for details.

Lemma 3.4. *Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of (3) in $\Omega \times (0, T_{\max})$. Then, we have*

$$\|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le C, \quad \|\nabla v_\varepsilon(\cdot,t)\|_{L^s(\Omega)} \le C, \quad t \in (0, T_{\max}),$$

where $s \in [1, \frac{n}{n-1})$; C is a constant independent of ε and t.

Proof. This is a simple conclusion of Lemmas 3.2 and 3.3.

The following Gagliardo-Nirenberg inequality (see Wang[2] and Winkler and Djie[32]) will be used in deriving the L^p estimates of u_ε and $|\nabla v_\varepsilon|$.

Lemma 3.5. Let $0 < s \le p \le \frac{2n}{(n-2)_+}$. There exists a positive constant C such that for all $u \in W^{1,2}(\Omega) \cap L^s(\Omega)$,

$$\|u\|_{L^p(\Omega)} \le C\left(\|\nabla u\|^a_{L^2(\Omega)} \|u\|^{1-a}_{L^s(\Omega)} + \|u\|_{L^s(\Omega)}\right)$$

is valid with $a = \frac{n/s - n/p}{1 - n/2 + n/s} \in (0, 1)$.

Lemma 3.6. *Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of (3) in $\Omega \times (0, T_{\max})$. Then, for any $r > 1$,*

$$\begin{aligned} &-\int_\Omega u_\varepsilon^r \nabla \cdot (u_\varepsilon^m \phi_w(u_\varepsilon) \nabla w_\varepsilon) dx \\ &\quad \le C\left(\int_\Omega u_\varepsilon^{m+r} dx + \int_\Omega u_\varepsilon^{m+r} v_\varepsilon dx + r\int_\Omega u_\varepsilon^{m+r-1} |\nabla u_\varepsilon| dx\right), \quad t \in (0, T_{\max}), \end{aligned}$$

with constant C *being independent of t, ε, and r.*

Proof. We denote $u_\varepsilon, v_\varepsilon, w_\varepsilon$ by u, v, w in this proof for the sake of simplicity. From the third equation of (3), we have

$$\begin{aligned} w(x,t) &= w_{0\varepsilon}(x) e^{-\int_0^t v(x,\tau)d\tau}, \\ \nabla w(x,t) &= \nabla w_{0\varepsilon}(x) e^{-\int_0^t v(x,\tau)d\tau} - w_{0\varepsilon}(x) e^{-\int_0^t v(x,\tau)d\tau} \int_0^t \nabla v(x,\tau) d\tau, \\ \Delta w(x,t) &\ge \Delta w_{0\varepsilon}(x) e^{-\int_0^t v(x,\tau)d\tau} - 2e^{-\int_0^t v(x,\tau)d\tau} \nabla w_{0\varepsilon}(x) \cdot \int_0^t \nabla v(x,\tau) d\tau \\ &\quad - w_{0\varepsilon}(x) e^{-\int_0^t v(x,\tau)d\tau} \int_0^t \Delta v(x,\tau) d\tau. \end{aligned}$$

According to the fact $\frac{\partial v}{\partial \nu} = \frac{\partial w_{0\varepsilon}}{\partial \nu} = 0$, we see that $\frac{\partial w}{\partial \nu} = 0$. For any r>1, we define

$$\Phi(s) = \int_0^s \tau^{m+r-1} \phi_w(\tau) d\tau.$$

Clearly, $0 \le \Phi(s) \le \frac{1}{m+r} s^{m+r}$. Integrating by parts yields

$$\begin{aligned} &-\int_\Omega u^r \nabla \cdot (u^m \phi_w(u) \nabla w) dx \\ &= \int_\Omega u^m \phi_w(u) \nabla w \cdot \nabla u^r dx = r\int_\Omega u^{m+r-1} \phi_w(u) \nabla w \cdot \nabla u dx \\ &= r\int_\Omega \nabla w \cdot \nabla \Phi(u) dx = -r\int_\Omega \Phi(u) \Delta w dx \\ &\le -r\int_\Omega \Phi(u) \cdot \left(\Delta w_{0\varepsilon}(x) e^{-\int_0^t v(x,\tau)d\tau} - 2e^{-\int_0^t v(x,\tau)d\tau} \nabla w_{0\varepsilon}(x) \cdot \int_0^t \nabla v(x,\tau) d\tau \quad - w_{0\varepsilon}(x) e^{-\int_0^t v(x,\tau)d\tau} \int_0^t \Delta v(x,\tau) d\tau\right) dx \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Now, we have the following estimates:

$$\begin{aligned} J_1 &= -r\int_\Omega \Phi(u) \Delta w_{0\varepsilon}(x) e^{-\int_0^t v(x,\tau)d\tau} dx \\ &\le \frac{r}{m+r} \|\Delta w_{0\varepsilon}\|_{L^\infty(\Omega)} \int_\Omega u^{m+r} dx \le 2\|\Delta w_0\|_{L^\infty(\Omega)} \int_\Omega u^{m+r} dx, \end{aligned}$$

and

$$
\begin{aligned}
J_2 &= 2r\int_\Omega \Phi(u)e^{-\int_0^t v(x,\tau)d\tau}\nabla w_{0\varepsilon}(x)\cdot\int_0^t \nabla v(x,\tau)d\tau dx\\
&= 2r\int_\Omega \Phi(u)\nabla w_{0\varepsilon}(x)\cdot\nabla e^{-\int_0^t v(x,\tau)d\tau}dx\\
&= -2r\int_\Omega \Phi(u)e^{-\int_0^t v(x,\tau)d\tau}\Delta w_{0\varepsilon}(x)dx - 2r\int_\Omega u^{m+r-1}\phi_w(u)e^{-\int_0^t v(x,\tau)d\tau}\nabla w_{0\varepsilon}(x)\cdot\nabla u dx\\
&\le \frac{2r}{m+r}\|\Delta w_{0\varepsilon}\|_{L^\infty(\Omega)}\int_\Omega u^{m+r}dx + 2r\|\nabla w_{0\varepsilon}\|_{L^\infty(\Omega)}\int_\Omega u^{m+r-1}|\nabla u|dx\\
&\le 4\|\Delta w_0\|_{L^\infty(\Omega)}\int_\Omega u^{m+r}dx + 4r\|\nabla w_0\|_{L^\infty(\Omega)}\int_\Omega u^{m+r-1}|\nabla u|dx,
\end{aligned}
$$

and

$$
\begin{aligned}
J_3 &= r\int_\Omega \Phi(u)w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau}\int_0^t \Delta v(x,\tau)d\tau dx\\
&= r\int_\Omega \Phi(u)w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau}\int_0^t (v_t + v - u)\, d\tau dx\\
&\le r\int_\Omega \Phi(u)w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau}v(x,t)dx + r\int_\Omega \Phi(u)w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau}\int_0^t v(x,\tau)d\tau dx\\
&\le \frac{r}{m+r}\|w_{0\varepsilon}\|_{L^\infty(\Omega)}\int_\Omega u^{m+r}v dx + \frac{r}{m+r}\|w_{0\varepsilon}\|_{L^\infty(\Omega)}\int_\Omega u^{m+r}dx\\
&\le (\|w_0\|_{L^\infty(\Omega)}+1)\int_\Omega u^{m+r}v dx + (\|w_0\|_{L^\infty(\Omega)}+1)\int_\Omega u^{m+r}dx.
\end{aligned}
$$

These complete the proof.

Lemma 3.7. *Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of (3) in $\Omega\times(0,T_{\max})$. If $\sigma = m$, then for any given $r\ge1$, there exists a constant $\varkappa>0$, such that if $\mu\ge\varkappa$, then we have*

$$\|u_\varepsilon\|_{L^r(\Omega)} \le C, \quad t\in(0,T_{\max}),$$

where $C>0$ is a constant independent of t and ε.

Proof. We denote $u_\varepsilon, v_\varepsilon, w_\varepsilon$ by u,v,w in this proof for the sake of simplicity. It is evidently sufficient to prove that for any $r_0>1$, we can find some $r>r_0$ and $C>0$ such that

$$\|u\|_{L^{r+1}(\Omega)} \le C, \quad t\in(0,T_{\max}).$$

Without loss of generality, we may assume that $\mu\ge\varkappa\ge1$. By a straightforward computation, testing the first equation in (3) by u^r for $r>1$ and integrating by parts, we find that

$$
\begin{aligned}
&\frac{1}{r+1}\frac{d}{dt}\int_\Omega u^{r+1}dx + \int_\Omega \nabla(u+\varepsilon)^m\cdot\nabla u^r dx\\
\le&\int_\Omega u^m\phi_v(u)\nabla v\cdot\nabla u^r dx + \int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla u^r dx\\
&+\mu\int_\Omega u^{m+r}dx - \mu\int_\Omega u^{m+r+1}dx + \int_\Omega u^r dx.
\end{aligned}
\tag{4}
$$

We note that

$$
\begin{aligned}
\mu\int_\Omega u^{m+r}dx &\le \frac{\mu}{8}\int_\Omega u^{m+r+1}dx + C_1,\\
\int_\Omega u^r dx &\le \frac{\mu}{8}\int_\Omega u^{m+r+1}dx + C_2,\\
\frac{m+r+1}{2(r+1)}\int_\Omega u^{r+1}dx &\le \frac{\mu}{8}\int_\Omega u^{m+r+1}dx + C_3,
\end{aligned}
\tag{5}
$$

where C_1, C_2, and C_3 are constants independent of t, as all subsequently appearing constants C_4,C_5, ..., in this proof, possibly depend on m,r,$|\Omega|$ and μ. Let

$$\Psi(s)=\int_0^s \tau^{m+r-1}\phi_v(\tau)d\tau.$$

It is easy to check that $|\Psi(s)|\leq s^{m+r}/(m+r)$. Then, integrating by parts, we can estimate

$$\begin{aligned}\int_\Omega u^m\phi_v(u)\nabla v\cdot\nabla u^r dx &= r\int_\Omega \nabla v\cdot\nabla\Psi(u)dx = r\int_\Omega \Psi(u)\Delta v dx\\ &\leq \int_\Omega u^{m+r}\Delta v dx \leq \frac{\mu}{8}\int_\Omega u^{m+r+1}dx+\left(\frac{8}{\mu}\right)^{m+r}\int_\Omega |\Delta v|^{m+r+1}dx\\ &\leq \frac{\mu}{8}\int_\Omega u^{m+r+1}dx + 8^{m+r}\int_\Omega |\Delta v|^{m+r+1}dx.\end{aligned}$$

According to Lemma 3.6 and the same argument as (5), we find

$$\begin{aligned}\int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla u^r dx &\leq C_4\left(\int_\Omega u^{m+r}dx+\int_\Omega u^{m+r}v dx + r\int_\Omega u^{m+r-1}|\nabla u|dx\right)\\ &\leq \frac{\mu}{8}\int_\Omega u^{m+r+1}dx + C_5\left(\int_\Omega v^{m+r+1}dx+1\right)+C_4 r\int_\Omega u^{m+r-1}|\nabla u|dx.\end{aligned}$$

We further have

$$\begin{aligned}C_4 r\int_\Omega u^{m+r-1}|\nabla u|dx &\leq mr\int_\Omega u^{m+r-2}|\nabla u|^2dx+\frac{C_4^2 r}{4m}\int_\Omega u^{m+r}dx\\ &\leq \int_\Omega \nabla(u+\varepsilon)^m\cdot\nabla u^r dx+\frac{\mu}{8}\int_\Omega u^{m+r+1}dx + C_6.\end{aligned}$$

Combining the above inequalities with (4), we infer that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{r+1}dx+\frac{m+r+1}{2}\int_\Omega u^{r+1}dx\\ &\leq -\frac{\mu(r+1)}{4}\int_\Omega u^{m+r+1}dx + C_7\left(\int_\Omega |\Delta v|^{m+r+1}dx+\int_\Omega v^{m+r+1}dx\right)+C_8,\end{aligned} \tag{6}$$

where $C_7=(r+1)\cdot\max\{8^{m+r},C_5\}$ and $C_8=(r+1)(C_1+C_2+C_3+C_5+C_6)$. Applying Gronwall inequality to the above inequality (6), we have

$$\begin{aligned}&e^{\frac{1}{2}(m+r+1)t}\int_\Omega u^{r+1}(\cdot,t)dx\\ &\leq \int_\Omega u_{0\varepsilon}^{r+1}dx-\frac{\mu(r+1)}{4}\int_0^t\int_\Omega e^{\frac{1}{2}(m+r+1)s}u^{m+r+1}(\cdot,s)dxds\\ &\quad + C_7\int_0^t\int_\Omega e^{\frac{1}{2}(m+r+1)s}\left(|\Delta v|^{m+r+1}(\cdot,s)+v^{m+r+1}(\cdot,s)\right)dxds + C_8\int_0^t e^{\frac{1}{2}(m+r+1)s}ds\\ &\leq \int_\Omega u_{0\varepsilon}^{r+1}dx-\frac{\mu(r+1)}{4}\int_0^t\int_\Omega e^{\frac{1}{2}(m+r+1)s}u^{m+r+1}(\cdot,s)dxds\\ &\quad + C_7C(m+r+1)\int_0^t\int_\Omega e^{\frac{1}{2}(m+r+1)s}u^{m+r+1}(\cdot,s)dxds\\ &\quad + \frac{2}{m+r+1}C_8e^{\frac{1}{2}(m+r+1)t}+C_9,\ t\in(0,T_{\max}),\end{aligned}$$

where $C(m+r+1)$ is the constant in Lemma 3.3. Thus,

$$\int_\Omega u^{r+1}(\cdot,t)dx \leq \int_\Omega (u_0+1)^{r+1}dx+\frac{2}{m+r+1}C_8+C_9,\quad t\in(0,T_{\max}),$$

provided that $\mu \geq \kappa$ with

$$\kappa = \frac{4C_7C(m+r+1)}{r+1}.$$

The proof is completed.

Lemma 3.8. *Let* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *be the solution of* (3) *in* $\Omega\times(0,T_{\max})$. *If* $\sigma>m$, *then for any given* $r\geq 1$, *we have*

$$\|u_\varepsilon\|_{L^r(\Omega)} \leq C, \quad t\in(0,T_{\max}),$$

where $C>0$ *is a constant independent of t and* ε.

Proof. This proof is quit similar to the proof of Lemma 3.7. We denote $u_\varepsilon, v_\varepsilon, w_\varepsilon$ by u,v,w in this proof for the sake of simplicity. It is evidently sufficient to prove that for any $r_0>1$, we can find some $r>r_0$ and $C>0$ such that

$$\|u\|_{L^{r+1}(\Omega)} \leq C, \quad t\in(0,T_{\max}).$$

By a straightforward computation, testing the first equation in (3) by u^r for $r>1$ and integrating by parts, we find that

$$\begin{aligned}
&\frac{1}{r+1}\frac{d}{dt}\int_\Omega u^{r+1}dx + \int_\Omega \nabla(u+\varepsilon)^m\cdot\nabla u^r dx\\
\leq &\int_\Omega u^m\phi_v(u)\nabla v\cdot\nabla u^r dx + \int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla u^r dx\\
&+\mu\int_\Omega u^{\sigma+r}dx - \mu\int_\Omega u^{\sigma+r+1}dx + \int_\Omega u^r dx.
\end{aligned} \tag{7}$$

Similar to the proof of Lemma 3.7, we have

$$\begin{aligned}
\mu\int_\Omega u^{\sigma+r}dx &\leq \frac{\mu}{8}\int_\Omega u^{\sigma+r+1}dx + C_1,\\
\int_\Omega u^r dx &\leq \frac{\mu}{8}\int_\Omega u^{\sigma+r+1}dx + C_2,\\
\frac{\sigma+r+1}{2(r+1)(\sigma+1-m)}\int_\Omega u^{r+1}dx &\leq \frac{\mu}{8}\int_\Omega u^{\sigma+r+1}dx + C_3,
\end{aligned} \tag{8}$$

where C_1, C_2, and C_3 are constants independent of t, as all subsequently appearing constants C_4, C_5, ..., in this proof, possibly depend on $m, r, |\Omega|, \sigma$ and μ. Let Ψ be the function defined in the proof of Lemma 3.7. Integrating by parts, we find

$$\begin{aligned}
&\int_\Omega u^m\phi_v(u)dx\nabla v\cdot\nabla u^r dx = r\int_\Omega \nabla v\cdot\nabla\Psi(u)dx = r\int_\Omega \Psi(u)\Delta v dx\\
\leq &\int_\Omega u^{m+r}\Delta v dx \leq \frac{\mu}{8}\int_\Omega u^{\sigma+r+1}dx + \left(\frac{8}{\mu}\right)^{\frac{m+r}{\sigma+1-m}}\int_\Omega |\Delta v|^{\frac{\sigma+r+1}{\sigma+1-m}}dx.
\end{aligned}$$

According to Lemma 3.6 and the same argument as (5), we find

$$\begin{aligned}
&\int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla u^r dx\\
\leq &C_4\left(\int_\Omega u^{m+r}dx + \int_\Omega u^{m+r}v dx + r\int_\Omega u^{m+r-1}|\nabla u|dx\right)\\
\leq &\frac{\mu}{8}\int_\Omega u^{\sigma+r+1}dx + C_5\left(\int_\Omega v^{\frac{\sigma+r+1}{\sigma+1-m}}dx + 1\right) + C_4 r\int_\Omega u^{m+r-1}|\nabla u|dx.
\end{aligned}$$

We also have

$$\begin{aligned}
C_4 r\int_\Omega u^{m+r-1}|\nabla u|dx &\leq mr\int_\Omega u^{m+r-2}|\nabla u|^2 dx + \frac{C_4^2 r}{4m}\int_\Omega u^{m+r}dx\\
&\leq \int_\Omega \nabla(u+\varepsilon)^m\cdot\nabla u^r dx + \frac{\mu}{8}\int_\Omega u^{\sigma+r+1}dx + C_6.
\end{aligned}$$

Combining the above inequalities with (7), we infer that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{r+1}dx+\frac{\sigma+r+1}{2(\sigma+1-m)}\int_\Omega u^{r+1}dx\\&\le-\frac{\mu(r+1)}{4}\int_\Omega u^{\sigma+r+1}dx+C_7\left(\int_\Omega|\Delta v|^{\frac{\sigma+r+1}{\sigma+1-m}}dx+\int_\Omega v^{\frac{\sigma+r+1}{\sigma+1-m}}dx\right)+C_8,\end{aligned}\tag{9}$$

where $C_7=(r+1)\cdot\max\{(8/\mu)^{m+r},C_5\}$ and $C_8=(r+1)(C_1+C_2+C_3+C_5+C_6)$. Applying Gronwall inequality to the above inequality (9), we have

$$\begin{aligned}&e^{\frac{\sigma+r+1}{2(\sigma+1-m)}t}\int_\Omega u^{r+1}(\cdot,t)dx\\&\le\int_\Omega u_{0\varepsilon}^{r+1}dx-\frac{\mu(r+1)}{4}\int_0^t\int_\Omega e^{\frac{\sigma+r+1}{2(\sigma+1-m)}s}u^{\sigma+r+1}(\cdot,s)dxds\\&\quad+C_7\int_0^t\int_\Omega e^{\frac{\sigma+r+1}{2(\sigma+1-m)}s}\left(|\Delta v|^{\frac{\sigma+r+1}{\sigma+1-m}}(\cdot,s)+v^{\frac{\sigma+r+1}{\sigma+1-m}}(\cdot,s)\right)dxds+C_8\int_0^t e^{\frac{\sigma+r+1}{2(\sigma+1-m)}s}ds\\&\le\int_\Omega u_{0\varepsilon}^{r+1}dx-\frac{\mu(r+1)}{4}\int_0^t\int_\Omega e^{\frac{\sigma+r+1}{2(\sigma+1-m)}s}u^{\sigma+r+1}(\cdot,s)dxds\\&\quad+C_7C\left(\frac{\sigma+r+1}{\sigma+1-m}\right)\int_0^t\int_\Omega e^{\frac{\sigma+r+1}{2(\sigma+1-m)}s}u^{\frac{\sigma+r+1}{\sigma+1-m}}(\cdot,s)dxds\\&\quad+\frac{2(\sigma+1-m)}{\sigma+r+1}C_8e^{\frac{\sigma+r+1}{2(\sigma+1-m)}t}+C_9,\qquad t\in(0,T_{\max}),\end{aligned}$$

where $C((\sigma+r+1)/(\sigma+1-m))$ is the constant in Lemma 3.3. Further, we note that

$$C_7C\left(\frac{\sigma+r+1}{\sigma+1-m}\right)u^{\frac{\sigma+r+1}{\sigma+1-m}}\le\frac{\mu(r+1)}{4}u^{\sigma+r+1}+C_{10},$$

since $\sigma>m$. Combining the above 2 inequalities, we find

$$\begin{aligned}e^{\frac{\sigma+r+1}{2(\sigma+1-m)}t}\int_\Omega u^{r+1}(\cdot,t)dx&\le\int_\Omega u_{0\varepsilon}^{r+1}dx+C_{10}\int_0^t\int_\Omega e^{\frac{\sigma+r+1}{2(\sigma+1-m)}s}ds+\frac{2(\sigma+1-m)}{\sigma+r+1}C_8e^{\frac{\sigma+r+1}{2(\sigma+1-m)}t}+C_9\\&\le\int_\Omega u_{0\varepsilon}^{r+1}dx+\frac{2(\sigma+1-m)}{\sigma+r+1}(C_8+C_{10}|\Omega|)e^{\frac{\sigma+r+1}{2(\sigma+1-m)}t}+C_9,\end{aligned}$$

which yields

$$\int_\Omega u^{r+1}(\cdot,t)dx\le\int_\Omega(u_0+1)^{r+1}dx+\frac{2(\sigma+1-m)}{\sigma+r+1}(C_8+C_{10}|\Omega|)+C_9,\qquad t\in(0,T_{\max}).$$

The proof is completed.

Lemma 3.9. *Let $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ be the solution of (3) in $\Omega\times(0,T_{max})$. Assume that $\sigma=m$ and μ is sufficiently large, or $\sigma>m$. Then, there exists a constant $C>0$ such that*

$$\|u_\varepsilon\|_{L^{n+1}(\Omega)}\le C,\quad\|v_\varepsilon\|_{L^\infty(\Omega)}\le C,\quad\|\nabla v_\varepsilon\|_{L^\infty(\Omega)}\le C,\qquad t\in(0,T_{\max}).$$

Proof. This follows from Lemmas 3.7, 3.8, and 3.3.

We now use the following Moser-type iteration to get the $L^\infty(\Omega)$ estimate of u.

Lemma 3.10. *Under the assumption of Lemma 3.9, there exists a constant $C>0$ independent of t and ε such that*

$$\|u_\varepsilon\|_{L^\infty(\Omega)}\le C,\quad t\in(0,T_{\max}).$$

Proof. We denote $u_\varepsilon,v_\varepsilon,w_\varepsilon$ by u,v,w in this proof for the sake of simplicity. We test the first equation in (3) by u^r for $r>1$, and integrating by parts, we find that

$$\begin{aligned}&\frac{1}{r+1}\frac{d}{dt}\int_\Omega u^{r+1}dx+\int_\Omega\nabla(u+\varepsilon)^m\cdot\nabla u^r dx\\&\le\int_\Omega u^m\phi_v(u)\nabla v\cdot\nabla u^r dx+\int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla u^r dx\\&\quad+\mu\int_\Omega u^{\sigma+r}dx-\mu\int_\Omega u^{\sigma+r+1}dx+\int_\Omega u^r dx.\end{aligned}\tag{10}$$

Using Young inequality, we can estimate

$$\begin{aligned}\mu\int_\Omega u^{\sigma+r}dx&\le\frac{\mu}{4}\int_\Omega u^{\sigma+r+1}dx+4^{\sigma+r}\mu|\Omega|,\\ \int_\Omega u^r dx&\le\frac{\mu}{4}\int_\Omega u^{\sigma+r+1}dx+\left(\frac{4}{\mu}\right)^{\frac{r}{\sigma+r}}|\Omega|,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega u^m\phi_v(u)\nabla v\cdot\nabla u^r dx&\le r\int_\Omega u^{m+r-1}|\nabla v\cdot\nabla u|dx\\&\le\frac{1}{4}mr\int_\Omega(u+\varepsilon)^{m-1}u^{r-1}|\nabla u|^2dx+\frac{r}{m}\int_\Omega u^{m+r}|\nabla v|^2dx\\&\le\frac{1}{4}\int_\Omega\nabla(u+\varepsilon)^m\cdot\nabla u^r dx+\frac{r}{m}\|\nabla v\|^2_{L^\infty(\Omega)}\int_\Omega u^{m+r}dx.\end{aligned}$$

Lemma 3.9 implies that $\|\nabla v\|_{L^\infty(\Omega)}$ and $\|v\|_{L^\infty(\Omega)}$ are uniformly bounded in $(0,T_{\max})$. According to Lemma 3.6, there exists a constant $C_0>0$ such that

$$\begin{aligned}\int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla u^r dx&\le C_0\left(\int_\Omega u^{m+r}dx+\int_\Omega u^{m+r}v dx+r\int_\Omega u^{m+r-1}|\nabla u|dx\right)\\&\le C_0(1+\|v\|_{L^\infty(\Omega)})\int_\Omega u^{m+r}dx+C_0r\int_\Omega u^{m+r-1}|\nabla u|dx.\end{aligned}$$

We also have

$$\begin{aligned}C_0r\int_\Omega u^{m+r-1}|\nabla u|dx&\le\frac{1}{4}mr\int_\Omega u^{m+r-2}|\nabla u|^2dx+\frac{C_0^2r}{m}\int_\Omega u^{m+r}dx\\&\le\frac{1}{4}\int_\Omega\nabla(u+\varepsilon)^m\cdot\nabla u^r dx+\frac{C_0^2r}{m}\int_\Omega u^{m+r}dx.\end{aligned}$$

Straightforward computations yield

$$\begin{aligned}\nabla(u+\varepsilon)^m\cdot\nabla u^r&=mr(u+\varepsilon)^{m-1}u^{r-1}|\nabla u|^2\\&\ge mru^{m+r-2}|\nabla u|^2=\frac{4mr}{(m+r)^2}|\nabla u^{\frac{m+r}{2}}|^2.\end{aligned}$$

Combining the above estimates with (10), we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{r+1}dx+\int_\Omega u^{r+1}dx+\frac{2mr(r+1)}{(m+r)^2}\int_\Omega|\nabla u^{\frac{m+r}{2}}|^2dx\\&\le\int_\Omega u^{r+1}dx-\frac{\mu(r+1)}{2}\int_\Omega u^{\sigma+r+1}dx\\&\quad+(r+1)\left(\frac{r}{m}\|\nabla v\|^2_{L^\infty(\Omega)}+C_0(1+\|v\|_{L^\infty(\Omega)})+\frac{C_0^2r}{m}\right)\int_\Omega u^{m+r}dx\\&\quad+4^{\sigma+r}(r+1)\mu|\Omega|+\left(\frac{4}{\mu}\right)^{\frac{r}{\sigma+r}}(r+1)|\Omega|\\&\le(r+1)\left(\frac{r}{m}\|\nabla v\|^2_{L^\infty(\Omega)}+C_0(1+\|v\|_{L^\infty(\Omega)})+\frac{C_0^2r}{m}\right)\int_\Omega u^{m+r}dx\\&\quad+4^{\sigma+r}(r+1)\mu|\Omega|+\left(\frac{4}{\mu}\right)^{\frac{r}{\sigma+r}}(r+1)|\Omega|+\left(\frac{2}{\mu(r+1)}\right)^{\frac{r+1}{\sigma}}|\Omega|,\end{aligned}\tag{11}$$

where we applied the Gagliardo-Nirenberg inequality, Lemma 3.5 and Young inequality to find a positive constant C_1 independent of r fulfilling

$$\begin{aligned}\int_\Omega u^{m+r}dx &= \|u^{\frac{m+r}{2}}\|^2_{L^2(\Omega)}\\ &\leq C_1\left(\|\nabla u^{\frac{m+r}{2}}\|^{\frac{2n}{n+2}}_{L^2(\Omega)}\|u^{\frac{m+r}{2}}\|^{\frac{4}{n+2}}_{L^1(\Omega)}+\|u^{\frac{m+r}{2}}\|^2_{L^1(\Omega)}\right)\\ &\leq \frac{2mr}{(m+r)^2(r\|\nabla v\|^2_{L^\infty(\Omega)}/m+C_0(1+\|v\|_{L^\infty(\Omega)})+C_0^2r/m)}\|\nabla u^{\frac{m+r}{2}}\|^2_{L^2(\Omega)}+C_2\|u^{\frac{m+r}{2}}\|^2_{L^1(\Omega)},\end{aligned}$$

where

$$C_2 = C_1^{\frac{n+2}{2}}\left(\frac{(m+r)^2(r\|\nabla v\|^2_{L^\infty(\Omega)}/m+C_0(1+\|v\|_{L^\infty(\Omega)})+C_0^2r/m)}{2mr}\right)^{\frac{n}{2}}+C_1.$$

For the sake of simplicity, we let

$$C_3 = (r+1)\left(\frac{r}{m}\|\nabla v\|^2_{L^\infty(\Omega)}+C_0(1+\|v\|_{L^\infty(\Omega)})+\frac{C_0^2r}{m}\right),$$

and

$$C_4 = 4^{\sigma+r}(r+1)\mu|\Omega|+\left(\frac{4}{\mu}\right)^{\frac{r}{\sigma+r}}(r+1)|\Omega|+\left(\frac{2}{\mu(r+1)}\right)^{\frac{r+1}{\sigma}}|\Omega|.$$

Therefore, according to (11), we have

$$\frac{d}{dt}\int_\Omega u^{r+1}dx+\int_\Omega u^{r+1}dx\leq C_2C_3\|u^{\frac{m+r}{2}}\|^2_{L^1(\Omega)}+C_4. \tag{12}$$

Now, we use the following Moser-type iteration. Let $r=r_j$ with $r_j=2^j+m-2$ for $j\in\mathbb{N}^+$; that is, $r_1=m$ and

$$r_{j-1}+1=\frac{r_j+m}{2},\quad j\in\mathbb{N}.$$

We can invoke Lemmas 3.7 and 3.8 to find C_* such that

$$\sup_{t\in(0,T_{\max})}\|u\|_{L^{r_1+1}(\Omega)}\leq C_*.$$

From (12) and an ODE comparison, we have

$$\sup_{t\in(0,T_{\max})}\|u\|^{r_j+1}_{L^{r_j+1}(\Omega)}\leq\max\left\{\int_\Omega(u_0+1)^{r_j+1}dx, C_2C_3\cdot\sup_{t\in(0,T_{\max})}\|u\|^{2(r_{j-1}+1)}_{L^{r_{j-1}+1}(\Omega)}+C_4\right\}. \tag{13}$$

A simple analysis shows that $C_2\leq a_1r^{b_1}, C_3\leq a_2r^{b_2}, C_4\leq a_3b_3^r$, for some positive constants a_1,a_2,a_3 and b_1,b_2,b_3 that all are greater than 1 and independent of r. Therefore, we can rewrite the above inequality (13) into

$$\sup_{t\in(0,T_{\max})}\|u\|^{r_j+1}_{L^{r_j+1}(\Omega)}\leq\max\left\{\int_\Omega(u_0+1)^{r_j+1}dx, a_1a_2r_j^{b_1+b_2}\cdot\sup_{t\in(0,T_{\max})}\|u\|^{2(r_{j-1}+1)}_{L^{r_{j-1}+1}(\Omega)}+a_3b_3^{r_j}\right\}. \tag{14}$$

Let

$$M_j=\max\left\{\sup_{t\in(0,T_{\max})}\int_\Omega u^{r_j+1}dx, 1\right\}.$$

Since boundedness of u in $L^\infty(\Omega)$ is evident in the case when $M_j \le \max\{\int_\Omega (u_0+1)^{r_j+1}dx, 1\}$ for infinitely many $j \geqslant 1$, we may assume that $M_j \ge \max\{\int_\Omega (u_0+1)^{r_j+1}dx, 1\}$ and thus, according to (14), there holds

$$M_j \le a_1 a_2 r_j^{b_1+b_2} M_{j-1}^2 + a_3 b_3^{r_j}. \tag{15}$$

We note that if $M_{j-1}^2 \le a_3 b_3^{r_j}$ for infinitely many $j \geqslant 1$, then

$$M_{j-1}^{\frac{1}{r_{j-1}+1}} \le (a_3 b_3^{r_j})^{\frac{1}{2(r_{j-1}+1)}} \le a_3^{\frac{1}{r_j+m}} b_3^{\frac{r_j}{r_j+m}} \le 2b_3,$$

for j sufficiently large, which shows the boundedness of u in $L^\infty(\Omega)$. Otherwise, $M_{j-1}^2 \ge a_3 b_3^{r_j}$ except for a finite number of $j \geqslant 1$. Thus, there exists a $j_0 \geqslant 1$ such that

$$M_{j-1}^2 \ge a_3 b_3^{r_j}, \qquad j \ge j_0.$$

Therefore, we can rewrite (15) into

$$M_j \le 2a_1 a_2 r_j^{b_1+b_2} \cdot M_{j-1}^2 \le D^j M_{j-1}^2 \tag{16}$$

for all $j \ge j_0$ with a constant D independent of j, whence upon enlarge D if necessary, we can achieve that (16) actually hold for all $j \geqslant 1$. By introduction, this yields

$$M_j \le D^{\sum_{i=0}^{j-2}(j-i)\cdot 2^i} \cdot M_1^{2^{j-1}} = D^{2^j+2^{j-1}-j-2} M_1^{2^{j-1}} \le D^{2^{j+1}} M_1^{2^{j-1}}$$

for all $j \geqslant 1$, and hence that

$$M_j^{\frac{1}{r_j+1}} \le D^{\frac{2^{j+1}}{2^j+m-1}} M_1^{\frac{2^{j-1}}{2^j+m-1}} \le D^2 M_1,$$

for all $j \geqslant 1$. This implies that u indeed belongs to $L^\infty(\Omega \times (0, T_{\max}))$.

Now we turn to the regularity estimates.

Lemma 3.11. *Under the assumption of Lemma 3.9, let* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *be the solution of (3) in* $\Omega \times (0, T_{\max})$*; then there exists a constant* $C>0$ *such that*

$$\int_0^T \int_\Omega |\Delta v_\varepsilon|^2 dxdt \le C(1+T), \quad T \in (0, T_{\max}).$$

Proof. We denote $u_\varepsilon, v_\varepsilon$ by u, v in this proof for the sake of simplicity. Multiplying the second equation in (3) by $-\Delta v$ and integrating over Ω yields

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \frac{\partial}{\partial t}|\nabla v|^2 dx + \int_\Omega |\Delta v|^2 + \int_\Omega |\nabla v|^2 dx \\
&= \int_\Omega \nabla v \cdot \nabla u dx \le \int_\Omega u|\Delta v| dx \\
&\le \frac{1}{2}\int_\Omega |\Delta v|^2 dx + \frac{1}{2}\|u\|_{L^\infty(\Omega)}^2 |\Omega|.
\end{aligned}$$

Since $\|u\|_{L^\infty(\Omega)}$ is uniformly bounded, integrating the above inequlity over $(0,T)$, we complete the proof.

Lemma 3.12. *Under the assumption of Lemma 3.9, let* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *be the solution of* (3) *in* $\Omega \times (0, T_{\max})$; then the third solution component w_ε fulfills

$$\|w_\varepsilon\|_{L^\infty(\Omega)} \le \|w_{0\varepsilon}\|_{L^\infty(\Omega)} \le \|w_0\|_{L^\infty(\Omega)} + 1, \quad t \in (0, T_{\max}),$$

and

$$\|\nabla w_\varepsilon\|_{L^\infty(\Omega)} \leqslant 2\|\nabla w_0\|_{L^\infty(\Omega)} + (\|w_0\|_{L^\infty(\Omega)}+1)\|\nabla v\|_{L^\infty(\Omega)}t, \quad t\in(0,T_{\max}).$$

Moreover, there exists a constant C > 0 *independent of* t *and* ε *such that*

$$\int_\Omega |\Delta w(x,t)|^2 dx \le C(1+t)^4.$$

Proof. We denote $u_\varepsilon, v_\varepsilon, w_\varepsilon$ by u,v,w in this proof for the sake of simplicity. From the third equation of (3), we have

$$\begin{aligned}
w(x,t) &= w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau},\\
\nabla w(x,t) &= \nabla w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau} - w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau}\int_0^t \nabla v(x,\tau)d\tau,\\
\Delta w(x,t) &= \Delta w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau} - 2e^{-\int_0^t v(x,\tau)d\tau}\nabla w_{0\varepsilon}(x)\cdot\int_0^t \nabla v(x,\tau)d\tau\\
&\quad - w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau}\int_0^t \Delta v(x,\tau)d\tau + w_{0\varepsilon}(x)e^{-\int_0^t v(x,\tau)d\tau}\left(\int_0^t \nabla v(x,\tau)d\tau\right)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\nabla w(x,t)| &\le |\nabla w_{0\varepsilon}(x)| + w_{0\varepsilon}(x)\|\nabla v\|_{L^\infty(\Omega)}t\\
&\leqslant 2\|\nabla w_0\|_{L^\infty(\Omega)} + (\|w_0\|_{L^\infty(\Omega)}+1)\|\nabla v\|_{L^\infty(\Omega)}t,
\end{aligned}$$

and

$$\begin{aligned}
|\Delta w(x,t)| &\le |\Delta w_{0\varepsilon}(x)| + 2|\nabla w_{0\varepsilon}(x)|\|\nabla v\|_{L^\infty(\Omega)}t + w_{0\varepsilon}(x)\int_0^t |\Delta v|ds + w_{0\varepsilon}(x)|\|\nabla v\|^2_{L^\infty(\Omega)}t^2\\
&\leqslant 2\|\Delta w_0\|_{L^\infty(\Omega)} + 4|\nabla w_0(x)|\|\nabla v\|_{L^\infty(\Omega)}t + (\|w_0\|_{L^\infty(\Omega)}+1)\int_0^t |\Delta v|ds\\
&\quad + (\|w_0\|_{L^\infty(\Omega)}+1)|\|\nabla v\|^2_{L^\infty(\Omega)}t^2.
\end{aligned}$$

Further, we have

$$\int_\Omega \left(\int_0^t |\Delta v|ds\right)^2 dx \le \int_\Omega\int_0^t |\Delta v|^2 dxdt\cdot t \le C(1+t)^2,$$

according to Lemma 3.11 with the constant C therein. Therefore,

$$\int_\Omega |\Delta w(x,t)|^2 dx \le C'(1+t)^4,$$

for some constant $C' > 0$.

Lemma 3.13. *Let the assumption of Lemma 3.9 holds, and let* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *be the solution of* (3) *in* $\Omega\times(0,T_{\max})$. Then, there exists a constant $C>0$ independent of ε and T, such that

$$\int_0^T\int_\Omega |\nabla u_\varepsilon^m|^2 dxdt \le C(1+T), \quad T\in(0,T_{\max}).$$

Proof. We denote $u_\varepsilon, v_\varepsilon, w_\varepsilon$ by u,v,w in this proof for the sake of simplicity. We test the first equation in (3) by $(u+\varepsilon)^m$ and get

$$\begin{aligned}
&\frac{1}{m+1}\frac{d}{dt}\int_\Omega (u+\varepsilon)^{m+1}dx + \int_\Omega |\nabla(u+\varepsilon)^m|^2 dx\\
\le&\int_\Omega u^m\phi_v(u)\nabla v\cdot\nabla(u+\varepsilon)^m dx + \int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla(u+\varepsilon)^m dx \qquad (17)\\
&+\mu\int_\Omega u^\sigma(u+\varepsilon)^m dx - \mu\int_\Omega u^{\sigma+1}(u+\varepsilon)^m dx + \int_\Omega (u+\varepsilon)^m dx.
\end{aligned}$$

According to Lemmas 3.9 and 3.10, ∇v and u are uniformly bounded. Thus,

$$\int_\Omega u^m \phi_v(u)\nabla v\cdot\nabla(u+\varepsilon)^m dx \le \frac{1}{4}\int_\Omega |\nabla(u+\varepsilon)^m|^2 dx + C_1,$$

where C_1 is a constant independent of t and ε, as all subsequently appearing constants C_2,C_3,... in this proof. A slight modification of the proof of Lemma 3.6 with $\Phi(s)$ being replaced by

$$\hat{\Phi}(s) = \int_0^s \tau^m(\tau+\varepsilon)^{m-1}\phi_w(\tau)d\tau$$

implies that

$$\begin{aligned}&\int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla(u+\varepsilon)^m dx\\ &\le C_2\left(\int_\Omega (u+\varepsilon)^{2m}dx + \int_\Omega (u+\varepsilon)^{2m} v dx + \int_\Omega |\nabla(u+\varepsilon)^{2m}|dx\right)\\ &\le \frac{1}{4}\int_\Omega |\nabla(u+\varepsilon)^m|^2 dx + C_3.\end{aligned}$$

Integrating (17) on $(0,T)$ yields

$$\int_\Omega (u+\varepsilon)^{m+1}dx + \int_0^T\int_\Omega |\nabla(u+\varepsilon)^m|^2 dxdt \le \int_\Omega (u_{0\varepsilon}+\varepsilon)^{m+1}dx + CT. \tag{18}$$

We note that

$$|\nabla u^m| = mu^{m-1}|\nabla u| \le m(u+\varepsilon)^{m-1}|\nabla(u+\varepsilon)| = |\nabla(u+\varepsilon)^m|.$$

This completes the proof.

Lemma 3.14. *Under the assumption of Lemma 3.9, let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of* (3) *in* $\Omega\times(0,T_{\max})$*; then there exists a constant $C>0$ independent of ε and T, such that*

$$\int_0^T\int_\Omega \left|\left(u_\varepsilon^{\frac{m+1}{2}}\right)_t\right|^2 dxdt + \int_\Omega |\nabla u_\varepsilon^m|^2 dx \le C(1+T)^5, \quad T\in(0,T_{\max}).$$

Moreover, $\displaystyle\int_0^T\int_\Omega |(u_\varepsilon^m)_t|^2 dxdt \le \frac{4m^2}{(m+1)^2}\|u_\varepsilon\|_{L^\infty(\Omega)}^{m-1}\int_0^T\int_\Omega\left|\left(u_\varepsilon^{\frac{m+1}{2}}\right)_t\right|^2 dxdt \le C(1+T)^5, \quad T\in(0,T_{\max}).$

Proof. We denote u_ε,v_ε,w_ε by u,v,w in this proof for the sake of simplicity. We multiply the first equation in (3) by $[(u+\varepsilon)^m]_t$, and then we have

$$\begin{aligned}&m\int_\Omega (u+\varepsilon)^{m-1}|u_t|^2dx + \int_\Omega \nabla(u+\varepsilon)^m\cdot\nabla[(u+\varepsilon)^m]_t dx\\ &\le \int_\Omega u^m\phi_v(u)\nabla v\cdot\nabla[(u+\varepsilon)^m]_t dx + \int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla[(u+\varepsilon)^m]_t dx\\ &\quad + \mu\int_\Omega u^\sigma[(u+\varepsilon)^m]_t dx - \mu\int_\Omega u^{\sigma+1}[(u+\varepsilon)^m]_t dx\\ &\quad -\mu\int_\Omega u^\sigma w[(u+\varepsilon)^m]_t dx + \int_\Omega |[(u+\varepsilon)^m]_t| dx.\end{aligned} \tag{19}$$

We note that $\|u\|_{L^\infty(\Omega)}$ is uniformly bounded according to Lemma 3.10, and then

$$\begin{aligned}&\mu\int_\Omega u^\sigma[(u+\varepsilon)^m]_t dx = m\mu\int_\Omega u^\sigma(u+\varepsilon)^{m-1}u_t dx \le \frac{m}{8}\int_\Omega (u+\varepsilon)^{m-1}|u_t|^2dx + C_1,\\ &-\mu\int_\Omega u^{\sigma+1}[(u+\varepsilon)^m]_t dx = -m\mu\int_\Omega u^{\sigma+1}(u+\varepsilon)^{m-1}u_t dx \le \frac{m}{8}\int_\Omega (u+\varepsilon)^{m-1}|u_t|^2dx + C_2,\\ &-\mu\int_\Omega u^\sigma w[(u+\varepsilon)^m]_t dx = -m\mu\int_\Omega u^\sigma w(u+\varepsilon)^{m-1}u_t dx \le \frac{m}{8}\int_\Omega (u+\varepsilon)^{m-1}|u_t|^2dx + C_3,\\ &\int_\Omega |[(u+\varepsilon)^m]_t|\ dx = m\int_\Omega (u+\varepsilon)^{m-1}|u_t|dx \le \frac{m}{8}\int_\Omega (u+\varepsilon)^{m-1}|u_t|^2dx + C_4,\end{aligned}$$

where C_1,C_2,C_3, and C_4 are constants independent of t and ε, as all subsequently appearing constants C_5,C_6,... in this proof. We also have

$$m\int_\Omega (u+\varepsilon)^{m-1}|u_t|^2dx = \frac{4m}{(m+1)^2}\int_\Omega \left|\left((u+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2 dx,$$

and

$$\int_\Omega \nabla(u+\varepsilon)^m\cdot\nabla[(u+\varepsilon)^m]_t dx = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla(u+\varepsilon)^m|^2dx.$$

There holds

$$\begin{aligned}
&\int_\Omega u^m\phi_v(u)\nabla v\cdot\nabla[(u+\varepsilon)^m]_t dx = -\int_\Omega [(u+\varepsilon)^m]_t\nabla\cdot(u^m\phi_v(u)\nabla v)dx\\
&= -\int_\Omega m(u+\varepsilon)^{m-1}u_t\cdot(mu^{m-1}\phi_v(u)\nabla u\cdot\nabla v + u^m\phi_v'(u)\nabla u\cdot\nabla v + u^m\phi_v(u)\Delta v)dx\\
&\le \frac{1}{8}\int_\Omega m(u+\varepsilon)^{m-1}|u_t|^2dx + C_5\int_\Omega (u+\varepsilon)^{2(m-1)}|\nabla u|^2dx + C_6\int_\Omega |\Delta v|^2dx\\
&\le \frac{1}{8}\int_\Omega m(u+\varepsilon)^{m-1}|u_t|^2dx + C_5\int_\Omega |\nabla(u+\varepsilon)^m|^2dx + C_6\int_\Omega |\Delta v|^2dx,
\end{aligned}$$

since the uniform boundedness of $\|\nabla v\|_{L^\infty(\Omega)}$. We also have

$$\begin{aligned}
&\int_\Omega u^m\phi_w(u)\nabla w\cdot\nabla[(u+\varepsilon)^m]_t dx = -\int_\Omega [(u+\varepsilon)^m]_t\nabla\cdot(u^m\phi_w(u)\nabla w)dx\\
&= -\int_\Omega m(u+\varepsilon)^{m-1}u_t\cdot(mu^{m-1}\phi_w(u)\nabla u\cdot\nabla w + u^m\chi_w'(u)\nabla u\cdot\nabla w + u^m\chi_w(u)\Delta w)dx\\
&\le \frac{1}{8}\int_\Omega m(u+\varepsilon)^{m-1}|u_t|^2dx + C_7(1+t)^2\int_\Omega (u+\varepsilon)^{2(m-1)}|\nabla u|^2dx + C_8\int_\Omega |\Delta w|^2dx\\
&\le \frac{1}{8}\int_\Omega m(u+\varepsilon)^{m-1}|u_t|^2dx + C_7(1+t)^2\int_\Omega |\nabla(u+\varepsilon)^m|^2dx + C_8(1+t)^4,
\end{aligned}$$

according to Lemma 3.12. Inserting the above inequalities into (19), and noticing the inequality (18) in the proof of Lemma 3.13, we find a constant C independent of t and ε such that

$$\begin{aligned}
&\int_0^T\int_\Omega \left|\left((u+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2 dxdt + \int_\Omega |\nabla(u+\varepsilon)^m|^2dx\\
&\le \int_\Omega |\nabla(u_{0\varepsilon}+\varepsilon)^m|^2dx + C_9(1+T)^5 \le C_{10}(1+T)^5.
\end{aligned}$$

Clearly, we have

$$\left|\left(u^{\frac{m+1}{2}}\right)_t\right|^2 = \frac{(m+1)^2}{4}u^{m-1}|u_t|^2 \le \frac{(m+1)^2}{4}(u+\varepsilon)^{m-1}|u_t|^2 = \left|\left((u+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2,$$

and

$$|(u^m)_t|^2 \le \frac{4m^2}{(m+1)^2}\|u_\varepsilon\|_{L^\infty(\Omega)}^{m-1}\left|\left(u^{\frac{m+1}{2}}\right)_t\right|^2 \le \frac{4m^2}{(m+1)^2}\|u_\varepsilon\|_{L^\infty(\Omega)}^{m-1}\left|\left((u+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2.$$

The proof is completed.

Proof of Theorem 2.1. According to the estimates, for any ε, *the approximation solution* (u_ε,v_ε,w_ε) *exists globally. The regularity estimates of* v_ε *and* w_ε *are trivial. For any* $T\in(0,\infty)$, *we see that* $u_\varepsilon^m\in L^\infty(Q_T)$, $\nabla u_\varepsilon^m\in L^2(Q_T)$, *and* $\partial u_\varepsilon^m/\partial t\in L^2(Q_T)$. *Thus, there exists a function* $\tilde{u}\in W^{1,2}(Q_T)$, *such that* u_ε^m *weakly in* $W^{1,2}(Q_T)$ *and strongly in* $L^2(Q_T)$ *converges to* $\tilde{u}$. *We denote* $u=\tilde{u}^{1/m}$ *since* $\tilde{u}\geqslant 0$. *Thus,* u_ε^m *converges almost everywhere to* u^m, *and* u_ε *converges almost everywhere to* u. *We can verify the integral identities in the definition of weak solutions. By taking a sequence of* $T\in(0,\infty)$ *and the diagonal subsequence procedure, we can find the existence of a global weak solution.*

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REFERENCES

1. Tao Y, Winkler M. A chemotaxis-haptotaxis model: the roles of nonlinear diffusion and logistic source. *SIAM J Math Anal*. 2011;43:685-705.
2. Wang YF. Boundedness in the higher-dimensional chemotaxis-haptotaxis model with nonlinear diffusion. *J Differ Equ*. 2016;260:1975-1989.
3. Zhigun A, Surulescu C, Uatay A. Global existence for a degenerate haptotaxis model of cancer invasion. *Zeit Ange Math Phys*. 2016;67:146.
4. Blanchet A, Carrillo JA, Laurençot P. Critical mass for a patlak-keller-segel model with degenerate diffusion in higher dimensions. *Cal Var Part Diff Equa*. 2009;35:133-168.
5. Eberl H, Efendiev M, Wrzosek D, Zhigun A. Analysis of a degenerate biofilm model with a nutrient taxis term. *Discr Cont Dyn Syst A*. 2014;34:99-119.
6. Jin H, Li J, Wang Z. Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity. *J Differ Equ*. 2013;255:193-219.
7. Laurencot P, Wrzosek D. *A Chemotaxis Model with Threshold Density and Degenerate Diffusion in Nonlinear Elliptic and Parabolic Problems*, Progr. Nonlinear Differential Equations Appl., Birkhäuser, Basel, 2005;64:273-290.
8. Li J, Li T, Wang Z. Stability of traveling waves of the KellerCSegel system with logarithmic sensitivity. *Math Models Methods Appl Sci*. 2014;24:2819-2849.
9. Wang ZA, Xiang Z, Wrzosek D. Global regularity vs. infinite-time singularity formation in chemotaxis model with volume-filling effect and degenerate diffusion. *SIAM J Math Anal*. 2012;44:3502-3525.
10. Szymańska Z, Rodrigo CM, Lachowicz M, Chaplain MA. Mathematical modelling of cancer invasion of tissue: the role and effect of nonlocal interactions. *Math Models Methods Appl Sci*. 2009;19:257-281.
11. Winkler M, Surulescu C. Global weak solutions to a strongly degenerate haptotaxis model. *Comm Math Sci*. 2017;6:1581-1616.
12. Zhigun A, Surulescu C, Hunt A. Global existence for a degenerate haptotaxis model of tumor invasion under the go-or-grow dichotomy hypothesis. *arXiv: 1605.09226*. 2016.
13. Ke Y, Wang Y. Large time behavior of solution to a fully parabolic chemotaxis-haptotaxis model in higher dimensions. *J Differ Equ*. 2016;260:6960-6988.
14. Li Y, Lankeit J. Boundedness in a chemotaxis-haptotaxis model with nonlinear diffusion. *Nonlinearity*. 2016;29:1564-1591.
15. Wang YF. Boundedness in a multi-dimensional chemotaxis-haptotaxis model with nonlinear diffusion. *Appl Math Lett*. 2016;59:122-126.
16. Zheng P, Mu C, Song X. On the boundedness and decay of solutions for a chemotaxis-haptotaxis system with nonlinear diffusion. *Discr Cont Dyn Syst A*. 2016;36:1737-1757.
17. Painter KJ, Hillen T. Volume-filling and quorum-sensing in models for chemosensitive movement. *Canad Appl Math Quart*. 2002;10: 501-543.
18. Lou Y, Winkler M. Global existence and uniform boundedness of smooth solutions to a cross-diffusion system with equal diffusion rates. *Comm Part Diff Equa*. 2015;40:1905-1941.
19. Painter KJ, Sherratt JA. Modelling the movement of interacting cell populations. *J Theor Biol*. 2003;225:327-339.
20. Xu T, Ji S, Jin C, Mei M, Yin J. Early and late stage profile for a new chemotaxis model with density-dependent jump probability and quorum-sensing mechanisms. arXiv: 1711.08114. 2017.
21. Ming GL, Song HJ, Berninger B, Holt CE, Tessier-Lavigne M, Poo MM. Camp-dependent growth cone guidance by netrin-1. *Neuron*. 1997;19:1225-1235.
22. Song H, Poo MM. The cell biology of neuronal navigation. *Nature Cell Biol*. 2001;3:81-88.
23. Chaplain MA, Lolas G. Mathematical modelling of cancer invasion of tissue: the role of the urokinase plasminogen activation system. *Math Models Methods Appl Sci*. 2005;15:1685-1734.

24. Stevens A, Othmer HG. Aggregation, blowup, and collapse: the ABC's of taxis in reinforced random walks. *SIAM J Appl Math*. 2001;57: 1044-1081.
25. Murry JD. *Mathematical Biology I: An Introduction*. New York, USA: Springer; 2002.
26. Okubo A, Levin SA. *Diffusion and Ecological Problems: Modern Perspectives*, Springer Science & Business Media, 2013.
27. Blumberg AA.. Logistic growth rate functions. *J Theor Biol*. 1968;21:42-44.
28. Turner ME, Blumenstein BA, Sebaugh JL. A generalization of the logistic law of growth. *Biometrics*. 1969;25:577-580.
29. Hillen T, Painter K. Global existence for a parabolic chemotaxis model with prevention of overcrowding. *Adv Appl Math*. 2001;26:280-301.
30. Fujie K, Ito A, Winkler M, Yokota T. Stabilization in a chemotaxis model for tumor invasion. *Discr Conti Dyam Syst*. 2016;36:151-166.
31. Cao X. Boundedness in a three-dimensional chemotaxis-haptotaxis model. *Zeit Ange Math Phys*. 2016;67:1-13.
32. Winkler M, Djie KC. Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect. *Nonl Anal*. 2010;72:1044-1064.

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