McGill University	December 6, 2018
Faculty of Science	Final Examination

Math 264: Advanced Calculus for Engineers

December 6, 2018, 9:00 am – 12:00 pm

Examiner: Prof. Ming Mei **Associate Examiner: Prof. Rustum Choksi**

Student Name:___________________________________ **Student ID:** ______________________________

INSTRUCTIONS

1. This is a **closed book exam, except** you are allowed **one double-sided 8.5 x 11 inches sheet of information**. Do not hand in this sheet of information.
2. Calculators and cell phones are **NOT** permitted.
3. Make sure you READ CAREFULLY the question before embarking on the solution.
4. Note the value of each question.
5. This exam consists of 12 pages (including the cover page). Please check that all pages are intact and provide all your answers on this exam.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Total
Mark											
Out of	10	10	10	10	10	10	10	10	10	10	100

1. Let $\boldsymbol{F}(x, y) = (4x^3y^2 - 2xy^3)\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\mathbf{j}.$

a). Show that $\boldsymbol{F}(x, y)$ is conservative by finding the potential curves.

Solution

Firstly we check with the necessary condition of conservative field: $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. In fact,

$$\frac{\partial F_1}{\partial y} = \frac{\partial(4x^3y^2 - 2xy^3)}{\partial y} = 8x^3y - 6xy^2 = \frac{\partial(2x^4y - 3x^2y^2 + 4y^3)}{\partial x} = \frac{\partial F_2}{\partial x}.$$

So, the vector field $\boldsymbol{F}(x, y)$ might be conservative. Now we are looking for a potential curve $\phi(x, y)$ such that $\nabla\phi = \boldsymbol{F}$, so then $\boldsymbol{F}(x, y)$ is conservative. Let ϕ be:

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = F_1 = 4x^3y^2 - 2xy^3 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1) \\ \dfrac{\partial \phi}{\partial y} = F_2 = 2x^4y - 3x^2y^2 + 4y^3 & \cdots\cdots\cdots\cdots\cdots\cdots (2) \end{cases}$$

Integrating (1) with respect to *x* yields

$$\phi(x, y) = \int (4x^3y^2 - 2xy^3)dx = x^4y^2 - x^2y^3 + C_1(y), \quad \text{for some integral constant } C_1(y).$$

Differentiating the above equation with respect to *y*, we have

$$\frac{\partial \phi}{\partial y} = 2x^4y - 3x^2y^2 + C_1'(y).$$

Comparing it with (2), we have $C_1'(y) = 4y^3$, which gives $C_1(y) = y^4 + C$. Thus, the potential curves are

$$\phi(x, y) = x^4y^2 - x^2y^3 + y^4 + C,$$

which are smooth in *x* and *y*, Therefore, the vector field is conservative.

b). Calculate $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$, where *C* is the curve given by $\boldsymbol{r}(t) = (\mathrm{t} + \sin \pi \mathrm{t})\ \mathbf{i} + (2t + \cos \pi t)\,\mathbf{j}$ for $0 \le t \le 1.$

Solution

Since the vector field is conservative, so the line integral is independent of the path, namely

$$\int_C \boldsymbol{F} \cdot \boldsymbol{dr} = \int_C \nabla\phi \; dt = \phi(x(1), y(1)) - \phi(x(0), y(0))$$
$$= (x^4y^2 - x^2y^3 + y^4)(t = 1) - (x^4y^2 - x^2y^3 + y^4)(t = 0)$$
$$= 0,$$

where $< x(1), y(1) > = < 1,1 >$ and $< x(0), y(0) > = < 0,1 >$.

2. a). Find curl **F** and div **F**, if $\boldsymbol{F}(x, y, z) = e^{-x}\sin y\, \mathbf{i} + e^{-y}\sin z\, \mathbf{j} + e^{-z}\sin x\, \mathbf{k}.$

Solution

$$\text{curl } \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ e^{-x}\sin y & e^{-y}\sin z & e^{-z}\sin x \end{bmatrix} = -e^{-y}\cos z\ \mathbf{i} - \mathrm{e}^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}\,.$$

$$\text{div } \mathbf{F} == \partial_x(e^{-x}\sin y) + \partial_y(e^{-y}\sin z) + \partial_z(e^{-z}\sin x) = -e^{-x}\sin y - \mathrm{e}^{-y}\sin z - e^{-z}\sin x.$$

b). Show that there is no vector field **G** such that curl $\mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}.$

Solution

If there is some vector field **G** such that curl $\mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}$, then

$$\nabla \cdot \text{curl } \mathbf{G} = \partial_x(2x) + \partial_y(3yz) + \partial_z(-xz^2) = 2 + 33z - 2xz \neq 0.$$

However, by the identity: $\nabla \cdot$ curl $\mathbf{G} = 0$ for any vector field **G**, it is a contradiction. So, there is no vector field **G** such that curl $\mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}.$

3. If f is a harmonic function, that is, $\nabla^2 f = f_{xx} + f_{yy} = 0$, show that the line integral $\int_C f_y dx - f_x dy$ is independent of path C in any simple region D.

Solution 1

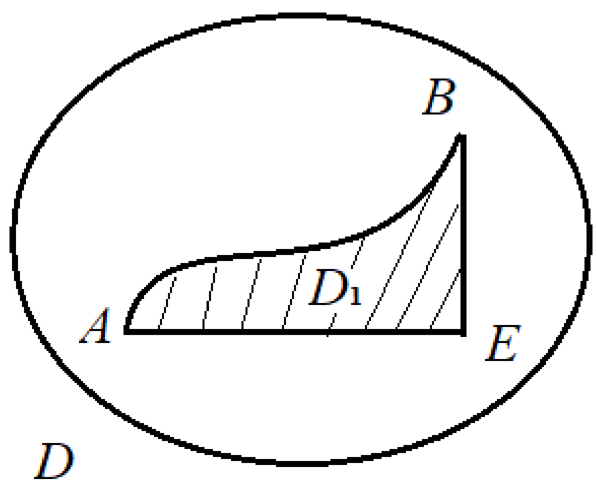
Let C be any path from point A(x_1, y_1) to point B(x_2, y_2) in any simple region D, and $C_2 = \overrightarrow{BE} + \overrightarrow{EA}$ be the lines with the vertical line $\overrightarrow{BE} = \{x = x_2, y_1 \le y \le y_2\}$ and the horizontal line $\overrightarrow{EA} = \{y = y_1, x_1 \le x \le x_2\}$, and D_1 be the region bounded by the curves C and C_2. See the graph below. Note that f is a harmonic function: $\nabla^2 f = f_{xx} + f_{yy} = 0$, by Green's Theorem, we have

$$\int_{C\cup C_2} f_y dx - f_x dy = \iint_{D_1} [\partial_x(-f_x) - \partial_y(f_y)]dA = -\iint_{D_1} [f_{xx} + f_{yy}]dA = 0.$$

This gives

$$\int_C f_y dx - f_x dy = \int_{-C_2} f_y dx - f_x dy = constant$$

Since C is arbitrarily given, the above line integral is independent of path C in any simple region D.



Solution 2

Let C be a given path in any simple region D. We rewrite

$$\int_C f_y dx - f_x dy = \int_C < f_y, -f_x > \cdot < dx, dy > = \int_C \boldsymbol{F} \cdot \boldsymbol{dr}$$

where $\boldsymbol{F} =< f_y,\ -f_x >$ is the vector field. Now we are going to prove it to be conservative by finding its potential curves, so then the line integral $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$ is independent of the path C.

First of all, we check with the necessary condition. Note that

$$\frac{\partial F_1}{\partial y} = \frac{\partial (f_y)}{\partial y} = f_{yy}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial(-f_x)}{\partial x} = -f_{xx}$$

and $f_{xx}+f_{yy}=0, \quad i.e., \quad f_{yy}=-f_{xx}$, then $\frac{\partial F_1}{\partial y}=\frac{\partial F_2}{\partial x}$.

Now we are looking for a smooth function $\phi(x,y)$ such that $\nabla\phi=\boldsymbol{F}$.

Let us solve the system

$$\begin{cases} \frac{\partial\phi}{\partial x}=F_1=f_y \\ \frac{\partial\phi}{\partial y}=F_2=-f_x \end{cases} \tag{1}$$

By integrating the first equation with respect to *x*, we have

$$\phi(x,y)=\int^x f_y(x,y)dx+C_1(y) \tag{2}$$

Differentiating (2) with respect to *y*, we have

$$\partial_y\phi=\int^x f_{yy}(x,y)dx+C_1{}'(y) \tag{3}$$

Comparing (3) with the second equation of (1), we have

$$\int^x f_{yy}(x,y)dx+C_1{}'(y)=-f_x=\int^x(-f_x)_x\,dx=\int^x -f_{xx}dx.$$

So,

$$C_1'(y)=-\int^x f_{yy}dx+\int^x(-f_x)_x\,dx=-\int^x\left(f_{yy}+f_{xx}\right)dx=0,$$

Namely, $C_1(y)=C_1=Constant$. Thus, we derive the potential curves of $\boldsymbol{F}$:

$$\phi(x,y)=\int^x f_y(x,y)dx+C_1=-\int^y f_x(x,y)dy+C_2.$$

Therefore, $\boldsymbol{F}$ is conservative, and the line integral $\int_C \boldsymbol{F}\cdot\boldsymbol{dr}$ is independent of the path *C*.

4. Evaluate $\int_C \sqrt{1+x^3}\,dx+2xy\,dy$, where *C* is the triangle with vertices (0,0), (1,0), (1,3).

<u>Solution 1</u>

Since $\int\sqrt{1+x^3}\,dx$ cannot be explicitly integrated, we have to use Green's Theorem to evaluate it. The region *D* of the triangle with vertices (0,0), (1,0) and (1,3) is expressed as

$$D=\{(x,y)|0\le x\le 1, 0\le y\le 3x.\}$$

So, the line integral is:

$$\int_C \sqrt{1+x^3}\,dx+2xy\,dy=\iint_D[\partial_x(2xy)-\partial_y\left(\sqrt{1+x^3}\right)]dA$$

$$=\int_0^1\int_0^{3x}2y\,dy\,dx=3.$$

Solution 2

Notice that $C = C_1 \cup C_2 \cup C_3$, where C_1 is the line from the point (0,0) to (1,0) with $y = 0, 0 \leq x \leq 1$,

C_2 is the line from the point (1,0) to (1,3) with $x = 1, 0 \leq y \leq 3$, and C_3 is the line from the point (1,3) to (0,0) with $y = 3x, 0 \leq x \leq 1$. So, the line integral is

$$\int_C \sqrt{1+x^3}\ dx + 2xy\ dy = \int_{C_1 \cup C_2 \cup C_3} \sqrt{1+x^3}\ dx + 2xy\ dy$$

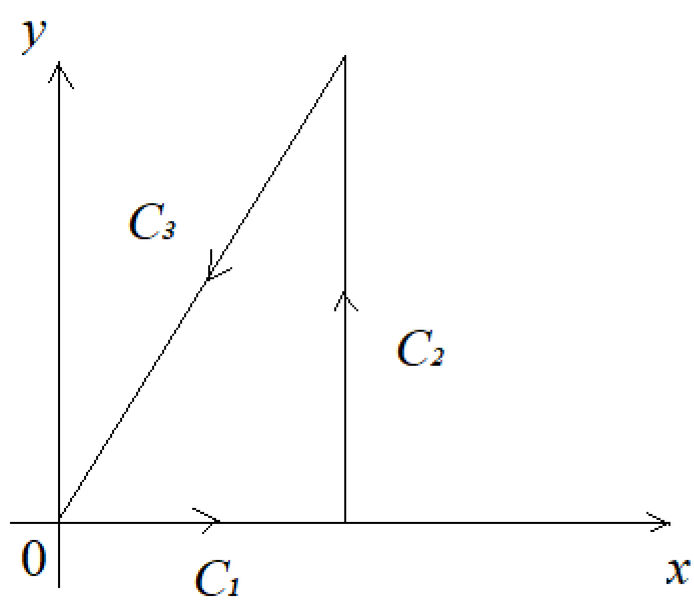
$$= \int_{C_1} \sqrt{1+x^3}\ dx + 2xy\ dy + \int_{C_1} \sqrt{1+x^3}\ dx + 2xy\ dy + \int_{C_3} \sqrt{1+x^3}\ dx + 2xy\ dy$$

$$= \int_0^1 \sqrt{1+x^3}\ dx + \int_0^3 2 \cdot 1 \cdot y\ dy + \int_1^0 \sqrt{1+x^3}\ dx + 2x(3x)\ 3dx$$

$$= \int_0^1 \sqrt{1+x^3}\ dx + y^2|\text{^}(y=3) - y\text{^}2|\_(y=0) \quad - \int_0^1 \sqrt{1+x^3}\ dx - \int_0^1 18x^2 dx$$

$$= \int_0^1 \sqrt{1+x^3}\ dx + 9 - \int_0^1 \sqrt{1+x^3}\ dx - 6$$

$$= 3.$$



5. Evaluate $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + \mathrm{z}\,\mathbf{k}$ and S is the part of paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation.

Solution

The 3-dimensioanl region D bounded by the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ is

$$D = \{(x, y, z)|z = x^2 + y^2, \quad 0 \le z \le 1\} = \{(r, \theta, z)|\ 0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad z = r^2\}.$$

By Divergence Theorem, we have

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_D div\, \boldsymbol{F}\, dV = \int_0^{2\pi}\int_0^1\int_0^{r^2} \left[\partial_x(x^2) + \partial_y(xy) + \partial_z(z)\right] rdzdrd\theta$$

$$= \int_0^{2\pi}\int_0^1\int_0^{r^2} [3x + 1] rdzdrd\theta = \int_0^{2\pi}\int_0^1\int_0^{r^2} [3r\cos\theta + 1] rdzdrd\theta$$

$$= \int_0^{2\pi}\int_0^1 [3r^4\cos\theta + r^3] drd\theta = \int_0^{2\pi} [\frac{3}{5}\cos\theta + \frac{1}{4}] d\theta = \frac{\pi}{2}.$$

6. Evaluate $\iint_S$ curl $\boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x,y,z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$ and S is the part of sphere $x^2+y^2+z^2=5$ that lies above the plane $z=1$ and S is oriented upward.

Solution

Let C be the boundary of the surface S: $x^2+y^2+z^2=5$ on the plane $z=1$, namely, $x^2+y^2=4,\ z=1$, which can be represented in the vector form $\boldsymbol{r}(\theta) =< 2\cos\theta, 2\sin\theta, 1 >$, and $d\boldsymbol{r} =< -2\sin\theta, 2\cos\theta, 0 > d\theta$. By Stokes' Theorem, we have

$$\iint_S \text{curl}\,\boldsymbol{F} \cdot d\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

$$= \int_0^{2\pi} < \mathrm{x}^2yz, yz^2, z^3e^{xy} > \cdot < -2\sin\theta, 2\cos\theta, 0 > d\theta$$

$$= \int_0^{2\pi} [-2^4\cos^2\theta\sin^2\theta + 4\sin\theta\cos\theta]d\theta$$

$$= \int_0^{2\pi} [-\,2^2\sin^2 2\theta + 2\sin 2\theta]d\theta$$

$$= \int_0^{2\pi} [2(\cos 4\theta - 1) + 2\sin\ 2\theta]d\theta$$

$$= -4\pi\ .$$

7. Evaluate $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + \mathrm{z}^3\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0,\ z = 2$.

Solution

The 3-dimensioanl region D bounded by cylinder $x^2 + y^2 = 1$ and the planes $z = 0,\ z = 2$ is

$$D = \{(x, y, z)|\ 0 \le x^2 + y^2 \le 1, 0 \le z \le 2\} = \{(r, \theta, z)|\ 0 \le \theta \le 2\pi, 0 \le r \le 1, 0 \le z \le 2\}.$$

By Divergence Theorem, we have

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_D div\,\boldsymbol{F}\,dV = \int_0^{2\pi}\int_0^1\int_0^2 \left[\partial_x(x^3) + \partial_y(y^3) + \partial_z(z^3)\right] r dzdrd\theta$$

$$= \int_0^{2\pi}\int_0^1\int_0^2 3(r^3 + z^2 r) dzdrd\theta = 11\pi\,.$$

8. a). Find the eigenvalues and eigenvectors: $y'' + \lambda y = 0, y'(0) = 0,\ y'(L) = 0.$

Solution

Let $y = e^{\gamma x}$, then we have the following characteristic equation: $\gamma^2 + \lambda = 0$.
When $\lambda < 0$, the solution is $y = C_1 e^{\sqrt{|\lambda|}x} + C_2 e^{-\sqrt{|\lambda|}x}$. From the boundary condition $y'(0) = 0,\ y'(L) = 0$, we have $C_1 = C_2 = 0$, namely $y = 0$, the trivial solution, which is not the case we look for.

When $\lambda = 0$, the solution is $y = C_1 x + C_2$. The boundary $y'(0) = 0,\ y'(L) = 0$ implies $C_1 = 0$, and $y = C_2$ for arbitrary constant C_2. So, $\lambda = 0$ is one of the eigenvalues, and the corresponding eigenfunction is $\phi_0(x) = 1$.

When $\lambda > 0$, the solution is $y = C_1 \cos\sqrt{\lambda}\ x + C_2 \sin\sqrt{\lambda}\ x$. From the boundary condition $y'(0) = 0,\ y'(L) = 0$, we have

$$-C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x = 0 \ \text{ for } x = 0\,,$$

$$-C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x = 0 \ \text{ for } x = L,$$

which give $C_2 = 0$, and

$$\lambda = \frac{n^2\pi^2}{L^2},\ \ \phi(x) = \cos\frac{n\pi}{L}x,\ \ n = 1,2,3,\cdots\cdots$$

Summary:

Eigenvalues: $\lambda_n = \frac{n^2\pi^2}{L^2},\ \ n = 0,1,2,3,\cdots\cdots$

Eigenfunctions : $\phi_0(x) = 1$, and $\phi_n(x) = \cos\frac{n\pi}{L}x,\ \ n = 1,2,3,\cdots\cdots$

8. b). Find the Fourier series for the function

$$f(x) = \begin{cases} x+2, & -2 \le x < 0, \\ 2-x, & 0 \le x < 2; \end{cases} \qquad f(x+4) = f(x).$$

Solution

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\frac{n\pi}{L} x + b_n \sin\frac{n\pi}{L} x].$$

Note that $f(x)$ is even, namely $f(-x) = f(x)$. So, $b_n = 0,$ for $n = 1,2,3,\cdots$, and

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{2}\int_0^2 (2-x)dx = 2,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L} x\, dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L} x\, dx = \frac{2}{2}\int_0^2 (2-x)\cos\frac{n\pi}{2} x\, dx$$

$$= (2-x)\frac{2}{n\pi}\sin\frac{n\pi}{2} x\ |\text{^}2\_(x=0) - \int_0^2 (-1)\frac{2}{n\pi}\sin\frac{n\pi}{2} x\, dx$$

$$= (2-x)\frac{2}{n\pi}\sin\frac{n\pi}{2} x\ |\text{^}2\_(x=0) - \frac{2^2}{n^2\pi^2}\cos\frac{n\pi}{2} x\ |\text{^}2\_(x=0)$$

$$= -\frac{2^2}{n^2\pi^2}[\cos n\pi - 1] = -\frac{2^2}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} 0, & \text{for } n = \text{even} \\ \dfrac{8}{n^2\pi^2}, & \text{for } n = \text{old.} \end{cases}$$

So,

$$f(x) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\frac{(2n-1)\pi}{2} x.$$

9. Solve the following initial-boundary value problem

$$\begin{cases} u_t = 5u_{xx}, \quad 0 \le x \le \pi, \qquad t > 0, \\ u(0,t) = 10, \quad u(\pi,t) = 20, \quad t > 0, \\ u(x,0) = \cos 2x - \cos 4x, \quad x \in [0,\pi]. \end{cases}$$

Solution

Since the boundary is non-homogeneous, we introduce a function

$$v(x) = \frac{10}{\pi}\, x + 10, \text{ satifying the boundary condition } v(0) = 10 \; and \; v(\pi) = 20.$$

Let $w = u - v$, then w satisfies

$$\begin{cases} w_t = 5w_{xx}, \quad 0 \le x \le \pi, \qquad t > 0, \\ w(0,t) = 0, \quad w(\pi,t) = 0, \quad t > 0, \\ w(x,0) = \cos 2x - \cos 4x - \dfrac{10}{\pi}\, x - 10 =: f(x), \quad x \in [0,\pi]. \end{cases}$$

Its solution is

$$w(x,t) = \sum_{n=1}^{\infty} c_n \; exp(-\frac{n^2\pi^2\alpha^2}{L^2} t) \; \sin\frac{n\pi}{L} x = \sum_{n=1}^{\infty} c_n \; e^{-5n^2 t} \; \sin n\, x,$$

with

$$c_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x \, dx = \frac{2}{\pi}\int_0^\pi \left[\cos 2x - \cos 4x - \frac{10}{\pi} x - 10\right] \sin nx \; dx$$
$$= -\frac{2}{\pi}\int_0^\pi \left[\frac{10}{\pi} x + 10\right] \sin nx \; dx = \frac{20}{n\pi}\,[2(-1)^n - 1].$$

So, the solution for the original IBVP is

$$u(x,t) = v(x) + w(x,t) = \frac{10}{\pi}\, x + 10 + \sum_{n=1}^{\infty} \frac{20}{n\pi}\,[2(-1)^n - 1]\; e^{-5n^2 t} \; \sin n\, x.$$

10. Consider the initial-value problem to the wave equation

$$\begin{cases} u_{tt} = a^2 u_{xx}\,, & -\infty < x < \infty, \qquad t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty, \\ u_t(x,0) = 0, & -\infty < x < \infty, \end{cases}$$

which can be reduced to the form $u_{\xi\eta} = 0$ by the change of variables $\xi = x - at,\ \eta = x + at$.

a). Show that the solution can be written as

$$u(x,t) = \phi(\xi) + \psi(\eta) = \phi(x - at) + \psi(x + at),$$

where ϕ and ψ are the functions satisfying

$$\phi(x) + \psi(x) = f(x), \qquad -\phi'(x) + \psi'(x) = 0.$$

Solution

Let $u = u(\xi, \eta)$ with $\xi = x - at,\ \eta = x + at$. Then

$$u_t = u_\xi \partial_t(\xi) + u_\eta \partial_t(\xi) = -a u_\xi + a u_\eta,$$
$$u_{tt} = \partial_t(-a u_\xi + a u_\eta) = \partial_\xi(-a u_\xi + a u_\eta)\partial_t \xi + \partial_\eta(-a u_\xi + a u_\eta)\partial_t \eta = a^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}),$$
$$u_x = u_\xi \partial_x(\xi) + u_\eta \partial_x(\xi) = u_\xi + u_\eta,$$
$$u_{xx} = \partial_x(u_\xi + u_\eta) = \partial_\xi(u_\xi + u_\eta)\partial_x \xi + \partial_\eta(u_\xi + u_\eta)\partial_x \eta = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

So, we have

$$0 = u_{tt} - a^2 u_{xx} = a^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) - a^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = -4a^2 u_{\xi\eta},$$

namely,

$$u_{\xi\eta} = 0.$$

Integrating it with respect to η, we have $u_\xi = C_1(\xi)$ for some integral constant $C_1(\xi)$. Integrating $u_\xi = C_1(\xi)$ both sides with respect to ξ, we then obtain

$$u(\xi, \eta) = \int C_1(\xi) d\xi = \phi(\xi) + \psi(\eta),$$

where $\phi'(\xi) = C_1(\xi)$ and $\psi(\eta)$ is the integral constant. Thus, we have

$$u(x,t) = u(\xi, \eta) = \phi(\xi) + \psi(\eta) = \phi(x - at) + \psi(x + at)$$

satisfying the initial value conditions

$$u(x,0) = \phi(x) + \psi(x) = f(x), \qquad u_t(x,0) = -a\phi'(x) + a\psi'(x) = 0.$$

10. b). By solving ϕ and ψ in part a), thereby show the following D'Alembert formula:

$$u(x,t) = \frac{1}{2}[f(x-at) + f(x+at)].$$

Solution

Since $-\phi'(x) + \psi'(x) = 0$, by integrating it with respect to x, we have

$$-\phi(x) + \psi(x) = C, \quad \text{for some constant } C.$$

Combining it with $\phi(x) + \psi(x) = f(x)$, we have

$$\phi(x) = \frac{1}{2}[f(x) - C], \quad \text{and} \quad \psi(x) = \frac{1}{2}[f(x) + C].$$

So, we have

$$u(x,t) = \phi(x-at) + \psi(x+at)$$

$$= \frac{1}{2}[f(x-at) - C] + \frac{1}{2}[f(x+at) + C]$$

$$= \frac{1}{2}[f(x-at) + f(x+at)].$$