McGill University Feb. 18, 2014
Midterm examination

Advanced Calculus for Engineers

Math 264

Feb. 18, 2014

Time: 10:05 - 11:25 AM

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Student name (last, first)	**Student number (McGill ID)**

INSTRUCTIONS

1. If you are not registered in this section, your grade will NOT count.
2. This is a closed book exam and calculators are NOT permitted.
3. Make sure you READ CAREFULLY the question before embarking on the solution.
4. SHOW ALL YOUR WORK.
5. This exam comprises 7 pages (including the cover page). Please provide all your answers on this exam.

Problem	1	2	3	4	5	6	Total
Mark							
Out of	10	10	10	10	10	10	60

Question 1 a) (5 pts) Consider the 2D gradient vector field $\nabla(x^2 + y^2)$. **Sketch** at least 3 field lines (curves) and at least 3 equipotential curves. Make clear which are which.

Answer: The vector field is $\langle 2x, 2y\rangle$. The field lines are lines through the origin. The equipotential curves are curves of the form $x^2 + y^2 = C$ for some constant C. These are circles centred at the origin. Note that these circles are orthogonal to the field lines.

b) (5 pts) Give an example of a **non constant** 2D vector field with the property that if $y = f(x)$ is a field line then so is $y = f(x) + C$ for any constant C.

Answer: What we want here is a vector field $\mathbf{F}$ with the property that $\mathbf{F}(x, y)$ is **independent** of y. Thus any vector field of the form $\langle g_1(x), g_2(x)\rangle$ for any functions g_1 and g_2 would work. For example,

$$\langle x, 0\rangle.$$

Question 2 a) (5 pts) Suppose a wire is bent into the shape of a unit circle which lies in the yz plane with center at the origin. Suppose the mass density at any point is given by $\rho(x, y, z) = y^2 z^3$. Write down an integral with respect to t whose value gives the mass of the wire (do **not** evaluate the integral).

Answer: The curve is easily parametrized by

$$x = 0 \qquad y = \cos t \qquad z = \sin t, \qquad t \in [0, 2\pi].$$

Hence

$$ds \;=\; \sqrt{0 + (-\sin t)^2 + (\cos t)^2}\, dt \;=\; 1\, dt,$$

and the mass is

$$\int_{\text{shape of wire}} \rho(x, y, z)\, ds \;=\; \int_0^{2\pi} \cos^2(t)\, \sin^3(t)\, dt.$$

b) (5 pts) Suppose the force field due to wind is given by the 2D vector field

$$\left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

Suppose we want to walk from the point $(-1, 0)$ to the point $(1, 0)$ and then back to the point $(-1, 0)$ in such a way that the total work we do against the wind is 0. **Sketch such a path and explain your reasoning**. Hint: this was one of our three important vector field examples. What was special about it?

Answer: This was the vector field which was conservative only on the plane minus a ray coming out of the origin. For example, the simply connected domain consisting of plane minus the positive y-axis. Hence the circulation over a closed curve will be zero if it lies entirely in this domain. Many possible choices here: e.g. go along the unit half circle in the $y < 0$ half plane, and then go back over the same curve.

Basically any piecewise smooth closed path which passes through the points $(-1, 0)$ and $(1, 0)$ will work **as long** as it does **not** enclose the origin.

Question 3 a) (5 pts) Let $\mathcal{C}$ be the **part** of the unit circle centered at the origin which lies in the plane $x+y+z=0$ and goes from the point $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ to $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Let

$$\mathbf{F} \;=\; \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2 \rangle.$$

Evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

Answer: This vector field is conservative with potential

$$\phi(x, y, z) \;=\; x^2y^2z^2.$$

Hence the integral is simply

$$\phi\left(\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right) \;-\; \phi\left(\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\right) \;=\; 0 - 0 \;=\; 0.$$

b) (5 pts) Let $\mathcal{C}$ be the curve which lies on the graph of $y = x^2$ from $x = 1$ to $x = 2$. Let $\mathbf{F} = \langle x, y \rangle$. Evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

Leave your answer as a single integral with respect to t.

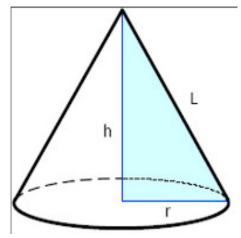
Answer: We can easily parametrize the curve with

$$\mathbf{r}(t) \;=\; \langle t, t^2 \rangle, \qquad t \in [1, 2].$$

Hence $d\mathbf{r} = \langle 1, 2t \rangle\, dt$ and

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_1^2 \langle t, t^2 \rangle \cdot \langle 1, 2t \rangle\, dt \;=\; \int_1^2 t + 2t^3\, dt.$$

Question 4 Consider the piece of a cone with base radius r, height h and side length L.



Use a surface integral to derive a formula for its surface area in terms of r, h and L (you will only need two of these). Here the surface area is just for the sides of the cone and not also for the bottom disc. Note that the usual equation for a cone (rotated by 180^o) $z = \sqrt{x^2+y^2}$ would correspond to the case $r = h$. Hence you need to first do a simple modification to this equation.

Answer: Note that r, h and L are **fixed** for this problem and $L^2 = r^2 + h^2$. The equation of the cone $z = \sqrt{x^2+y^2}$ corresponds to the case where the angle between the side L makes with the positive z axis is 45^o. This would be the case if $r = h$. In general this side line has slope h/r, and hence the equation of the above cone (rotated by 180^o) is

$$z = \frac{h}{r}\sqrt{x^2+y^2}.$$

To surface, whose area we wish to compute, can be naturally described as the graph of function $g(x,y) = \frac{h}{r}\sqrt{x^2+y^2}$ over the **projected disk** D of radius r and centre $(0,0)$ in the xy plane.
We compute

$$\begin{aligned} dS &= \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy \\ &= \sqrt{1 + \frac{h^2}{r^2}\frac{x^2}{x^2+y^2} + \frac{h^2}{r^2}\frac{y^2}{x^2+y^2}}\,dx\,dy \\ &= \frac{\sqrt{r^2+h^2}}{r}\,dx\,dy \\ &= \frac{L}{r}\,dx\,dy. \end{aligned}$$

Thus

$$\begin{aligned} \text{surface area} &= \iint_{\text{cone}} 1\,dS \\ &= \iint_D \frac{L}{r}\,dx\,dy \\ &= (\text{area of } D)\,\frac{L}{r} \\ &= \pi r^2\,\frac{L}{r} \\ &= \pi r L. \end{aligned}$$

Question 5 a) Let $\mathbf{F} = \langle xy, -y, yz^2 \rangle$. Compute the flux

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

out of of $\mathcal{S}$, the part of the sphere $x^2 + y^2 + z^2 = 9$ which lies in the first octant. **You may leave your answer** as an iterated double integral with respect to x and y.

Answer: The surface can be viewed as the part of the level set $G(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$ which lies over R, the quarter disc centred at the origin of radius 3 lying in the first quadrant of the xy plane. Thus

$$d\mathbf{S} \;=\; \frac{\nabla G}{G_3}\, dx\, dy \;=\; \frac{\langle 2x, 2y, 2z \rangle}{2z}\, dx\, dy.$$

Note here that in the first octant, this normal points up which is indeed out of the sphere. Thus

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \;=\; \int_0^3 \int_0^{\sqrt{9-x^2}} \left\langle xy, -y, y(9 - x^2 - y^2) \right\rangle \,\cdot\, \frac{\left\langle 2x, 2y, 2\sqrt{9 - x^2 - y^2} \right\rangle}{2\sqrt{9 - x^2 - y^2}}\, dx\, dy.$$

b) Suppose the velocity of the fluid particle at point (x, y, z) is given by $\mathbf{v}(x, y, z) = \langle x, y, z \rangle$. What is the rate of fluid (volume per unit time) **out of** the sphere of radius 3?

Answer: If $\mathcal{S}$ denotes the sphere of radius 3 oriented with the outer normal, we want to know

$$\iint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S} \;=\; \iint_{\mathcal{S}} \mathbf{v} \cdot \mathbf{N}\, dS,$$

where $\mathbf{N}$ denotes the unit outer normal to the sphere. At any point (x, y, z) on the sphere,

$$\mathbf{N} \;=\; \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} \;=\; \frac{\langle x, y, z \rangle}{3}.$$

Hence at any point (x, y, z) on the sphere,

$$\mathbf{v} \cdot \mathbf{N} \;=\; \langle x, y, z \rangle \cdot \frac{\langle x, y, z \rangle}{3} \;=\; \frac{x^2 + y^2 + z^2}{3} \;=\; \frac{9}{3} \;=\; 3.$$

In other words, constant on the sphere!!! So the flux integral is very easy:

$$\iint_{\text{sphere}} \mathbf{v} \cdot \mathbf{N}\, dS \;=\; 3\,(\text{area of sphere}) \;=\; 3\,(4\,\pi\,3^2) \;=\; 108\pi.$$

Note that we used the fact that at any point (x, y, z) on the sphere, we have $x^2 + y^2 + z^2 = 9$.

Question 6 (10 pts) Consider the boundary surface of the unit cube lying in the first octant with one vertex at the origin. Orient this boundary cube with the **outer normal**. Let

$$\mathbf{F} = \left\langle e^x yz \cos x\,,\, 4\,,\, \sin xy \right\rangle.$$

The surface consists of 6 square pieces – the six sides of the cube with the appropriate orientation.
(a) Consider the **flux out of** the top square piece. **Write down** a double iterated integral with respect to x and y which gives this flux.

Answer: The top square corresponds to $z = 1$, i.e the piece of the graph of $z = 1$ above the unit square in the xy plane. Hence

$$d\mathbf{S} = \langle 0, 0, 1 \rangle \, dx\, dy,$$

and

$$\iint_{\text{top}} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \sin xy \, dx\, dy.$$

(b) Evaluate (i.e. I want a number) the flux integral of $\mathbf{F}$ out of **any two** of sides. You choose but make sure your choice is clearly indicated.

Answer: If you look at the vector field, you see that the second component (the y component) is the constant 4. Hence let us choose the sides parallel to the xz plane with normals $\pm\langle 0, 1, 0 \rangle$. These are the sizes corresponding to $y = 0$ and $y = 1$. On the $y = 0$ side, the outer normal to the cube is $-\langle 0, 1, 0 \rangle$ whereas on the side $y = 1$ the outer normal to the cube is $\langle 0, 1, 0 \rangle$. Thus on the the $y = 0$ side, the flux is

$$\iint_{y=0 \text{ side}} \mathbf{F} \cdot d\mathbf{S} = \iint_{y=0 \text{ side}} \mathbf{F} \cdot \mathbf{N}\, dS = \iint_{y=0 \text{ side}} -4\, dS = -4.$$

On the $y = 1$ side, the flux is 4. If you add them up you get 0.

Note that the sum being 0 results from the symmetry of the y component of $\mathbf{F}$. It would NOT be the case that the total flux out of any other two opposite sides is 0.