McGill University April 20, 2016

Faculty of Science Final examination

Advanced Calculus for Engineers

Math 264

April 20, 2016

Time: 2PM-5PM

Examiner: Prof. R. Choksi Associate Examiner: Prof. A. Hundemer

Student name (last, first)	**Student number (McGill ID)**

INSTRUCTIONS

1. This is a closed book exam **except** you are allowed **one double-sided 8.5 x 11 inches sheet of information**. Do **not** hand in this sheet of information.
2. **Calculators** are **not** permitted.
3. Make sure you **carefully read** the question **before** embarking on the solution.
4. Note the value of each question.
5. This exam consists of 13 pages (including the cover page). Please check that **all** pages are intact and provide all your answers on this exam.

Problem	1	2	3	4	5	6	7	8	9	10	Total
Mark											
Out of	10	10	10	10	10	10	10	10	10	10	100

Question 1a (5 pts). Let
$$\mathbf{F} \;=\; \big\langle y, -x \big\rangle \;=\; y\mathbf{i} \;-\; x\mathbf{j}.$$
Evaluate
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r},$$
where $\mathcal{C}$ is the part of the unit circle centred at the origin which goes clockwise from the point $(1,0)$ to the point $(0,-1)$, i.e., one quarter of the full circle.

Solution:
$$x = \cos t \qquad y = -\sin t \qquad t \in \left[0, \frac{\pi}{2}\right].$$
Then
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_0^{\frac{\pi}{2}} \langle -\sin t, -\cos t\rangle \cdot \langle -\sin t, -\cos t\rangle \, dt \;=\; \int_0^{\frac{\pi}{2}} 1 \, dt \;=\; \frac{\pi}{2}.$$

1b (5 pts). Let
$$\mathbf{F} = \big\langle 1, 2, x+y+z \big\rangle \;=\; \mathbf{i} \;+\; 2\mathbf{j} \;+\; (x+y+z)\mathbf{k}.$$
Consider the **triangle** lying **in the plane** $x+y+z=3$ with **vertices** $(3,0,0), (0,3,0), (0,0,3)$. Let $\mathcal{S}$ be the part of the plane which lies **inside** this triangle with normal pointing **upwards**. Evaluate the flux
$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

Solution: Plane is given by the equation $G(x,y,z) = x+y+z-3 = 0$. Base is the triangle in the xy plane.
$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{base} \mathbf{F} \cdot \frac{\nabla G}{G_3} \, dx\, dy \;=\; \iint_{base} 1+2+3 \, dx\, dy \;=\; 6\,\frac{9}{2} \;=\; 27.$$

Question 2a (5 pts): Let

$$\mathbf{F}(x,y) \,=\, (x^2+y^2+1)\,\big\langle 1,2\big\rangle \,=\, (x^2+y^2+1)\mathbf{i} \,+\, 2(x^2+y^2+1)\mathbf{j}.$$

In words or with a picture, describe the **field lines** of $\mathbf{F}$.

Solution: Lines parallel to $\langle 1,2\rangle$ or lines with slope 2.

2b (5 pts): Give an example of a 3D vector field $\mathbf{F}(x,y,z)$ which at any point on the surface defined by the equation

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 \,=\, 1,$$

is **normal** to the surface.

Solution:

$$\mathbf{F} \,=\, \nabla\left(\frac{x^2}{4} + \frac{y^2}{9} + z^2\right) \,=\, \left\langle \frac{x}{2}\,,\, \frac{2y}{9}\,,\, 2z \right\rangle.$$

Question 3 (10 pts). Assume the surface of the earth is a perfect sphere with radius ρ_0. Suppose the **population density** of a virus at any point on the earth is **proportional** to the **distance squared** (along the surface) **to the equator**, with proportionality constant k. **Find a double integral** with respect to two variables which gives the the **total population of the virus on the earth**. Here, **"distance"** refers to the **closest** distance **along** the surface of the earth.

Solution:

$$Density\,(\theta,\phi) \;=\; k\rho_0^2\left(\frac{\pi}{2}-\phi\right)^2.$$

Hence

$$Population \;=\; \int_0^{2\pi}\int_0^{\pi} k\left(\frac{\pi}{2}-\phi\right)^2 \rho_0^4\sin\phi\,d\phi\,d\theta.$$

Question 4. Let $\mathbf{F}$ be a 3D vector field such that

- div $\mathbf{F}(x, y, z) = 3$ for all (x, y, z) in the ball (inside of the sphere) centred at the origin with radius 2.
- curl $\mathbf{F}(x, y, z) = 5\mathbf{k} = \langle 0, 0, 5\rangle$ in the ball (inside of the sphere) centred at the point $(10, 10, 10)$ with radius 2.

a) (5 pts) Evaluate

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathcal{S}$ is the sphere centred at the origin of radius 1, oriented with the outer normal.

Solution:

$$3 \times \textit{volume inside sphere} \;=\; 2\,\frac{4}{3}\pi 1^3 \;=\; 4\pi.$$

b) (5 pts) Evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ is the circle centred at the point $(10, 10, 10)$ with radius 1 which lies in the plane $x+y+z = 30$. Here the orientation of $\mathcal{C}$ is the positive induced orientation from the upward normal to the plane.

Solution:

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{N} \;=\; \frac{5}{\sqrt{3}}.$$

Hence by Stokes' Theorem

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \frac{5}{\sqrt{3}}\,(\textit{area inside circle}) \;=\; \frac{5\pi}{\sqrt{3}}.$$

Question 5. Recall (in case you forgot to write it down!) the **gravitational vector field** generated by a mass m at the origin:

$$\mathbf{F}(x, y, z) = \mathbf{F}(\mathbf{r}) = -\frac{km}{|\mathbf{r}|^3}\mathbf{r} = -\frac{km}{\left(x^2 + y^2 + z^2\right)^{3/2}} \langle x, y, z \rangle.$$

Also recall the **gravitational potential**

$$\phi(x, y, z) = \frac{km}{|\mathbf{r}|} = \frac{km}{\sqrt{x^2 + y^2 + z^2}}.$$

a) (3 pts) Find the **gravitational flux** out of the sphere centred at the origin of radius 1.

Solution: On sphere, $\mathbf{F} \cdot \mathbf{N}$ is constant indeed,

$$\mathbf{F} \cdot \mathbf{N} = -\frac{km}{|\mathbf{r}|^3}\mathbf{r} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = -\frac{km}{|\mathbf{r}|^2} = -km.$$

Hence the flux out is imply $-km$ times the surface area of the sphere

$$-4\pi km.$$

b) (3 pts) Find the **gravitational flux** out of the sphere centred at the point $(5, 5, 5)$ of radius 1.

Solution: Since div $\mathbf{F} = 0$ away form the origin, by the divergence theorem the flus out is 0.

c) (4 pts) Find the **work done by gravity** in moving a unit mass along the following path: We start at the point $(1, 1, 1)$ and go in a straight line directly to the point $(10, 10, 10)$, and then along a straight line directly to the point $(2, 3, 1)$.

Solution: Difference in potentials at the final and initial point:

$$\phi(2, 3, 1) - \phi(1, 1, 1) = \frac{km}{\sqrt{4 + 9 + 1}} - \frac{km}{\sqrt{3}} = km\left(\frac{1}{\sqrt{14}} - \frac{1}{\sqrt{3}}\right).$$

Question 6. Let $\mathcal{S}$ be the boundary surface of the unit cube which is centered at the origin. This is the cube which lies between the planes $z = 1/2$ and $z = -1/2$, the planes $x = 1/2$ and $x = -1/2$, and the planes $y = 1/2$ and $y = -1/2$. Consider the vector field

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}..$$

a) (5 pts) Write down a double integral with respect to two variables which gives the **flux** of **F** **out of the top of the cube**.

Solution:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}\, dx\, dy.$$

Which by the way is just $\frac{1}{2}$.

b) (5 pts) Using the **Divergence Theorem**, compute this flux (the one from part a). **Hint:** think of symmetry.

Solution:

$$\iint_{boundary\ of\ cube} \mathbf{F} \cdot d\mathbf{S} = \iiint_{inside\ of\ cube} \text{div } \mathbf{F}\ dx\, dy\, dz = 3.$$

Hence by symmetry, the flux is

$$\frac{3}{6} = \frac{1}{2}.$$

Question 7 (10 pts). **Use Stokes' Theorem** to evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F}(x, y, z) \;=\; \big\langle -y^2, x, z^2 \big\rangle \;=\; -y^2\mathbf{i} \;+\; x\mathbf{j} \;+\; z^2\mathbf{k}.$$

and $\mathcal{C}$ is the intersection of the plane $y+z = 2$ and the cylinder $x^2+y^2 = 1$, oriented counterclockwise when viewed from a point high on the z-axis.

Solution:

$$\operatorname{curl}\mathbf{F} \;=\; \langle 0, 0, 1+2y\rangle.$$

Hence by Stokes' Theorem

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_{\mathcal{S}} \text{curl } \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{unit\ disc} \langle 0, 0, 1+2y\rangle \cdot \langle 0, 1, 1\rangle\, dx\, dy \;=\; \iint_{unit\ disc} 1 + 2y\, dx\, dy.$$

In polar coordinates this gives

$$\int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta \;=\; \pi.$$

Question 8a (3 pts). Write down the **Fourier sine series** for the function $\phi(x) = x^2$ **on** $(0, 1)$. You may leave the coefficients as integrals.

Solution:

$$\sum_{n=1}^{\infty} a_n \sin n\pi x \qquad \text{where} \qquad a_n = 2\int_0^1 x^2 \sin n\pi x\, dx.$$

b) (3 pts). The Fourier sine series from part a) will converge to some function on $(-2, 2)$. **Sketch this function**, making sure you indicate the values at ± 1 and 0.

Solution:

c) (4pts). What are all the eigenvalues and eigenfunctions for the problem for $X(x)$ on $[0, 5]$

$$X'' + \lambda X = 0 \qquad X'(0) = 0 = X(5).$$

Solution:

$$\lambda_n = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{25} \qquad\qquad \cos\left(n + \frac{1}{2}\right)\frac{\pi}{5}.$$

Question 9. Consider a bar of length π which is insulated on its sides and has an initial temperature distribution over $x \in [0, \pi]$ given by

$$\sin 3x - \sin 5x.$$

Suppose at $t > 0$, the temperature at the left end is held fixed at 5 while at the right end it is held fixed at 10. You can take the numerical value for the thermal diffusivity to be $k = 3$.

a) (3pts). Write down a boundary value problem (BVP) for the temperature $u(x,t)$ at $x \in [0, \pi]$ in the bar and time $t \geq 0$.

Solution:

$$u_t = 3u_{xx} \qquad x \in [0, \pi], t > 0$$
$$u(0,t) = 5 \qquad u(\pi, t) = 10 \qquad t > 0$$
$$u(x,0) = \sin 3x \, - \, \sin 5x \qquad x \in [0, \pi].$$

b) (7 pts). Solve this BVP (more space on the next page).

Solution: Steady state is

$$v(x) = 5 \, + \, \frac{5x}{\pi}.$$

Let $u(x,t) = w(x,t) + v(x)$. The transient solution $w(x,t)$ solves

$$w_t = 3w_{xx} \qquad x \in [0, \pi], t > 0$$
$$w(0,t) = 0 \qquad w(\pi, t) = 0 \qquad t > 0$$
$$u(x,0) = \sin 3x \, - \, \sin 5x \, - \, \left(5 \, + \, \frac{5x}{\pi}\right) \qquad x \in [0, \pi].$$

Solution is

$$w(x,t) \; = \; \sum_{n=1}^{\infty} a_n \sin nx e^{-3n^2 t}$$

where

$$\begin{aligned} a_n \;&=\; \frac{2}{\pi} \int_0^{\pi} (\sin 3x \, - \, \sin 5x) \, \sin nx \, dx \; - \; \frac{2}{\pi} \int_0^{\pi} \left(5 \, + \, \frac{5x}{\pi}\right) \, \sin nx \, dx \\ &=\; \begin{cases} 1 - \frac{2}{\pi} \int_0^{\pi} \left(5 \, + \, \frac{5x}{\pi}\right) \, \sin 3x \, dx & \text{if} \;\; n = 3 \\ -1 - \frac{2}{\pi} \int_0^{\pi} \left(5 \, + \, \frac{5x}{\pi}\right) \, \sin 5x \, dx & \text{if} \;\; n = 5 \\ -\frac{2}{\pi} \int_0^{\pi} \left(5 \, + \, \frac{5x}{\pi}\right) \, \sin nx \, dx & \text{all other n.} \end{cases} \end{aligned}$$

Finally we add $5 \, + \, \frac{5x}{\pi}$ to $w(x,t)$ to get $u(x,t)$.

Question 9 – Additional space

Question 10. Consider the following boundary value problem (BVP) for the wave equation.

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < 2,\ t > 0 \\ u_x(0,t) = 0 \quad u_x(2,t) = 0 & t > 0 \\ u(x,0) = x(1-x) \qquad u_t(x,0) = 0. & 0 < x < 2. \end{cases}$$

a) (3pts). Describe a physical situation which this BVP models? This means: briefly describe **physical interpretations** of x, t, u, the PDE, the boundary conditions and the initial conditions.

Solution: This models the vertical vibrations (displacement from equilibrium) of a taut string of length 2 which has rings at the left and right end which are attached to a frictionless vertical rod. Hence no vertical component of the tension at the ends. The initial displacement of the string is given by $x(1-x)$ and it is released from rest. So this is like plucking a string.

b) (7 pts). Solve this BVP (more space on the next page). You can **leave** the Fourier coefficients **as integrals**.

Solution:

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi t}{2} \cos\frac{n\pi x}{2},$$

where

$$c_n = \int_0^2 (x(1-x)) \cos\frac{n\pi x}{2}\, dx \qquad n = 0, 1, 2, \ldots$$

Question 10 – Additional space