MATH 203/2 FALL 2006
ASSIGNMENT 9 (WEEK 10) SOLUTIONS

Section 4.2

4. $f(x)=x\sqrt{x+6}$, $[-6,0]$. f is continuous on its domain, $[-6,\infty)$, and differentiable on $(-6,\infty)$, so it is continuous on $[-6,0]$ and differentiable on $(-6,0)$. Also, $f(-6)=0=f(0)$. $f'(c)=0 \Leftrightarrow \dfrac{3c+12}{2\sqrt{c+6}}=0 \Leftrightarrow$ $c=-4$, which is in $(-6,0)$.

12. $f(x)=x^3+x-1$, $[0,2]$. f is continuous on $[0,2]$ and differentiable on $(0,2)$. $f'(c)=\dfrac{f(2)-f(0)}{2-0} \Leftrightarrow$ $3c^2+1=\dfrac{9-(-1)}{2} \Leftrightarrow 3c^2=5-1 \Leftrightarrow c^2=\dfrac{4}{3} \Leftrightarrow c=\pm\dfrac{2}{\sqrt{3}}$, but only $\dfrac{2}{\sqrt{3}}$ is in $(0,2)$.

18. Let $f(x)=2x-1-\sin x$. Then $f(0)=-1<0$ and $f(\pi/2)=\pi-2>0$. f is the sum of the polynomial $2x-1$ and the scalar multiple $(-1)\cdot\sin x$ of the trigonometric function $\sin x$, so f is continuous (and differentiable) for all x. By the Intermediate Value Theorem, there is a number c in $(0,\pi/2)$ such that $f(c)=0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a<b$, then $f(a)=f(b)=0$. Since f is continuous on $[a,b]$ and differentiable on (a,b) , Rolle's Theorem implies that there is a number r in (a,b) such that $f'(r)=0$. But $f'(r)=2-\cos r>0$ since $\cos r\leq 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one real root.

24. If $3\leq f'(x)\leq 5$ for all x , then by the Mean Value Theorem, $f(8)-f(2)=f'(c)\cdot(8-2)$ for some c in $[2,8]$.
(f is differentiable for all x , so, in particular, f is differentiable on $(2,8)$ and continuous on $[2,8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8)-f(2)=6f'(c)$ and $3\leq f'(c)\leq 5$, it follows that $6\cdot 3\leq 6f'(c)\leq 6\cdot 5 \Rightarrow 18\leq f(8)-f(2)\leq 30$.

Section 4.3

8. **(a)** f is increasing on the intervals where $f'(x)>0$, namely, $(2,4)$ and $(6,9)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x=4$). Similarly, where f' changes from negative to positive, f has a local minimum (at $x=2$ and at $x=6$).

(c) When f' is increasing, its derivative f'' is positive and hence, f is concave upward. This happens on $(1,3)$, $(5,7)$, and $(8,9)$. Similarly, f is concave downward when f' is decreasing ––– that is, on $(0,1)$, $(3,5)$, and $(7,8)$.

(d) f has inflection points at $x=1$, 3, 5, 7, and 8, since the direction of concavity changes at each of these values.

18. **(a)** $y=f(x)=x^2e^x \Rightarrow f'(x)=x^2e^x+2xe^x=x(x+2)e^x$. So $f'(x)>0 \Leftrightarrow x(x+2)>0 \Leftrightarrow$ either $x<-2$ or $x>0$. Therefore f is increasing on $(-\infty,-2)$ and $(0,\infty)$, and decreasing on $(-2,0)$.

(b) f changes from increasing to decreasing at $x=-2$, so $f(-2)=4e^{-2}$ is a local maximum value. f changes from decreasing to increasing at $x=0$, so $f(0)=0$ is a local minimum value.

(c) $f'(x)=\left(x^2+2x\right)e^x \Rightarrow f''(x)=\left(x^2+2x\right)e^x+e^x(2x+2)=e^x\left(x^2+4x+2\right)$. $f''(x)=0 \Leftrightarrow x^2+4x+2=0 \Leftrightarrow x=-2\pm\sqrt{2}$. $f''(x)<0 \Leftrightarrow -2-\sqrt{2}<x<-2+\sqrt{2}$, so f is concave downward on $\left(-2-\sqrt{2},-2+\sqrt{2}\right)$ and concave upward on $\left(-\infty,-2-\sqrt{2}\right)$ and $\left(-2+\sqrt{2},\infty\right)$. There are inflection points at $\left(-2-\sqrt{2},\mathrm{f}\left(-2-\sqrt{2}\right)\right)\approx(-3.41,0.38)$ and $\left(-2+\sqrt{2},\mathrm{f}\left(-2+\sqrt{2}\right)\right)\approx(-0.59,0.19)$.

22. $f(x)=\dfrac{x}{x^2+4} \Rightarrow f'(x)=\dfrac{\left(x^2+4\right)\cdot 1-x(2x)}{\left(x^2+4\right)^2}=\dfrac{4-x^2}{\left(x^2+4\right)^2}=\dfrac{(2+x)(2-x)}{\left(x^2+4\right)^2}$.

First Derivative Test: $f'(x)>0 \Rightarrow -2<x<2$ and $f'(x)<0 \Rightarrow x>2$ or $x<-2$. Since f' changes from positive to negative at $x=2$, $f(2)=\frac{1}{4}$ is a local maximum value; and since f' changes from negative to positive at $x=-2$, $f(-2)=-\frac{1}{4}$ is a local minimum value.

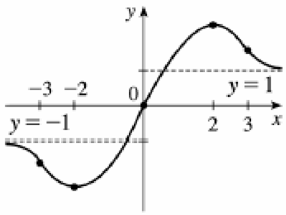
Second Derivative Test:

$$f''(x) = \frac{\left(x^2+4\right)^2(-2x)-\left(4-x^2\right)\cdot 2\left(x^2+4\right)(2x)}{\left[\left(x^2+4\right)^2\right]^2}$$

$$= \frac{-2x\left(x^2+4\right)\left[\left(x^2+4\right)+2\left(4-x^2\right)\right]}{\left(x^2+4\right)^4}=\frac{-2x\left(12-x^2\right)}{\left(x^2+4\right)^3}$$

$f'(x)=0 \Leftrightarrow x=\pm 2.\ f''(-2)=\frac{1}{16}>0 \Rightarrow f(-2)=-\frac{1}{4}$ is a local minimum value.
$f''(2)=-\frac{1}{16}<0 \Rightarrow f(2)=\frac{1}{4}$ is a local maximum value.
Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

30.



$f'(x)>0$ if $|x|<2 \Rightarrow f$ is increasing on $(-2,2)$. $f'(x)<0$ if $|x|>2 \Rightarrow f$ is decreasing on $(-\infty,-2)$ and $(2,\infty)$. $f'(2)=0$, so f has a horizontal tangent (and local maximum) at $x=2$. $\lim_{x\to\infty} f(x)=1 \Rightarrow y=1$ is a horizontal asymptote. $f(-x)=-f(x) \Rightarrow f$ is an odd function (its graph is symmetric about the origin). Finally, $f''(x)<0$ if $0<x<3$ and $f''(x)>0$ if $x>3$, so f is CD on $(0,3)$ and CU on $(3,\infty)$.

36. **(a)** $g(x)=200+8x^3+x^4 \Rightarrow g'(x)=24x^2+4x^3=4x^2(6+x)=0$ when $x=-6$ and when $x=0$. $g'(x)>0 \Leftrightarrow x>-6$ ($x\neq 0$) and $g'(x)<0 \Leftrightarrow x<-6$, so g is decreasing on $(-\infty,-6)$ and g is increasing on $(-6,\infty)$, with a horizontal tangent at $x=0$.
(b) $g(-6)=-232$ is a local minimum value. There is no local maximum value.
(c) $g''(x)=48x+12x^2=12x(4+x)=0$ when $x=-4$ and when $x=0$. $g''(x)>0 \Leftrightarrow x<-4$ or $x>0$ and $g''(x)<0 \Leftrightarrow -4<x<0$, so g is CU on $(-\infty,-4)$ and $(0,\infty)$, and g is CD on $(-4,0)$. Inflection points at $(-4,-56)$ and $(0,200)$
(d)