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EXISTENCE OF MULTIPLE NONTRIVIAL SOLUTIONS FOR A p-KIRCHHOFF TYPE ELLIPTIC PROBLEM INVOLVING SIGN-CHANGING WEIGHT FUNCTIONS

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Abstract. This paper deals with a p-Kirchhoff type problem involving sign-changing weight functions. It is shown that under certain conditions, by means of variational methods, the existence of multiple nontrivial nonnegative solutions for the problem with the subcritical exponent are obtained. Moreover, in the case of critical exponent, we establish the existence of the solutions and prove that the elliptic equation possesses at least one nontrivial nonnegative solution.

1. **Introduction and main theorems.** The purpose of this article is to investigate the existence of multiple nontrivial nonnegative solutions to the following nonlocal boundary value problem of the p-Kirchhoff type

$$\begin{cases} -M\Big(\displaystyle\int_\Omega |\nabla u|^p \mathrm{d}x\Big)\Delta_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u = \mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u})$, Ω is a bounded domain in R^N with a smooth boundary $\partial\Omega$, $1 < q < p < r \le p^*$ where $p^* = \frac{Np}{N-p}$ if $N > p$ and $p^* = \infty$ if $N \le p$, $M(s) = as + b$ and the parameters $a,\ b,\ \lambda > 0$, the weight functions f, g satisfy the following conditions:

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$(A1)$ f, $g \in C(\overline{\Omega})$, and $f^{\pm} = \max\{\pm f, 0\} \not\equiv 0$, $g^{\pm} = \max\{\pm g, 0\} \not\equiv 0$;
$(A2)$ $\int_\Omega f|u|^q \mathrm{d}x > 0$ and $\int_\Omega g|u|^r \mathrm{d}x > 0$ for $u \in W_0^{1,p}(\Omega) \setminus \{0\}$.

Such problems are called nonlocal problems because of the expression of $M(\int_\Omega |\nabla u|^p \mathrm{d}x)$, which implies that the equation contains an integral over Ω, and is no longer pointwise identities. In the case $p = 2$, if we replace $\lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u$ by function $h(x,u)$, the problem (1) reduces to the following nonlocal Kirchhoff elliptic problem

$$\begin{cases} -M\Big(\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x\Big)\Delta u = h(x,u) & in\ \Omega, \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{2}$$

This is related to the stationary analogue of the Kirchhoff problem

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2 \mathrm{d}x\Big)\Delta u = h(x,u),$$

such a model was first proposed by Kirchhoff [19] in 1883 to describe transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations. Nonlocal problems also arise in other fields, for example, physical and biological systems where u describes a process which depends on the average of itself. For more details of background, we refer to [1, 6, 7].

The study of Kirchhoff type problems is one of hot spots in nonlocal partial differential equations. The first frame work was given by Lions [20]. Since then, the study of Kirchhoff type problems have been paid more attention. In [24], Ma and Muñoz Rivera proved the existence of positive solutions for the Kirchhoff elliptic problem (2) by the variational method and minimization arguments, under some restrictions on $M(s)$ and $h(x,u)$. Subsequently, by the truncation argument and uniform *a priori* estimates of Gidas and Spruck type [15], Alves, Corrêa and Ma [2] proved the existence of positive solutions if the nonlinear $h(x,u)$ satisfies the so-called Ambrosetti-Rabinowitz condition, where $M(s)$ is nonincreasing and does not grow too fast in a suitable interval near zero. When $M(s)$ is increasing, the existence of positive solutions is also obtained by Ma [25] and Perera and Zhang [27], where in [27] the nontrivial solutions was established by the Yang index. Furthermore, Chen, Kuo and Wu [8] considered the problem (2) with $h(x,u) = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u$, where $1 < q < 2 < r < 2^*$. By using the Nehari manifold and fibering map methods, they examined the multiplicity of positive solutions for the exponent r satisfying $r > 4$, $r = 4$, and $r < 4$, respectively. If the nonlinearity is critical, Figueiredo [14] obtained the existence of solutions by using the truncation argument. For more results, we refer to [3, 4, 9, 16, 28].

With regard to p-Kirchhoff type elliptic problems, Corrêa and Figueiredo [10] proved a result of existence and multiplicity of solutions by the Krasnoselskii's genus when the nonlinear term is nonnegative function and satisfies subcritical growth condition. Liu [22] established the existence of infinite many solutions by the Fountain theorem and Dual Fountain theorem. According to Morse theory and the local linking, Liu and Zhao [23] further proved the existence of two nontrivial solutions if $M(s)$ is bounded. Recently, Huang, Chen and Xiu [17] studied the following quasilinear elliptic problem with concave-convex nonlinearities

$$\begin{cases} -M(\|u\|^p)\Delta_p u = \lambda h(x)|u|^{q-2}u + H(x)|u|^{r-2}u & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

where $M(s) = as^k + b$, $1 < q < p < r < p^*$, and proved that the problem has at least one positive solution when $r > p(k+1)$ and the functions $h(x)$, $H(x)$ are nonnegative. The approach adopted is the mountain pass lemma. In [18], Hamydy, Massar and Tsouli considered the following problem with critical exponent

$$\begin{cases} -M(\|u\|^p)\Delta_p u = \lambda f(x,u) + |u|^{p^*-2}u & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

By the variational method, they obtained a nontrivial solution when the parameter λ is sufficiently large. After that, Ourraoui [26] showed the existence of at least one solution when the parameter $\lambda = 1$.

However, when the weight functions $f(x)$ and $g(x)$ change their signs, the existence of solutions to the p-Kirchhoff elliptic equations is open, as we know. To attach this problem will be the main target of the present paper. Motivated by the results of above-mentioned papers, in this paper, we will discuss the existence of multiple nontrivial nonnegative solutions to the problem (1) by a variational method. There are three special features of this study. Firstly, the corresponding energy functional $J_{\lambda,M}(u)$ of the problem (1) is not bounded in $W_0^{1,p}(\Omega)$ for $r \geq 2p$, then we cannot take advantage of the standard variational argument directly. In order to overcome this difficulty and obtain the existence of nontrivial nonnegative solutions, we will adopt a variational method on the Nahari manifold which is similar to the fibering method (see [5, 12] for details). Secondly, the problem (1) involves the p-Laplacian operator, which makes the uniform *a prior* estimates of Gidas and Spruck type for the case $p = 2$ failed. To overcome this shortage and to get the existence of two nontrivial nonnegative solutions for $r < 2p$, we need to compare the min-max levels of energy and use the truncation arguments. Such an idea originally comes from Corrêa and Figueiredo [11]. Finally, when $r = p^*$, due to the lack of compactness of the embedding of $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we prove the compactness of the extraction of the Palais-Smale sequences in the Nehari manifold by the Lions concentration-compactness principle.

Before stating our main theorems, let us have the following notations. Let $W_0^{1,p}(\Omega)$ be the Sobolev space with norm $\|u\| = \big(\int_\Omega |\nabla u|^p \mathrm{d}x\big)^{\frac{1}{p}}$, and we denote by S_l the best Sobolev constant for the embedding of $W_0^{1,p}(\Omega)$ in $L^l(\Omega)$ with $1 < l \leq p^*$, in particular,

$$\|u\|_{L^l} \leq S_l^{\frac{-1}{p}} \|u\| \text{ for all } u \in W_0^{1,p}(\Omega) \setminus \{0\},$$

where $\|u\|_{L^l} = \big(\int_\Omega |u|^l \mathrm{d}x\big)^{\frac{1}{l}}$.

Firstly, we give the definition of the weak solution to the problem (1).

Definition 1.1. We say that a function $u \in W_0^{1,p}(\Omega)$ is a weak solution of the problem (1) if

$$M(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \mathrm{d}x - \lambda\int_\Omega f|u|^{q-2}u\varphi\mathrm{d}x - \int_\Omega g|u|^{r-2}u\varphi\mathrm{d}x = 0$$

for all $\varphi \in W_0^{1,p}(\Omega)$. Thus, the corresponding energy functional of the problem (1) is defined by

$$J_{\lambda,M}(u) = \frac{1}{p}\hat{\mathrm{M}}(\|u\|^p) - \frac{\lambda}{q}\int_\Omega f|u|^q\mathrm{d}x - \frac{1}{r}\int_\Omega g|u|^r\mathrm{d}x,$$

where $\hat{\mathrm{M}}(s) = \int_0^s M(t)\mathrm{d}t$. It is well known that the weak solutions to the problem (1) are the critical points of the energy functional $J_{\lambda,M}(u)$. However, from the expression of functional $J_{\lambda,M}(u)$, we know that it is not bounded in $W_0^{1,p}(\Omega)$ when $r \geq 2p$, so it is useful to discuss the functional $J_{\lambda,M}(u)$ on the Nehari manifold

$$N_{\lambda,M} = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} | \langle J'_{\lambda,M}(u), u \rangle = 0\}.$$

Moreover, $u \in N_{\lambda,M}$ if and only if

$$M(\|u\|^p)\|u\|^p - \lambda \int_\Omega f|u|^q \mathrm{d}x - \int_\Omega g|u|^r \mathrm{d}x = 0,$$

and $N_{\lambda,M}$ contains every nonzero solution of the problem (1).

Define

$$K_{u,M}(t) = J_{\lambda,M}(tu) = \frac{1}{p}\hat{\mathrm{M}}(t^p\|u\|^p) - \frac{\lambda t^q}{q}\int_\Omega f|u|^q \mathrm{d}x - \frac{t^r}{r}\int_\Omega g|u|^r \mathrm{d}x, \ t > 0,$$

we have

$$\begin{aligned}
K'_{u,M}(t) =& t^{p-1}M(\|tu\|^p)\|u\|^p - \lambda t^{q-1}\int_\Omega f|u|^q \mathrm{d}x - t^{r-1}\int_\Omega g|u|^r \mathrm{d}x,\\
K''_{u,M}(t) =& (p-1)t^{p-2}M(\|tu\|^p)\|u\|^p + pt^{2p-2}M'(\|tu\|^p)\|u\|^{2p}\\
&- \lambda(q-1)t^{q-2}\int_\Omega f|u|^q \mathrm{d}x - (r-1)t^{r-2}\int_\Omega g|u|^r \mathrm{d}x,
\end{aligned}$$

and $K'_{u,M}(t) = 0$ for $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, $t > 0$ if and only if $tu \in N_{\lambda,M}$. In particular, $K'_{u,M}(1) = 0$ if and only if $u \in N_{\lambda,M}$. Now, we split $N_{\lambda,M}$ into three parts:

$$\begin{aligned}
N^+_{\lambda,M} &= \{u \in N_{\lambda,M} | K''_{u,M}(1) > 0\};\\
N^0_{\lambda,M} &= \{u \in N_{\lambda,M} | K''_{u,M}(1) = 0\};\\
N^-_{\lambda,M} &= \{u \in N_{\lambda,M} | K''_{u,M}(1) < 0\}.
\end{aligned}$$

Thus, for each $u \in N_{\lambda,M}$, one has

$$K''_{u,M}(1) = (p-q)M(\|u\|^p)\|u\|^p + pM'(\|u\|^p)\|u\|^{2p} - (r-q)\int_\Omega g|u|^r \mathrm{d}x \tag{3}$$

$$= (p-r)M(\|u\|^p)\|u\|^p + pM'(\|u\|^p)\|u\|^{2p} + \lambda(r-q)\int_\Omega f|u|^q \mathrm{d}x. \tag{4}$$

The main results of this paper are the following theorems:

Theorem 1.2. *Assume $2p < r < p^*$ and $N < 2p$. Then for each $a > 0$, there exists a positive number $\lambda^* = \max\{\frac{q}{\sqrt{2p}}\lambda_1(a), \frac{q}{2p}\lambda_2(a), \frac{q}{p}\lambda_3\}$ such that the problem (1) has at least two nontrivial nonnegative solutions $u^+_{\lambda,M} \in N^+_{\lambda,M}$ and $u^-_{\lambda,M} \in N^-_{\lambda,M}$ for $0 < \lambda < \lambda^*$, where*

$$\begin{aligned}
\lambda_1(a) &= \frac{2\sqrt{ab(r-2p)(r-p)}S_q^{\frac{q}{p}}}{(r-q)\|f\|_\infty}\Big(\frac{2\sqrt{ab(2p-q)(p-q)}S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{3p-2q}{2r-3p}},\\
\lambda_2(a) &= \frac{a(r-2p)S_q^{\frac{q}{p}}}{(r-q)\|f\|_\infty}\Big(\frac{a(2p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{2p-q}{r-2p}},\\
\lambda_3 &= \frac{b(r-p)S_q^{\frac{q}{p}}}{(r-q)\|f\|_\infty}\Big(\frac{b(p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{p-q}{r-p}}.
\end{aligned}$$

Define

$$\Lambda = \inf\Big\{\|u\|^{2p} \mid u \in W_0^{1,p}(\Omega),\ \int_\Omega g|u|^{2p}\mathrm{d}x = 1\Big\}, \tag{5}$$

then $\Lambda > 0$ is achieved by some $\phi_\Lambda \in W_0^{1,p}(\Omega)$ with $\int_\Omega g|u|^{2p}\mathrm{d}x = 1$. In particular,

$$\Lambda \int_\Omega g|u|^{2p}\mathrm{d}x \leq \|u\|^{2p}.$$

Theorem 1.3. *Assume $r = 2p$ and $N < 2p$. Then*

(i) for each $a \geq \frac{1}{\Lambda}$ and $\lambda > 0$, the problem (1) has at least one nontrivial nonnegative solution $u_{\lambda,M} \in N^+_{\lambda,M} = N_{\lambda,M}$;

(ii) for each $a < \frac{1}{\Lambda}$ and $0 < \lambda < \frac{q}{p}\lambda_0(a)$, where

$$\lambda_0(a) = \frac{bpS_q^{\frac{q}{p}}}{(2p-q)\|f\|_\infty}\Big(\frac{b(p-q)\Lambda}{(2p-q)(1-a\Lambda)}\Big)^{\frac{p-q}{p}},$$

the problem (1) has at least two nontrivial nonnegative solutions $u^+_{\lambda,M} \in N^+_{\lambda,M}$, $u^-_{\lambda,M} \in N^-_{\lambda,M}$ and

$$\lim_{a\to\frac{1}{\Lambda}^-}\ \inf_{u\in N^-_{\lambda,M}} J_{\lambda,M}(u) = \infty.$$

Theorem 1.4. *Assume $r = p^*$.*

Then for each $a > 0$ and $0 < \lambda < \frac{b(p^*-p)S_q^{\frac{q}{p}}}{(p^*-q)\|f\|_\infty}\Big(\frac{b(p-q)S_{p^*}^{\frac{p^*}{p}}}{(p^*-q)\|g\|_\infty}\Big)^{\frac{p-q}{p^*-p}}$, the problem (1) has at least one nontrivial nonnegative solution $u_{\lambda,M} \in N^+_{\lambda,M}$.

Theorem 1.5. *Assume $p < \frac{2p^2}{2p-q} < r < 2p$. Then*

(i) for each $a > 0$ and $\lambda > 0$, the problem (1) has at least one nontrivial nonnegative solution $u_{\lambda,M} \in W_0^{1,p}(\Omega)$. Moreover, if $a > A$ and $\lambda > 0$, then $u_{\lambda,M} \in N^+_{\lambda,M} = N_{\lambda,M}$.

(ii) for each $\vartheta > 0$ and $0 < a < \frac{b^2(r-p)}{rL(\vartheta)}$, there exists a positive number $\lambda_* = \min\{\vartheta, \lambda_4(a), \lambda_5(a)\}$ such that the problem (1) has at least one nontrivial nonnegative solution $u^{(1)}_{\lambda,M} \in N^+_{\lambda,M}$ for $0 < \lambda < \lambda_*$, and

$$\|u^{(1)}_{\lambda,M}\|^p < \frac{b(r-p)}{pa},$$

where

$$A = \frac{\big((r-q)\|g\|_\infty S_r^{-\frac{r}{p}}\big)^{\frac{p}{r-p}}}{\big(b(p-q)\big)^{\frac{2p-r}{r-p}}(2p-q)},$$

$$\lambda_4(a) = \frac{(b(r-p)-a(2p-r)k)S_q^{\frac{q}{p}}}{(r-q)\|f\|_\infty}\Big(\frac{b(p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{p-q}{r-p}},\ k \in \Big(\frac{b(r-p)}{ar}, \frac{b(r-p)}{pa}\Big),$$

$$\lambda_5(a) = \Big(\frac{b(r-p)}{ar}\Big)^{\frac{p-q}{p}}\frac{b(r-p)S_q^{\frac{q}{p}}}{(r-q)\|f\|_\infty},$$

$$L(\vartheta) = \vartheta\|f\|_\infty S_q^{-\frac{q}{p}}\tilde{C}^q + \|g\|_\infty S_r^{-\frac{r}{p}}\tilde{C}^r.$$

Theorem 1.6. *Assume $p < \frac{2p^2}{2p-q} < r < 2p$. Then for each $\vartheta > 0$ and $0 < a < \min\{\frac{b^2(r-p)}{rL(\vartheta)}, A_*\}$, there exists a positive number $\tilde{\lambda}^* \leq \min\{\vartheta, \hat{\Lambda}, \lambda_*\}$ such that the* *problem (1) has at least two nontrivial nonnegative solutions* $u^{(1)}_{\lambda,M}$, $u^{(2)}_{\lambda,M} \in N^{+}_{\lambda,M}$ *for* $0<\lambda<\tilde{\lambda}^*$. *Moreover,*

$$\|u^{(1)}_{\lambda,M}\|^p < \frac{b(r-p)}{pa} < \|u^{(2)}_{\lambda,M}\|^p,$$

where

$$\hat{\Lambda} = a(\frac{b(r-p)}{a(2p-r)})^{\frac{2p-q}{p}}\|f\|^{-1}_{\infty}S_q^{\frac{q}{p}},$$
$$A_* = \frac{p^{\frac{r}{p-r}}(r-p)^2}{Sr}(\frac{2p-r}{b})^{\frac{2p-r}{r-p}}.$$

The outline of this paper is as follows. In Section 2, we present some necessary preliminaries and some properties of Nehari manifold. Section 3 will be devoted to the proofs of Theorems 1.2, 1.3 and 1.4. In Section 4 and Section 5, we will prove Theorem 1.5 and Theorem 1.6, respectively.

2. **Preliminaries.**

We present some important properties of Nehari manifold.

Lemma 2.1. *Assume that* u_0 *is a local minimizer for* $J_{\lambda,M}(u)$ *on* $N_{\lambda,M}$ *and* $u_0 \notin N^0_{\lambda,M}$. *Then* u_0 *is a critical point of functional* $J_{\lambda,M}(u)$.

Proof. The proof is similar to the proof of Theorem 2.3 in [5], we omit the details here. □

Lemma 2.2. (i) *If* $r \geq 2p$, *then the energy functional* $J_{\lambda,M}(u)$ *is coercive and bounded in* $N_{\lambda,M}$;

(ii) *If* $r < 2p$, *then the energy functional* $J_{\lambda,M}(u)$ *is coercive and bounded in* $W^{1,p}_0(\Omega)$.

Proof. (i) By the definition of $N_{\lambda,M}$, the Sobolev imbedding theorem and Young's inequality, we find that

$$\begin{aligned} J_{\lambda,M}(u) &= \frac{1}{p}\hat{\mathrm{M}}(\|u\|^p) - \frac{\lambda}{q}\int_\Omega f|u|^q \mathrm{d}x - \frac{1}{r}\int_\Omega g|u|^r \mathrm{d}x \\ &\geq \frac{a(r-2p)}{2pr}\|u\|^{2p} + \frac{b(r-p)}{pr}\|u\|^p - \frac{\lambda(r-q)}{qr}\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q \\ &\geq -(\frac{b(r-p)}{pr})^{-\frac{q}{p-q}}(\frac{\lambda(r-q)}{qr}\|f\|_\infty S_q^{-\frac{q}{p}})^{\frac{p}{p-q}}. \end{aligned}$$

Thus, $J_{\lambda,M}(u)$ is coercive and bounded in $N_{\lambda,M}$.

(ii) Using the Sobolev imbedding theorem, we have

$$J_{\lambda,M}(u) \geq \frac{a}{2p}\|u\|^{2p} + \frac{b}{p}\|u\|^p - \frac{\lambda}{q}\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q - \frac{1}{r}\|g\|_\infty S_r^{-\frac{r}{p}}\|u\|^r,$$

then the energy functional $J_{\lambda,M}(u)$ is coercive and bounded in $W^{1,p}_0(\Omega)$ by the Young's inequality. The proof of Lemma 2.2 is complete. □

Lemma 2.3. *If* $r < p^*$, *then each Palais-Smale sequence for* $J_{\lambda,M}(u)$ *in* $W^{1,p}_0(\Omega)$ *has a strongly convergent subsequence.*

Proof. First, we need to show that Palais-Smale sequence $\{u_n\}$ for $J_{\lambda,M}(u)$ in $W_0^{1,p}(\Omega)$ is bounded. Due to $J_{\lambda,M}(u_n) \to c$, $J'_{\lambda,M}(u_n) \to 0$ as $n \to \infty$, we see that

$$\begin{aligned} c + o(1) &= J_{\lambda,M}(u_n) - \frac{1}{r}\langle J'_{\lambda,M}(u_n), u_n\rangle \\ &= \frac{a(r-2p)}{2pr}\|u_n\|^{2p} + \frac{b(r-p)}{pr}\|u_n\|^p - \frac{\lambda(r-q)}{qr}\int_\Omega f|u_n|^q \mathrm{d}x \\ &\geq \frac{a(r-2p)}{2pr}\|u_n\|^{2p} + \frac{b(r-p)}{pr}\|u_n\|^p - \frac{\lambda(r-q)}{qr}\|f\|_\infty S_q^{-\frac{q}{p}}\|u_n\|^q, \end{aligned}$$

using the Young's inequality, we can conclude that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

Next, we prove that each Palais-Smale sequence for $J_{\lambda,M}(u)$ in $W_0^{1,p}(\Omega)$ has a strongly convergent subsequence. Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, we know that there exists a subsequence, still denoted by $\{u_n\}$ and $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega), \\ &u_n \to u \text{ strongly in } L^r(\Omega) \text{ for } 1 < r < p^*, \\ &u_n \to u \text{ almost everywhere in } \Omega. \end{aligned}$$

Denote $P_n = \langle J'_{\lambda,M}(u_n), u_n - u\rangle$ and $Q_n = M(\|u_n\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_n - u)\mathrm{d}x$, we have

$$\begin{aligned} &\lim_{n\to\infty} P_n = 0 \quad\text{and}\quad \lim_{n\to\infty} Q_n = 0, \\ &\lim_{n\to\infty}\int_\Omega f(x)|u_n|^{q-2}u_n(u_n - u)\mathrm{d}x = 0, \\ &\lim_{n\to\infty}\int_\Omega g(x)|u_n|^{r-2}u_n(u_n - u)\mathrm{d}x = 0. \end{aligned}$$

Noting that

$$\begin{aligned} P_n - Q_n = &M(\|u_n\|^p)\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u)\mathrm{d}x \\ &- \lambda\int_\Omega f|u_n|^{q-2}u_n(u_n - u)\mathrm{d}x - \int_\Omega g|u_n|^{r-2}u_n(u_n - u)\mathrm{d}x, \end{aligned}$$

we can derive that

$$\lim_{n\to\infty} M(\|u_n\|^p)\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u)\mathrm{d}x = 0.$$

Moreover, using the standard inequality in R^n given by

$$\begin{aligned} &\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta,\ \xi - \eta\rangle \geq C_p|\xi - \eta|^p, && p \geq 2, \\ &\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta,\ \xi - \eta\rangle \geq C_p|\xi - \eta|^2(|\xi| + |\eta|)^{p-2}, && 1 < p < 2, \end{aligned}$$

we get $\|u_n - u\|_{W_0^{1,p}(\Omega)} \to 0$ as $n \to \infty$. Then, $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. □

Lemma 2.4. *(i) If $2p < r < p^*$ and $0 < \lambda < \max\{\lambda_1(a), \lambda_2(a), \lambda_3\}$, then the submanifold $N^0_{\lambda,M} = \emptyset$ for all $a > 0$;*

(ii) If $r = 2p$ and $a \geq \frac{1}{\Lambda}$, then the submanifold $N^+_{\lambda,M} = N_{\lambda,M}$ for all $\lambda > 0$;

(iii) If $r = 2p$, $a < \frac{1}{\Lambda}$ and $0 < \lambda < \lambda_0(a)$, then the submanifold $N^0_{\lambda,M} = \emptyset$;

(iv) If $r < 2p$ and $a > A$, then the submanifold $N^+_{\lambda,M} = N_{\lambda,M}$ for all $\lambda > 0$.

Proof. (i) Suppose that $N^0_{\lambda,M} \neq \emptyset$, then for $u \in N^0_{\lambda,M}$, by (3), (4), arithmetic-geometric and Sobolev inequality, one has

$$\left.\begin{aligned} 2\sqrt{ab(2p-q)(p-q)}\|u\|^{\frac{3p}{2}} \\ a(2p-q)\|u\|^{2p} \\ b(p-q)\|u\|^{p} \end{aligned}\right\} \leq a(2p-q)\|u\|^{2p} + b(p-q)\|u\|^p$$

$$\leq (r-q)\|g\|_\infty S_r^{-\frac{r}{p}}\|u\|^r,$$

$$\left.\begin{aligned} 2\sqrt{ab(r-2p)(r-p)}\|u\|^{\frac{3p}{2}} \\ a(r-2p)\|u\|^{2p} \\ b(r-p)\|u\|^{p} \end{aligned}\right\} \leq a(r-2p)\|u\|^{2p} + b(r-p)\|u\|^p$$

$$\leq \lambda(r-q)\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q,$$

it follows that

$$\Big(\frac{2\sqrt{ab(2p-q)(p-q)}S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{2}{2r-3p}} \leq \|u\| \leq \Big(\frac{\lambda(r-q)\|f\|_\infty}{2\sqrt{ab(r-2p)(r-p)}S_q^{\frac{q}{p}}}\Big)^{\frac{2}{3p-2q}},$$

$$\Big(\frac{a(2p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{1}{r-2p}} \leq \|u\| \leq \Big(\frac{\lambda(r-q)\|f\|_\infty}{a(r-2p)S_q^{\frac{q}{p}}}\Big)^{\frac{1}{2p-q}},$$

$$\Big(\frac{b(p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{1}{r-p}} \leq \|u\| \leq \Big(\frac{\lambda(r-q)\|f\|_\infty}{b(r-p)S_q^{\frac{q}{p}}}\Big)^{\frac{1}{p-q}}.$$

This implies $\lambda \geq \max\{\lambda_1(a), \lambda_2(a), \lambda_3\}$, which is a contradiction. Therefore, we can get the submanifold $N^0_{\lambda,M} = \emptyset$ if $0 < \lambda < \max\{\lambda_1(a), \lambda_2(a), \lambda_3\}$.

(ii) If $r = 2p$ and $a \geq \frac{1}{\Lambda}$, then combing (3) with (5), we have

$$\begin{aligned} K''_{\lambda,M}(1) =& a(2p-q)\|u\|^{2p} + b(p-q)\|u\|^p - (2p-q)\int_\Omega g|u|^{2p}\mathrm{d}x \\ \geq& \frac{(a\Lambda-1)(2p-q)}{\Lambda}\|u\|^{2p} + b(p-q)\|u\|^p > 0 \end{aligned}$$

for all $u \in N_{\lambda,M}$. Thus, $N^+_{\lambda,M} = N_{\lambda,M}$ for all $\lambda > 0$.

(iii) Suppose that $N^0_{\lambda,M} \neq \emptyset$, then for $u \in N^0_{\lambda,M}$, using (3), (4), (5) and Sobolev inequality, we get

$$b(p-q)\|u\|^p = (2p-q)(\int_\Omega g|u|^{2p}\mathrm{d}x - a\|u\|^{2p}) \leq (2p-q)\frac{1-a\Lambda}{\Lambda}\|u\|^{2p},$$

$$bp\|u\|^p = \lambda(2p-q)\int_\Omega f|u|^q\mathrm{d}x \leq \lambda(2p-q)\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q,$$

that is,

$$\Big(\frac{b(p-q)\Lambda}{(2p-q)(1-a\Lambda)}\Big)^{\frac{1}{p}} \leq \|u\| \leq \Big(\frac{\lambda(2p-q)\|f\|_\infty}{bpS_q^{\frac{q}{p}}}\Big)^{\frac{1}{p-q}}.$$

This implies $\lambda \geq \lambda_0(a)$, which is a contradiction. Thus if $0 < \lambda < \lambda_0(a)$, then the submanifold $N^0_{\lambda,M} = \emptyset$.

(iv) If $r < 2p$ and $u \in N_{\lambda,M}$, then by (3) and Sobolev inequality, we find that

$$K''_{\lambda,M}(1) \geq \|u\|^p[a(2p-q)\|u\|^p + b(p-q) - (r-q)\|g\|_\infty S_r^{-\frac{r}{p}}\|u\|^{r-p}].$$

Since $a > A = \frac{\left((r-g)\|g\|_\infty S_r^{-\frac{r}{p}}\right)^{\frac{p}{r-p}}}{\left(b(p-q)\right)^{\frac{2p-r}{r-p}}(2p-q)}$, then we have

$$a(2p-q)\|u\|^p + b(p-q) - (r-q)\|g\|_\infty S_r^{-\frac{r}{p}}\|u\|^{r-p} > 0$$

and $K''_{\lambda,M}(1) > 0$. Hence, $N^+_{\lambda,M} = N_{\lambda,M}$ for all $\lambda > 0$. $\square$

Lemma 2.5. *(i) If $2p < r < p^*$ and $0 < \lambda < \max\{\lambda_1(a), \lambda_2(a), \lambda_3\}$, then $N_{\lambda,M} = N^+_{\lambda,M}\bigcup N^-_{\lambda,M}$ and $N^\pm_{\lambda,M} \neq \emptyset$ for all $a > 0$;*

(ii) If $r = 2p$ and $a \geq \frac{1}{\Lambda}$, then $N^+_{\lambda,M} = N_{\lambda,M} \neq \emptyset$ for all $\lambda > 0$;

(iii) If $r = 2p$, $a < \frac{1}{\Lambda}$ and $0 < \lambda < \lambda_0(a)$, then $N_{\lambda,M} = N^+_{\lambda,M}\bigcup N^-_{\lambda,M}$ and $N^\pm_{\lambda,M} \neq \emptyset$;

(iv) If $r < 2p$ and $a > A$, then $N^+_{\lambda,M} = N_{\lambda,M} \neq \emptyset$ for all $\lambda > 0$.

To prove Lemma 2.5, we require the following lemmas.

Lemma 2.6. *Assume that $2p < r \leq p^*$ and $0 < \lambda < \max\{\lambda_1(a), \lambda_2(a), \lambda_3\}$. Then for each $u \in W_0^{1,p}(\Omega)$, there are unique $0 < t^+ < t_{a,max} < t^-$ such that $t^+u \in N^+_{\lambda,M}$, $t^-u \in N^-_{\lambda,M}$ and*

$$J_{\lambda,M}(t^+u) = \inf_{0\leq t\leq t_{a,max}} J_{\lambda,M}(tu), \quad J_{\lambda,M}(t^-u) = \sup_{t\geq t_{a,max}} J_{\lambda,M}(tu).$$

Proof. Fix $u \in W_0^{1,p}(\Omega)$, we define

$$h_a(t) = at^{2p-q}\|u\|^{2p} + bt^{p-q}\|u\|^p - t^{r-q}\int_\Omega g|u|^r \mathrm{d}x \quad \text{ for } a,\ t \geq 0,$$

then it is easy to see that $h_a(0) = 0$, $\lim_{t\to+\infty} h_a(t) = -\infty$, $h_a(t)$ achieves its maximum at $t = t_{a,max}$, increasing for $t \in [0, t_{a,max})$ and decreasing for $t \in (t_{a,max}, +\infty)$.

Now, we divide the proof into three cases:

Case (i). If $\max\{\lambda_1(a), \lambda_2(a), \lambda_3\} = \lambda_1(a)$. Let

$$m_a(t) = 2\sqrt{ab}t^{\frac{3p-2q}{2}}\|u\|^{\frac{3p}{2}} - t^{r-q}\int_\Omega g|u|^r \mathrm{d}x \quad \text{ for } a,\ t \geq 0,$$

then, it is easy to see that $m_a(t) \leq h_a(t)$, $m_a(0) = 0$, $\lim_{t\to+\infty} m_a(t) = -\infty$ and there is a unique $\bar{t}_{\max} = \left(\frac{(3p-2q)\sqrt{ab}\|u\|^{\frac{3p}{2}}}{(r-q)\int_\Omega g|u|^r \mathrm{d}x}\right)^{\frac{2}{2r-3p}}$ such that $m_a(t)$ achieves its maximum at $t = \bar{t}_{\max}$, increasing for $t \in [0, \bar{t}_{\max})$ and decreasing for $t \in (\bar{t}_{\max}, +\infty)$. Moreover,

$$m_a(\bar{t}_{\max}) \geq \frac{(2r-3p)\sqrt{ab}}{r-q}\left(\frac{\sqrt{ab}(3p-2q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\right)^{\frac{3p-2q}{2r-3p}}\|u\|^q.$$

Since

$$\begin{aligned}
h_a(0) = 0 < \lambda\int_\Omega f|u|^q \mathrm{d}x &\leq \lambda\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q \\
&< \frac{(2r-3p)\sqrt{ab}}{r-q}\left(\frac{\sqrt{ab}(3p-2q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\right)^{\frac{3p-2q}{2r-3p}}\|u\|^q \\
&= m_a(\bar{t}_{max}) \leq h_a(t_{a,max}),
\end{aligned}$$

therefore, there are unique t^+ and t^- such that $0 < t^+ < t_{a,max} < t^-$, $h_a(t^+) = \lambda\int_\Omega f|u|^q\mathrm{d}x = h_a(t^-)$ and $h_a'(t^+) > 0 > h_a'(t^-)$. A bunch of computations yield

$$\begin{aligned}
&\frac{d}{dt}J_{\lambda,M}(t^\pm u) = \frac{1}{t}\big\langle J_{\lambda,M}(t^\pm u), t^\pm u\big\rangle = (t^\pm u)^{q-1}(h_a(t^\pm u) - \lambda\int_\Omega f|u|^q\mathrm{d}x) = 0,\\
&\quad K''_{t^+u,M}(1) = (t^+)^2K''_{u,M}(t^+) = (t^+)^{q+1}h_a'(t^+) > 0,\\
&\quad K''_{t^-u,M}(1) = (t^-)^2K''_{u,M}(t^-) = (t^-)^{q+1}h_a'(t^-) < 0,
\end{aligned}$$

thus, $t^+u \in N^+_{\lambda,M}$, $t^-u \in N^-_{\lambda,M}$, and

$$J_{\lambda,M}(t^+u) = \inf_{0\le t\le t_{a,\max}} J_{\lambda,M}(tu), \quad J_{\lambda,M}(t^-u) = \sup_{t\ge t_{a,\max}} J_{\lambda,M}(tu).$$

Case *(ii)***.** If $\max\{\lambda_1(a), \lambda_2(a), \lambda_3\} = \lambda_2(a)$. Let

$$n_a(t) = at^{2p-q}\|u\|^{2p} - t^{r-q}\int_\Omega g|u|^r\mathrm{d}x \quad \text{ for } a,\ t \ge 0,$$

then $n_a(t) \le h_a(t)$, $n_a(0) = 0$, $\lim_{t\to+\infty} n_a(t) = -\infty$ and there is a unique $\underline{t}_{\max} = \big(\frac{a(2p-q)\|u\|^{2p}}{(r-q)\int_\Omega g|u|^r\mathrm{d}x}\big)^{\frac{1}{r-2p}}$ such that $n_a(t)$ achieves its maximum at $t = \underline{t}_{\max}$, increasing for $t \in [0, \underline{t}_{\max})$ and decreasing for $t \in (\underline{t}_{\max}, +\infty)$. Moreover,

$$n_a(\underline{t}_{\max}) \ge a^{\frac{r-q}{r-2p}}\big(\frac{r-2p}{r-q}\big)\big(\frac{(2p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\big)^{\frac{2p-q}{r-2p}}\|u\|^q.$$

Since

$$\begin{aligned}
h_a(0) = 0 < \lambda\int_\Omega f|u|^q\mathrm{d}x &\le \lambda\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q\\
&< a^{\frac{r-q}{r-2p}}\big(\frac{r-2p}{r-q}\big)\big(\frac{(2p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\big)^{\frac{2p-q}{r-2p}}\|u\|^q\\
&\le n_a(\underline{t}_{max}) \le h_a(t_{a,max}),
\end{aligned}$$

then, there are unique t^+ and t^- such that $0 < t^+ < t_{a,max} < t^-$, $h_a(t^+) = \lambda\int_\Omega f|u|^q\mathrm{d}x = h_a(t^-)$ and $h_a'(t^+) > 0 > h_a'(t^-)$. Repeating the same argument of Case (i), we conclude that $t^+u \in N^+_{\lambda,M}$, $t^-u \in N^-_{\lambda,M}$, and

$$J_{\lambda,M}(t^+u) = \inf_{0\le t\le t_{a,\max}} J_{\lambda,M}(tu), \quad J_{\lambda,M}(t^-u) = \sup_{t\ge t_{a,\max}} J_{\lambda,M}(tu).$$

Case *(iii)***.** If $\max\{\lambda_1(a), \lambda_2(a), \lambda_3\} = \lambda_3$. Let

$$h_0(t) = bt^{p-q}\|u\|^p - t^{r-q}\int_\Omega g|u|^r\mathrm{d}x \quad \text{ for } \quad t \ge 0,$$

then, we see that $h_0(t) \le h_a(t)$, $h_0(0) = 0$, $\lim_{t\to+\infty} h_0(t) = -\infty$ and there exists a unique $t_{0,\max} = \big(\frac{b(p-q)\|u\|^p}{(r-q)\int_\Omega g|u|^r\mathrm{d}x}\big)^{\frac{1}{r-p}}$ such that $h_0(t)$ achieves its maximum at $t = t_{0,\max}$, increasing for $t \in [0, t_{0,\max})$ and decreasing for $t \in (t_{0,\max}, +\infty)$. Moreover,

$$h_0(t_{0,\max}) \ge b^{\frac{r-q}{r-p}}\big(\frac{r-p}{r-q}\big)\big(\frac{(p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\big)^{\frac{p-q}{r-p}}\|u\|^q.$$

On the other hand, since

$$\begin{aligned} h_a(0) = 0 &< \lambda \int_\Omega f|u|^q \mathrm{d}x \le \lambda \|f\|_\infty S_q^{-\frac{q}{p}} \|u\|^q \\ &< b^{\frac{r-q}{r-p}} \Big(\frac{r-p}{r-q}\Big) \Big(\frac{(p-q) S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{p-q}{r-p}} \|u\|^q \\ &\le h_0(t_{0,max}) < h_a(t_{a,max}), \end{aligned}$$

therefore, there are unique t^+ and t^- such that $0 < t^+ < t_{a,max} < t^-$, $h_a(t^+) = \lambda \int_\Omega f|u|^q \mathrm{d}x = h_a(t^-)$ and $h_a'(t^+) > 0 > h_a'(t^-)$. Repeating the same argument of Case (i), we conclude that $t^+u \in N^+_{\lambda,M}$, $t^-u \in N^-_{\lambda,M}$, and

$$J_{\lambda,M}(t^+u) = \inf_{0\le t\le t_{a,\max}} J_{\lambda,M}(tu), \quad J_{\lambda,M}(t^-u) = \sup_{t\ge t_{a,\max}} J_{\lambda,M}(tu).$$

This completes the proof of Lemma 2.6. □

Lemma 2.7. *Assume that $r = 2p$, $a \ge \frac{1}{\Lambda}$. Then for each $u \in W_0^{1,p}(\Omega)$, there is a uniquely determined number $t^+ > 0$ such that $t^+u \in N^+_{\lambda,M}$ and $J_{\lambda,M}(t^+u) = \inf\limits_{t\ge 0} J_{\lambda,M}(tu)$.*

Proof. Fix $u \in W_0^{1,p}(\Omega)$, let

$$\overline{h}_a(t) = t^{2p-q}(a\|u\|^{2p} - \int_\Omega g|u|^{2p}\mathrm{d}x) + bt^{p-q}\|u\|^p \quad \text{ for } t \ge 0,$$

we see that $\overline{h}_a(0) = 0$ and $\lim\limits_{t\to+\infty} \overline{h}_a(t) = +\infty$. Since

$$\overline{h}_a'(t) = (2p-q)t^{2p-q-1}(a\|u\|^{2p} - \int_\Omega g|u|^{2p}\mathrm{d}x) + (p-q)bt^{p-q-1}\|u\|^p,$$

we can conclude that $\overline{h}_a(t)$ is increasing for $t \in [0, +\infty)$. Thus, there is a unique $t^+ > 0$ such that $\overline{h}_a(t^+) = \lambda \int_\Omega f|u|^q \mathrm{d}x$ and $\overline{h}_a'(t^+) > 0$. Repeating the same argument of Lemma 2.6, we conclude that $t^+u \in N^+_{\lambda,M}$ and $J_{\lambda,M}(t^+u) = \inf\limits_{t\ge0} J_{\lambda,M}(tu)$. □

Lemma 2.8. *Assume that $r = 2p$, $a < \frac{1}{\Lambda}$ and $0 < \lambda < \lambda_0(a)$. Then for each $u \in W_0^{1,p}(\Omega)$, there is a uniquely determined number $0 < t^+ < t_{max}$ such that $t^+u \in N^+_{\lambda,M}$ and $J_{\lambda,M}(t^+u) = \inf\limits_{0\le t\le t_{max}} J_{\lambda,M}(tu)$.*

Proof. Fix $u \in W_0^{1,p}(\Omega)$, let

$$\overline{h}(t) = bt^{-p}\|u\|^p - \lambda t^{q-2p} \int_\Omega f|u|^q \mathrm{d}x \quad \text{ for } t > 0,$$

it is not difficult to see that $\lim\limits_{t\to 0^+} \overline{h}(t) = -\infty$ and $\lim\limits_{t\to+\infty} \overline{h}(t) = 0$. Since

$$\overline{h}'(t) = -pbt^{-p-1}\|u\|^p - (q-2p)\lambda t^{q-2p-1} \int_\Omega f|u|^q \mathrm{d}x,$$

we can conclude that there exists a unique $t_{\max} = (\frac{\lambda(2p-q)\int_\Omega f|u|^q \mathrm{d}x}{pb\|u\|^p})^{\frac{1}{p-q}}$ such that $\overline{h}(t)$ reaches its maximum at $t = t_{\max}$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, +\infty)$. Furthermore, due to

$$\begin{aligned}\overline{h}(t_{\max}) &\geq \frac{b(p-q)}{2p-q}\Big(\frac{bpS_q^{\frac{q}{p}}}{\lambda(2p-q)\|f\|_\infty}\Big)^{\frac{p}{p-q}}\|u\|^{2p}\\ &> \frac{1-a\Lambda}{\Lambda}\|u\|^{2p},\end{aligned}$$

and

$$\int_\Omega g|u|^{2p}\mathrm{d}x - a\|u\|^{2p} \leq \frac{1-a\Lambda}{\Lambda}\|u\|^{2p} < \overline{h}(t_{\max}),$$

thus, there is a unique t^+ such that $0 < t^+ < t_{max}$, $\overline{h}(t^+) = \int_\Omega g|u|^{2p}\mathrm{d}x - a\|u\|^{2p}$ and $\overline{h}'(t^+) > 0$. Repeating the same argument of Lemma 2.6, we conclude that $t^+u \in N^+_{\lambda,M}$ and $J_{\lambda,M}(t^+u) = \inf\limits_{0\leq t\leq t_{max}} J_{\lambda,M}(tu)$. □

Lemma 2.9. *Assume that $r = 2p$, $a < \frac{1}{\Lambda}$ and $0 < \lambda < \lambda_0(a)$, and let $\phi_\Lambda > 0$ as in (5). Then there exit two uniquely determined numbers t^+ and t^- satisfying $0 < t^+ < t_{\phi,max} < t^-$, such that $t^+\phi_\Lambda \in N^+_{\lambda,M}$, $t^-\phi_\Lambda \in N^-_{\lambda,M}$ and*

$$J_{\lambda,M}(t^+\phi_\Lambda) = \inf_{0\leq t\leq t_{\phi,max}} J_{\lambda,M}(t\phi_\Lambda), \quad J_{\lambda,M}(t^-\phi_\Lambda) = \sup_{t\geq t_{\phi,max}} J_{\lambda,M}(t\phi_\Lambda).$$

Proof. Let

$$h_\phi(t) = bt^{p-q}\|\phi_\Lambda\|^p - t^{2p-q}\Big(\int_\Omega g|\phi_\Lambda|^{2p}\mathrm{d}x - a\|\phi_\Lambda\|^{2p}\Big) \quad \text{for } a,\ t\geq 0,$$

combing (5) with $a < \frac{1}{\Lambda}$, it follows that $\int_\Omega g|\phi_\Lambda|^{2p}\mathrm{d}x - a\|\phi_\Lambda\|^{2p} = 1 - a\Lambda > 0$. Then, we have $h_\phi(0) = 0$, $\lim\limits_{t\to+\infty} h_\phi(t) = -\infty$ and there is a unique $t_{\phi,\max} = \big(\frac{b(p-q)\Lambda^{\frac{1}{2}}}{(2p-q)(1-a\Lambda)}\big)^{\frac{1}{p}}$ such that $h_\phi(t)$ achieves its maximum at $t = t_{\phi,\max}$, increasing for $t \in [0, t_{\phi,\max})$ and decreasing for $t \in (t_{\phi,\max}, +\infty)$. Moreover,

$$h_\phi(t_{\phi,\max}) = \Big(\frac{b(p-q)\Lambda^{\frac{1}{2}}}{(2p-q)(1-a\Lambda)}\Big)^{\frac{p-q}{p}}\frac{bp\Lambda^{\frac{1}{2}}}{2p-q},$$

and

$$\begin{aligned}h_\phi(0) = 0 < \lambda\int_\Omega f|\phi_\Lambda|^q\mathrm{d}x &\leq \lambda\|f\|_\infty S_q^{-\frac{q}{p}}\|\phi_\Lambda\|^q\\ &< \Big(\frac{b(p-q)\Lambda^{\frac{1}{2}}}{(2p-q)(1-a\Lambda)}\Big)^{\frac{p-q}{p}}\frac{bp\Lambda^{\frac{1}{2}}}{2p-q}\\ &= h_\phi(t_{\phi,max}).\end{aligned}$$

The rest of proof is similar to the proof of Lemma 2.6, we omit the details here. □

Lemma 2.10. *Assume that $r < 2p$ and $a > A$. Then for each $u \in W_0^{1,p}(\Omega)$ and $\lambda > 0$, there is a unique $t_\lambda > 0$ such that $t_\lambda u \in N^+_{\lambda,M}$ and $J_{\lambda,M}(t_\lambda u) = \inf\limits_{t\geq 0} J_{\lambda,M}(tu)$.*

Proof. Similar to the argument in Lemma 2.6, we can prove Lemma 2.10. Here, the details are omitted. □

3. **Proofs of Theorems 1.2, 1.3 and 1.4.** In this section, we give the proofs of Theorems 1.2, 1.3 and 1.4. Applying Lemma 2.5(i), we write $N_{\lambda,M} = N^+_{\lambda,M} \cup N^-_{\lambda,M}$, and define

$$\theta^+_{\lambda,M} = \inf_{u\in N^+_{\lambda,M}} J_{\lambda,M}(u), \quad \theta^-_{\lambda,M} = \inf_{u\in N^-_{\lambda,M}} J_{\lambda,M}(u).$$

To prove Theorem 1.2 and Theorem 1.3, we need the following results.

Lemma 3.1. *If $2p < r < p^*$ and $0 < \lambda < \lambda^* = \max\{\frac{q}{\sqrt{2p}}\lambda_1(a), \frac{q}{2p}\lambda_2(a), \frac{q}{p}\lambda_3\}$, then*

(i) $\theta^+_{\lambda,M} < 0$;

(ii) $\theta^-_{\lambda,M} > k_0 > 0$ for some k_0 depending on $\lambda, a, b, r, p, q, S_r, S_q, \|f\|_\infty, \|g\|_\infty$.

In particular, $\theta^+_{\lambda,M} = \inf_{u\in N_{\lambda,M}} J_{\lambda,M}(u)$.

Proof. (i) Let $u \in N^+_{\lambda,M}$, it follows from (4) that

$$\lambda(r-q)\int_\Omega f|u|^q \mathrm{d}x > a(r-2p)\|u\|^{2p} + b(r-p)\|u\|^p,$$

substituting it into $J_{\lambda,M}(u)$, we obtain

$$\begin{aligned} J_{\lambda,M}(u) &= \frac{a(r-2p)}{2pr}\|u\|^{2p} + \frac{b(r-p)}{pr}\|u\|^p - \frac{\lambda(r-q)}{qr}\int_\Omega f|u|^q \mathrm{d}x \\ &< \frac{a(r-2p)(q-2p)}{2pqr}\|u\|^{2p} + \frac{b(r-p)(q-p)}{pqr}\|u\|^p < 0, \end{aligned}$$

so $\theta^+_{\lambda,M} = \inf_{u\in N^+_{\lambda,M}} J_{\lambda,M}(u) < 0$.

(ii) Let $u \in N^-_{\lambda,M}$, we divide the proof into the following three cases.

Case (i)**.** $\lambda^* = \frac{q}{\sqrt{2p}}\lambda_1(a)$. From (3), arithmetic-geometric and the Sobolev imbedding theorem, we find that

$$2\sqrt{ab(2p-q)(p-q)}\|u\|^{\frac{3p}{2}} < (r-q)\|g\|_\infty S_r^{-\frac{r}{p}}\|u\|^r,$$

which implies

$$\|u\| > \Big(\frac{2\sqrt{ab(2p-q)(p-q)}S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{2}{2r-3p}},$$

this show that

$$\begin{aligned} J_{\lambda,M}(u) \geq &\frac{\sqrt{2ab(r-2p)(r-p)}}{pr}\|u\|^{\frac{3p}{2}} - \frac{\lambda(r-q)}{qr}\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q \\ > &\Big(\frac{2\sqrt{ab(2p-q)(p-q)}S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{2q}{2r-3p}}\Big[\frac{\sqrt{2ab(r-2p)(r-p)}}{pr} \\ &\times\Big(\frac{2\sqrt{ab(2p-q)(p-q)}S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{3p-2q}{2r-3p}} - \frac{\lambda(r-q)}{qr}\|f\|_\infty S_q^{-\frac{q}{p}}\Big] = k_0. \end{aligned}$$

Thus, we have $\theta^-_{\lambda,M} > k_0 > 0$ for $0 < \lambda < \frac{q}{\sqrt{2p}}\lambda_1(a)$, where k_0 depending on λ, a, b, r, p, q, S_r, S_q, $\|f\|_\infty$, $\|g\|_\infty$.

Case (ii)**.** $\lambda^* = \frac{q}{2p}\lambda_2(a)$. Using (3) and the Sobolev imbedding theorem, we see that

$$a(2p-q)\|u\|^{2p} < (r-q)\|g\|_\infty S_r^{-\frac{r}{p}}\|u\|^r,$$

which implies

$$\|u\| > \Big(\frac{a(2p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{1}{r-2p}}.$$

Then, we have

$$\begin{aligned} J_{\lambda,M}(u) \geq & \frac{a(r-2p)}{2pr}\|u\|^{2p} - \frac{\lambda(r-q)}{qr}\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q \\ > & \Big(\frac{a(2p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{q}{r-2p}}\Big[\frac{a(r-2p)}{2pr}\Big(\frac{a(2p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{2p-q}{r-2p}} \\ & - \frac{\lambda(r-q)}{qr}\|f\|_\infty S_q^{-\frac{q}{p}}\Big] = k_0 > 0 \end{aligned}$$

for $0 < \lambda < \frac{q}{2p}\lambda_2(a)$.

Case (iii)**.** $\lambda^* = \frac{q}{p}\lambda_3$. Combining (3) with Sobolev's imbedding theorem, we get

$$b(p-q)\|u\|^p < (r-q)\|g\|_\infty S_r^{-\frac{r}{p}}\|u\|^r,$$

which indicates

$$\|u\| > \Big(\frac{b(p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{1}{r-p}}.$$

Then, one has

$$\begin{aligned} J_{\lambda,M}(u) \geq & \frac{b(r-p)}{pr}\|u\|^{p} - \frac{\lambda(r-q)}{qr}\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q \\ > & \Big(\frac{b(p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{q}{r-p}}\Big[\frac{b(r-p)}{pr}\Big(\frac{b(p-q)S_r^{\frac{r}{p}}}{(r-q)\|g\|_\infty}\Big)^{\frac{p-q}{r-p}} \\ & - \frac{\lambda(r-q)}{qr}\|f\|_\infty S_q^{-\frac{q}{p}}\Big] = k_0 > 0 \end{aligned}$$

for $0 < \lambda < \frac{q}{p}\lambda_3$. This completes the proof of Lemma 3.1. □

Lemma 3.2. *If $r = 2p$, $a < \frac{1}{\Lambda}$ and $0 < \lambda < \frac{q}{p}\lambda_0(a)$, then*

(i) $\theta^+_{\lambda,M} < 0$;

(ii) $\theta^-_{\lambda,M} > K_0 > 0$ for K_0 depending on $\lambda, a, b, p, q, \Lambda, S_q, \|f\|_\infty$.

In particular, $\theta^+_{\lambda,M} = \inf_{u\in N_{\lambda,M}} J_{\lambda,M}(u)$.

Proof. (i) Let $u \in N^+_{\lambda,M}$, by (4), we see that

$$\lambda(2p-q)\int_\Omega f|u|^q \mathrm{d}x > bp\|u\|^p,$$

and

$$\begin{aligned} J_{\lambda,M}(u) = & \frac{b}{2p}\|u\|^p - \frac{\lambda(2p-q)}{2pq}\int_\Omega f|u|^q\mathrm{d}x \\ < & \frac{b}{2p}\|u\|^p - \frac{b}{2q}\|u\|^p < 0. \end{aligned}$$

Hence, we have $\theta^+_{\lambda,M} = \inf_{u\in N^+_{\lambda,M}} J_{\lambda,M}(u) < 0$.

(ii) Let $u \in N^-_{\lambda,M}$, from (3), we find that

$$b(p-q)\|u\|^p < (2p-q)[\int_\Omega g|u|^r \mathrm{d}x - a\|u\|^{2p}] \leq \frac{(2p-q)(1-a\Lambda)}{\Lambda}\|u\|^{2p},$$

which implies that

$$\|u\| > \left(\frac{b\Lambda(p-q)}{(2p-q)(1-a\Lambda)}\right)^{\frac{1}{p}}. \tag{6}$$

On the other hand, since

$$\begin{aligned} J_{\lambda,M}(u) \geq & \frac{b}{2p}\|u\|^p - \frac{\lambda(2p-q)}{2pq}\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q \\ > & \left(\frac{b\Lambda(p-q)}{(2p-q)(1-a\Lambda)}\right)^{\frac{q}{p}}\Big[\frac{b}{2p}\left(\frac{b\Lambda(p-q)}{(2p-q)(1-a\Lambda)}\right)^{\frac{p-q}{p}} \\ & - \frac{\lambda(2p-q)}{2qp}\|f\|_\infty S_q^{-\frac{q}{p}}\Big] = K_0. \end{aligned} \tag{7}$$

Thus, we have $\theta^-_{\lambda,M} > K_0 > 0$ for $0 < \lambda < \frac{q}{p}\lambda_0(a)$, where K_0 depending on $\lambda, a, b, p, q, \Lambda, S_q, \|f\|_\infty$. The proof of Lemma 3.2 is complete. □

Proof of Theorem 1.2. Applying Lemma 2.2 (i), Lemma 2.5 (i), Lemma 3.1 and the Ekeland variational principle [13], we obtain that there exist two minimizing sequences $\{u_n^\pm\}$ for $J_{\lambda,M}(u)$ in $N^\pm_{\lambda,M}$ such that

$$J_{\lambda,M}(u_n^\pm) = \theta^\pm_{\lambda,M} + o_n(1), \quad J'_{\lambda,M}(u_n^\pm) = o_n(1).$$

Then, it follows from Lemma 2.3 that there exist subsequences still denoted by $\{u_n^\pm\} \subset N^\pm_{\lambda,M}$ and $u^\pm_{\lambda,M} \in W_0^{1,p}(\Omega)$ such that

$$u_n^\pm \to u^\pm_{\lambda,M} \text{ strongly in } \ W_0^{1,p}(\Omega),$$

hence, $u^\pm_{\lambda,M} \in N^\pm_{\lambda,M}$ are solutions of the problem (1) and $J_{\lambda,M}(u^\pm_{\lambda,M}) = \theta^\pm_{\lambda,M}$. On the other hand, since $J_{\lambda,M}(u^\pm_{\lambda,M}) = J_{\lambda,M}(|u^\pm_{\lambda,M}|)$ and $|u^\pm_{\lambda,M}| \in N^\pm_{\lambda,M}$, we get that $u^\pm_{\lambda,M} \in N^\pm_{\lambda,M}$ are nontrivial nonnegative solutions of the problem (1). Moreover, $N^+_{\lambda,M} \cap N^-_{\lambda,M} = \emptyset$ show that $u^+_{\lambda,M} \neq u^-_{\lambda,M}$. Thus, the problem (1) has at least two nontrivial nonnegative solutions. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. (i) By Lemma 2.5 (ii), we write $N_{\lambda,M} = N^+_{\lambda,M}$ and define

$$\theta_{\lambda,M} = \inf_{u \in N^+_{\lambda,M}} J_{\lambda,M}(u).$$

Similar to Lemma 3.2, we can conclude that $\theta_{\lambda,M} < 0$. Applying Lemma 2.2 (i) and the Ekeland variational principle [13], we obtain that there exists a minimizing sequence $\{u_n\}$ for $J_{\lambda,M}(u)$ on $N^+_{\lambda,M}$ such that

$$J_{\lambda,M}(u_n) = \theta_{\lambda,M} + o_n(1), \quad J'_{\lambda,M}(u_n) = o_n(1).$$

Then by Lemma 2.3, there exists a subsequence still denoted by $\{u_n\} \subset N^+_{\lambda,M}$ and $u_{\lambda,M} \in W_0^{1,p}(\Omega)$ such that

$$u_n \to u_{\lambda,M} \text{ strongly in } \ W_0^{1,p}(\Omega).$$

Thus, $u_{\lambda,M} \in N^+_{\lambda,M}$ is a solution of the problem (1) and $J_{\lambda,M}(u) = \theta_{\lambda,M}$. On the other hand, since $J_{\lambda,M}(u_{\lambda,M}) = J_{\lambda,M}(|u_{\lambda,M}|)$ and $|u_{\lambda,M}| \in N^+_{\lambda,M}$, we get that $u_{\lambda,M} \in N^+_{\lambda,M}$ is nontrivial nonnegative solution of the problem (1).

(ii) Similar to the proof of Theorem 1.2, we know that the problem (1) has at least two nontrivial nonnegative solutions $u^+_{\lambda,M} \in N^+_{\lambda,M}, u^-_{\lambda,M} \in N^-_{\lambda,M}$. Moreover, combining (6) with (7), we see that

$$\begin{aligned} &\|u^-_{\lambda,M}\| \to \infty \text{ as } \ a \to \frac{1}{\Lambda}^-, \\ &\lim_{a\to\frac{1}{\Lambda}^-} \inf_{u\in N^-_{\lambda,M}} J_{\lambda,M}(u) = \infty. \end{aligned}$$

This completes the proof of Theorem 1.3. □

Before giving the proof of Theorem 1.4, we introduce the following lemmas.

Lemma 3.3. *If $r = p^*$ and $0 < \lambda < \frac{b(p^*-p)S_q^{\frac{q}{p}}}{(p^*-q)\|f\|_\infty}\left(\frac{b(p-q)S_{p^*}^{\frac{p^*}{p}}}{(p^*-q)\|g\|_\infty}\right)^{\frac{p-q}{p^*-p}}$, then the submanifold $N_{\lambda,M} = N^+_{\lambda,M} \cup N^-_{\lambda,M}$ and $N^\pm_{\lambda,M} \neq \emptyset$.*

Proof. The proof is similar to Lemma 2.5(i), we omit the details here. □

Let $\theta^+_{\lambda,M} = \inf\limits_{u\in N^+_{\lambda,M}} J_{\lambda,M}(u)$, then we have

Lemma 3.4. *If $r = p^*$ and $0 < \lambda < \frac{b(p^*-p)S_q^{\frac{q}{p}}}{(p^*-q)\|f\|_\infty}\left(\frac{b(p-q)S_{p^*}^{\frac{p^*}{p}}}{(p^*-q)\|g\|_\infty}\right)^{\frac{p-q}{p^*-p}}$, then $\theta^+_{\lambda,M} < 0$.*

Proof. Let $u \in N^+_{\lambda,M}$, it follows from (4) that

$$\lambda(p^*-q)\int_\Omega f|u|^q \mathrm{d}x > a(p^*-2p)\|u\|^{2p} + b(p^*-p)\|u\|^p,$$

and

$$\begin{aligned} J_{\lambda,M}(u) =& \frac{a(p^*-2p)}{2pp^*}\|u\|^{2p} + \frac{b(p^*-p)}{pp^*}\|u\|^p - \frac{\lambda(p^*-q)}{qp^*}\int_\Omega f|u|^q\mathrm{d}x \\ <& \frac{a(p^*-2p)(q-2p)}{2pqp^*}\|u\|^{2p} + \frac{b(p^*-p)(q-p)}{pqp^*}\|u\|^p < 0, \end{aligned}$$

so $\theta^+_{\lambda,M} = \inf\limits_{u\in N^+_{\lambda,M}} J_{\lambda,M}(u) < 0$. □

Lemma 3.5. *If $r = p^*$ and $0 < \lambda < \frac{b(p^*-p)S_q^{\frac{q}{p}}}{(p^*-q)\|f\|_\infty}\left(\frac{b(p-q)S_{p^*}^{\frac{p^*}{p}}}{(p^*-q)\|g\|_\infty}\right)^{\frac{p-q}{p^*-p}}$, then $J_{\lambda,M}$ satisfies the $(PS)_{\theta^+_{\lambda,M}}$-condition.*

Proof. Let $\{u_n\} \subset N^+_{\lambda,M}$ be a $(PS)_{\theta^+_{\lambda,M}}$-sequence satisfiying

$$J_{\lambda,M}(u_n) = \theta^+_{\lambda,M} + o_n(1), \quad J'_{\lambda,M}(u_n) = o_n(1).$$

Similarly to the proof of Lemma 2.3, we know that $\{u_n\}$ is bounded in $N^+_{\lambda,M}$, and there exists a subsequence, still denoted by $\{u_n\}$ and $u \in W^{1,p}_0(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } \ W^{1,p}_0(\Omega), \\ &u_n \to u \text{ strongly in } \ L^r(\Omega) \text{ for } 1 < r < p^*, \\ &u_n \rightharpoonup u \text{ weakly in } \ L^{p^*}(\Omega), \\ &u_n \to u \text{ almost everywhere in } \Omega. \end{aligned}$$

By concentration-compactness principle [21], there exists at most set J, a set of different points $\{x_j\}_{j\in J}\subset\Omega$, sets of nonnegative real numbers $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$ such that

$$\begin{aligned}|\nabla u_n|^p\mathrm{d}x &\rightharpoonup \mathrm{d}\mu \geq |\nabla u|^p\mathrm{d}x+\sum_{j\in J}\mu_j\delta_{x_j},\\ |u_n|^{p^*}\mathrm{d}x &\rightharpoonup \mathrm{d}\nu = |u|^{p^*}\mathrm{d}x+\sum_{j\in J}\nu_j\delta_{x_j},\end{aligned}\tag{8}$$

where δ_x is the Dirac mass at x, and the constants μ_j, ν_j satisfying

$$\mu_j\geq S_{p^*}\nu_j^{\frac{p}{p^*}},\quad \text{where } x_j\in\Omega.\tag{9}$$

Following, we claim that J is finite for any $j\in J$, either $\nu_j=0$ or $\nu_j\geq\left(\frac{S_{p^*}b}{\|g\|_\infty}\right)^{\frac{p^*}{p^*-p}}$. In fact, choosing $\varepsilon>0$ sufficiently small such that $B_\varepsilon(x_i)\cap B_\varepsilon(x_j)=\emptyset$ for $i\neq j, i,j\in J$. Let $\phi_\varepsilon^j(x)$ be a smooth cut off function centered at x_j such that

$$0\leq\phi_\varepsilon^j(x)\leq 1 \text{ for } |x-x_j|<\varepsilon,\quad \phi_\varepsilon^j(x)=\begin{cases}1, & |x-x_j|\leq\frac{\varepsilon}{2},\\ 0, & |x-x_j|\geq\varepsilon,\end{cases}\quad \text{and } |\nabla\phi_\varepsilon^j|\leq\frac{4}{\varepsilon}.$$

Noting that

$$\begin{aligned}\left\langle J'_{\lambda,M}(u_n),u_n\phi_\varepsilon^j(x)\right\rangle =& M(\|u_n\|^p)\int_\Omega|\nabla u_n|^p\phi_\varepsilon^j(x)\mathrm{d}x\\ &+M(\|u_n\|^p)\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\varepsilon^j(x)u_n\mathrm{d}x\\ &-\lambda\int_\Omega f(x)|u_n|^q\phi_\varepsilon^j(x)\mathrm{d}x-\int_\Omega g(x)|u_n|^{p^*}\phi_\varepsilon^j(x)\mathrm{d}x,\end{aligned}$$

and by (8), we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}M(\|u_n\|^p)\int_\Omega|\nabla u_n|^p\phi_\varepsilon^j(x)\mathrm{d}x\geq b\mu_j,\\ &\lim_{\varepsilon\to0}\lim_{n\to\infty}M(\|u_n\|^p)\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\varepsilon^j(x)u_n\mathrm{d}x=0,\\ &\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_\Omega f(x)|u_n|^q\phi_\varepsilon^j(x)\mathrm{d}x=0,\\ &\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_\Omega g(x)|u_n|^{p^*}\phi_\varepsilon^j(x)\mathrm{d}x=g(x_j)\nu_j\leq\|g\|_\infty\nu_j.\end{aligned}$$

Thus,

$$0=\lim_{\varepsilon\to0}\lim_{n\to\infty}\left\langle J'_\lambda(u_n),u_n\phi_\varepsilon^j(x)\right\rangle\geq b\mu_j-\|g\|_\infty\nu_j.\tag{10}$$

It follows from (9) and (10) that

$$\nu_j=0\quad\text{or}\quad\nu_j\geq\left(\frac{bS_{p^*}}{\|g\|_\infty}\right)^{\frac{p^*}{p^*-p}},$$

which implies that J is finite. If $\nu_j\neq0$,

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_\Omega|\nabla u_n|^p\mathrm{d}x\geq\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_\Omega|\nabla u_n|^p\phi_\varepsilon^j(x)\mathrm{d}x$$

$$\geq \lim_{\varepsilon\to 0}\Big(\int_\Omega |\nabla u|^p \phi_\varepsilon^j(x)\mathrm{d}x + \mu_j\Big)$$

$$\geq S_{p^*}\nu_j^{\frac{p}{p^*}} \geq \Big(\frac{bS_{p^*}^{\frac{p^*}{p}}}{\|g\|_\infty}\Big)^{\frac{p}{p^*-p}}.$$

On the other hand, since $u_n \in N_{\lambda,M}^+$, we have

$$\|u_n\|^p < \Big(\frac{\lambda(p^*-q)\|f\|_\infty}{b(p^*-p)S_q^{\frac{q}{p}}}\Big)^{\frac{p}{p-q}}.$$

This implies

$$\lambda \geq \frac{b(p^*-p)S_q^{\frac{q}{p}}}{(p^*-q)\|f\|_\infty}\Big(\frac{bS_{p^*}^{\frac{p^*}{p}}}{\|g\|_\infty}\Big)^{\frac{p-q}{p^*-p}} > \frac{b(p^*-p)S_q^{\frac{q}{p}}}{(p^*-q)\|f\|_\infty}\Big(\frac{b(p-q)S_{p^*}^{\frac{p^*}{p}}}{(p^*-q)\|g\|_\infty}\Big)^{\frac{p-q}{p^*-p}},$$

which is a contradiction. Hence, $\mu_j = \nu_j = 0$ and we can obtain that $u_n \to u$ strongly in $L^{p^*}(\Omega)$ and $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. Moreover, since $u_n \in N_{\lambda,M}$, we deduce

$$\frac{\lambda(p^*-q)}{qp^*}\int_\Omega f|u_n|^q\mathrm{d}x = \frac{a(p^*-2p)}{2pp^*}\|u_n\|^{2p} + \frac{b(p^*-p)}{pp^*}\|u_n\|^p - J_{\lambda,M}(u_n),$$

letting $n\to\infty$, we have

$$\frac{\lambda(p^*-q)}{qp^*}\int_\Omega f|u|^q\mathrm{d}x \geq -\theta_{\lambda,M}^+ > 0,$$

which yields u is nonzero and $u \in N_{\lambda,M}$.

Next, we need show that $u \in N_{\lambda,M}^+$. Due to

$$a(p^*-2p)\|u_n\|^{2p} + b(p^*-p)\|u_n\|^p - \lambda(p^*-q)\int_\Omega f|u_n|^q\mathrm{d}x < 0,$$

let $n\to\infty$, It is clear that

$$a(p^*-2p)\|u\|^{2p} + b(p^*-p)\|u\|^p - \lambda(p^*-q)\int_\Omega f|u|^q\mathrm{d}x \leq 0.$$

If $a(p^*-2p)\|u\|^{2p} + b(p^*-p)\|u\|^p - \lambda(p^*-q)\int_\Omega f|u|^q\mathrm{d}x = 0$, we have $u \in N_{\lambda,M}^0$, which is a contradiction with $N_{\lambda,M}^0 = \emptyset$ for $0<\lambda<\frac{b(p^*-p)S_q^{\frac{q}{p}}}{(p^*-q)\|f\|_\infty}\Big(\frac{b(p-q)S_{p^*}^{\frac{p^*}{p}}}{(p^*-q)\|g\|_\infty}\Big)^{\frac{p-q}{p^*-p}}$. Hence, we have $u \in N_{\lambda,M}^+$. □

Proof of Theorem 1.4. Applying Lemma 2.2 (i), Lemma 3.3, Lemma 3.4 and the Ekeland variational principle [13], we obtain that there exist a minimizing sequence $\{u_n\}$ for $J_{\lambda,M}(u)$ on $N_{\lambda,M}^+$ such that

$$J_{\lambda,M}(u_n) = \theta_{\lambda,M}^+ + o_n(1), \quad J_{\lambda,M}'(u_n) = o_n(1).$$

Then, it follows from Lemma 3.5 that there exist subsequence still denoted by $\{u_n\} \subset N_{\lambda,M}^+$ and $u_{\lambda,M} \in W_0^{1,p}(\Omega)$ such that

$$u_n \to u_{\lambda,M} \text{ strongly in } \ W_0^{1,p}(\Omega),$$

hence, $u_{\lambda,M} \in N_{\lambda,M}^+$ is a solution of the problem (1) and $J_{\lambda,M}(u_{\lambda,M}) = \theta_{\lambda,M}^+$. On the other hand, since $J_{\lambda,M}(u_{\lambda,M}) = J_{\lambda,M}(|u_{\lambda,M}|)$ and $|u_{\lambda,M}| \in N_{\lambda,M}^+$, we get that $u_{\lambda,M} \in N_{\lambda,M}^+$ is nontrivial nonnegative solution of the problem (1). This completes the proof of Theorem 1.4. □

4. **Proof of Theorem 1.5.** First, we consider the following truncated problem:

$$\begin{cases} -M_k(\int_\Omega |\nabla u|^p \mathrm{d}x)\Delta_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where $k \in (\frac{b(r-p)}{ar}, \frac{b(r-p)}{pa})$ and

$$M_k(s) = \begin{cases} M(s), & s \le k, \\ M(k), & s > k \end{cases}$$

is a truncated function of $M(s)$. Then the solutions of truncated problem (11) are critical points of the energy functional

$$J_{\lambda,M_k}(u) = \frac{1}{p}\hat{\mathrm{M}}_k(\|u\|^p) - \frac{\lambda}{q}\int_\Omega f|u|^q \mathrm{d}x - \frac{1}{r}\int_\Omega g|u|^r \mathrm{d}x,$$

where $\hat{\mathrm{M}}_k(t) = \int_0^t M_k(s)\mathrm{d}s$. Thus, we have the following lemma about the functional $J_{\lambda,M_k}(u)$.

Lemma 4.1. *The energy functional $J_{\lambda,M_k}(u)$ is coercive and bounded in N_{λ,M_k}.*

Proof. If $u \in N_{\lambda,M_k}$, then by the definition of N_{λ,M_k} and the Sobolev imbedding theorem, we find that

$$J_{\lambda,M_k}(u) \ge (\frac{b}{p} - \frac{M(k)}{r})\|u\|^p - \lambda\frac{r-q}{rq}\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q,$$

since $k < \frac{b(r-p)}{pa}$, this gives $\frac{b}{p} - \frac{M(k)}{r} > 0$. Thus, $J_{\lambda,M_k}(u)$ is coercive and bounded in N_{λ,M_k} by the Young's inequality. The proof of Lemma 4.1 is complete. □

Note that by (3) and (4), if $u \in N_{\lambda,M_k}$ with $\|u\|^p \le k$, we see that

$$\begin{aligned} K''_{u,M_k}(1) =&[a(2p-q)\|u\|^p + b(p-q)]\|u\|^p - (r-q)\int_\Omega g|u|^r \mathrm{d}x \\ =&[a(2p-r)\|u\|^p - b(r-p)]\|u\|^p + \lambda(r-q)\int_\Omega f|u|^q \mathrm{d}x, \end{aligned} \tag{12}$$

and if $u \in N_{\lambda,M_k}$ with $\|u\|^p > k$, we have

$$\begin{aligned} K''_{u,M_k}(1) =&M(k)(p-q)\|u\|^p - (r-q)\int_\Omega g|u|^r \mathrm{d}x \\ =&-M(k)(r-p)\|u\|^p + \lambda(r-q)\int_\Omega f|u|^q \mathrm{d}x. \end{aligned} \tag{13}$$

Subsequently, we have the following lemmas.

Lemma 4.2. *If $r < 2p$ and $0 < \lambda < \lambda_4(a)$, then the submanifold $N^0_{\lambda,M_k} = \emptyset$.*

Proof. The proof is similar to Lemma 2.4, again we omit its details. □

Lemma 4.3. *If $r < 2p$ and $0 < \lambda < \min\{\lambda_4(a), \lambda_5(a)\}$, then the manifold $N_{\lambda,M_k} = N^+_{\lambda,M_k} \cup N^-_{\lambda,M_k}$ and $N^+_{\lambda,M} \neq \emptyset$.*

Proof. Fix $u \in W_0^{1,p}(\Omega)$, we define

$$l_a(t) = at^{2p-r}\|u\|^{2p} + bt^{p-r}\|u\|^p - \lambda t^{q-r}\int_\Omega f|u|^q \mathrm{d}x \quad \text{for } a,\ t > 0,$$

it is easy to see that $\lim\limits_{t\to 0^+} l(t) = -\infty$, and $\lim\limits_{t\to+\infty} l(t) = +\infty$. Let

$$l_0(t) = bt^{p-r}\|u\|^p - \lambda t^{q-r}\int_\Omega f|u|^q \mathrm{d}x \qquad \text{for } t > 0,$$

then $l_0(t) < l_a(t)$, $\lim\limits_{t\to 0^+} l(t) = -\infty$, $\lim\limits_{t\to+\infty} l(t) = 0$ and there is a unique $t^* = \big(\frac{\lambda(r-q)\int_\Omega f|u|^q\mathrm{d}x}{(r-p)b\|u\|^p}\big)^{\frac{1}{p-q}}$ such that $l_0(t)$ achieves its maximum at $t = t^*$, increasing for $t \in (0, t^*)$ and decreasing for $t \in (t^*, +\infty)$. Moreover,

$$\begin{aligned} l_0(t^*) =& \frac{\lambda(p-q)\int_\Omega f|u|^q\mathrm{d}x}{r-p}\big(\frac{b(r-p)\|u\|^p}{\lambda(r-q)\int_\Omega f|u|^q\mathrm{d}x}\big)^{\frac{r-q}{p-q}} \\ \geq& \frac{p-q}{r-p}\big(\frac{b(r-p)}{(r-q)}\big)^{\frac{r-q}{p-q}}\big(\frac{S_q^{\frac{q}{p}}}{\lambda\|f\|_\infty}\big)^{\frac{r-p}{p-q}}\|u\|^r \\ >& \|g\|_\infty S_r^{-\frac{r}{p}}\|u\|^r \\ \geq& \int_\Omega g|u|^r\mathrm{d}x, \end{aligned}$$

and

$$\|t^*u\|^p = \big(\frac{\lambda(r-q)\int_\Omega f|u|^q\mathrm{d}x}{(r-p)b\|u\|^p}\big)^{\frac{p}{p-q}}\|u\|^p \leq \big(\frac{\lambda(r-q)\|f\|_\infty}{(r-p)bS_q^{\frac{q}{p}}}\big)^{\frac{p}{p-q}} < k$$

for $0 < \lambda < \min\{\lambda_4(a), \lambda_5(a)\}$.

Therefore, we can obtain a $0 < t^+ < t^*$ such that $l_a(t^+) = \int_\Omega g|u|^r\mathrm{d}x$, $\|t^+u\|^p < k$ and $t^+u \in N^+_{\lambda,M_k}$. Thus, we have $N^+_{\lambda,M_k} \neq \emptyset$. □

Lemma 4.4. *If $p < \frac{2p^2}{2p-q} < r < 2p$ and $0 < \lambda < \min\{\lambda_4(a), \lambda_5(a)\}$, then we have $\theta^+_{\lambda,M_k} = \inf\limits_{u\in N^+_{\lambda,M_k}} J_{\lambda,M_k}(u) < 0$, In particular, $\theta_{\lambda,M_k} = \inf\limits_{u\in N_{\lambda,M_k}} J_{\lambda,M_k}(u) \leq \theta^+_{\lambda,M_k}$.*

Proof. If $\|u\|^p \leq k$, it follows from (12) that

$$\begin{aligned} J_{\lambda,M_k}(u) =& \frac{a(r-2p)}{2pr}\|u\|^{2p} + \frac{b(r-p)}{pr}\|u\|^p - \frac{\lambda(r-q)}{qr}\int_\Omega f|u|^q\mathrm{d}x \\ <& \frac{\|u\|^p}{pqr}\big[\frac{a(2p-r)(2p-q)}{2}k - b(r-p)(p-q)\big], \end{aligned}$$

since $k < \frac{b(r-p)}{pa}$ and $\frac{2p^2}{2p-q} < r < 2p$, then we have $J_{\lambda,M_k}(u) < 0$.

If $\|u\|^p > k$, from (13), we find that

$$\begin{aligned} J_{\lambda,M_k}(u) =& -\frac{ak^2}{2p} + \frac{M(k)(r-p)}{pr}\|u\|^p - \frac{\lambda(r-q)}{qr}\int_\Omega f|u|^q\mathrm{d}x \\ <& -\frac{ak^2}{2p} + \frac{M(k)(r-p)(q-p)}{pqr}\|u\|^p < 0. \end{aligned}$$

Therefore, $\theta^+_{\lambda,M_k} = \inf\limits_{u\in N^+_{\lambda,M_k}} J_{\lambda,M_k}(u) < 0$ and $\theta_{\lambda,M_k} \leq \theta^+_{\lambda,M_k}$. □

Lemma 4.5. *If $u \in N^+_{\lambda,M_k}$ is a solution of truncated problem (11), then there exists a constant $\tilde{C}$ such that $\|u\| \leq \tilde{C}$ and $\|u\|^p \leq \frac{\lambda\|f\|_\infty S_q^{-\frac{q}{p}}\tilde{C}^q + \|g\|_\infty S_r^{-\frac{r}{p}}\tilde{C}^r}{M(\|u\|^p)}$.*

Proof. If $\|u\|^p \le k$, we choose $\tilde{\mathrm{C}} = k^{\frac{1}{p}}$. Applying

$$M(\|u\|^p)\|u\|^p = \lambda \int_\Omega f|u|^q \mathrm{d}x + \int_\Omega g|u|^r \mathrm{d}x,$$

we find that

$$\|u\|^p \le \frac{\lambda\|f\|_\infty S_q^{-\frac{q}{p}}\tilde{\mathrm{C}}^q + \|g\|_\infty S_r^{-\frac{r}{p}}\tilde{\mathrm{C}}^r}{M(\|u\|^p)}.$$

If $\|u\|^p > k$, we choose $\tilde{\mathrm{C}}^p = \frac{ark^2}{br-pM(k)} + (\frac{\lambda 2pr(r-q)\|f\|_\infty}{rq(br-M(k)p)S_q^{\frac{q}{p}}})^{\frac{p}{p-q}}$. by $u \in N^+_{\lambda,M_k}$ and the Young's inequality, we have

$$\begin{aligned}
0 >&\theta^+_{\lambda,M_k}\\
=&J_{\lambda,M_k}(u) = -\frac{a}{2p}k^2 + \frac{(r-p)M(k)}{pr}\|u\|^p - \frac{\lambda(r-q)}{qr}\int_\Omega f|u|^q\mathrm{d}x\\
>&-\frac{a}{2p}k^2 + \frac{br-pM(k)}{pr}\|u\|^p - \frac{\lambda(r-q)\|f\|_\infty}{qrS_q^{\frac{q}{p}}}\|u\|^q\\
\ge&-\frac{a}{2p}k^2 + \frac{br-pM(k)}{2pr}\|u\|^p - (\frac{br-pM(k)}{2pr})^{-\frac{q}{p-q}}(\frac{\lambda(r-q)}{rq}\|f\|_\infty S_q^{-\frac{q}{p}})^{\frac{p}{p-q}},
\end{aligned}$$

which implies

$$\|u\|^p < \frac{ark^2}{br-pM(k)} + (\frac{\lambda 2pr(r-q)\|f\|_\infty}{rq(br-M(k)p)S_q^{\frac{q}{p}}})^{\frac{p}{p-q}} = \tilde{\mathrm{C}}^p.$$

Furthermore, we can get

$$\|u\|^p \le \frac{\lambda\|f\|_\infty S_q^{-\frac{q}{p}}\tilde{\mathrm{C}}^q + \|g\|_\infty S_r^{-\frac{r}{p}}\tilde{\mathrm{C}}^r}{M(\|u\|^p)}.$$

□

Proof of Theorem 1.5. (i) By Lemma 2.2 (ii) and the Ekeland variational principle [13], we obtain that there exists a minimizing sequence $\{u_n\}$ for $J_{\lambda,M}(u)$ on $W_0^{1,p}(\Omega)$ such that

$$J_{\lambda,M}(u_n) = c_{\lambda,M} + o_n(1), \quad J'_{\lambda,M}(u_n) = o_n(1),$$

where $c_{\lambda,M} = \inf_{u\in W_0^{1,p}(\Omega)} J_{\lambda,M}(u_n) < 0$. Then, utilizing Lemma 2.3, there exists a subsequence still denoted by $\{u_n\} \subset W_0^{1,p}(\Omega)$ and $u_{\lambda,M} \in W_0^{1,p}(\Omega)$ such that

$$u_n \to u_{\lambda,M} \text{ strongly in } W_0^{1,p}(\Omega),$$

so $u_{\lambda,M} \in W_0^{1,p}(\Omega)$ is a nonzero solution of the problem (1) and $J_{\lambda,M}(u_{\lambda,M}) = c_{\lambda,M}$. On the other hand, since $J_{\lambda,M}(u_{\lambda,M}) = J_{\lambda,M}(|u_{\lambda,M}|)$ and $|u_{\lambda,M}| \in W_0^{1,p}(\Omega)$, we get that $u_{\lambda,M} \in W_0^{1,p}(\Omega)$ is a nontrivial nonnegative solution of the problem (1). Similarly, we can prove that the problem (1) has at least one nontrivial nonnegative solution $u_{\lambda,M} \in N^+_{\lambda,M} = N_{\lambda,M}$ for $a > A$ and $\lambda > 0$.

(ii) Let $\vartheta > 0$ and choose $0 < \lambda < \lambda_* = \min\{\vartheta, \lambda_4(a), \lambda_5(a)\}$. By Lemma 4.1, Lemma 4.3, Lemma 4.4 and the Ekeland variational principle [13], we obtain that there exists a minimizing sequence $\{u_n\}$ for $J_{\lambda,M_k}(u)$ on N^+_{λ,M_k} such that

$$J_{\lambda,M_k}(u_n) = \theta^+_{\lambda,M_k} + o_n(1), \quad J'_{\lambda,M_k}(u_n) = o_n(1).$$

Applying the Lemma 2.3, there exists a subsequence still denoted by $\{u_n\} \subset N^+_{\lambda,M_k}$ and $u^{(1)}_{\lambda,M} \in W^{1,p}_0(\Omega)$ such that

$$u_n \to u^{(1)}_{\lambda,M} \text{ strongly in } W^{1,p}_0(\Omega),$$

thus, $u^{(1)}_{\lambda,M} \in N^+_{\lambda,M_k}$ is a solution of the problem (11) and $J_{\lambda,M_k}(u^{(1)}_{\lambda,M}) = \theta^+_{\lambda,M_k}$. Moreover, since $J_{\lambda,M_k}(u^{(1)}_{\lambda,M}) = J_{\lambda,M_k}(|u^{(1)}_{\lambda,M}|)$ and $|u^{(1)}_{\lambda,M}| \in N^+_{\lambda,M}$, we get that $u^{(1)}_{\lambda,M} \in N^+_{\lambda,M_k}$ is a nontrivial nonnegative solution of the problem (11).

Next, we proof $\|u^{(1)}_{\lambda,M}\|^p \leq k$. If $\|u^{(1)}_{\lambda,M}\|^p > k$, using $k \in (\frac{b(r-p)}{ar}, \frac{b(r-p)}{pa})$ and Lemma 4.5, we have

$$\begin{aligned}\frac{b(r-p)}{arL(\vartheta)} =& \frac{b(r-p)}{ar(\vartheta\|f\|_\infty S_q^{-\frac{q}{p}}\tilde{\mathrm{C}}^q + \|g\|_\infty S_r^{-\frac{r}{p}}\tilde{\mathrm{C}}^r)} \\ <& \frac{k}{\lambda\|f\|_\infty S_q^{-\frac{q}{p}}\tilde{\mathrm{C}}^q + \|g\|_\infty S_r^{-\frac{r}{p}}\tilde{\mathrm{C}}^r} < \frac{1}{b},\end{aligned}$$

then $a > \frac{b^2(r-p)}{rL(\vartheta)}$, which is a contradiction. Thus, $\|u^{(1)}_{\lambda,M}\|^p \leq k < \frac{b(r-p)}{pa}$ and $u^{(1)}_{\lambda,M}$ is also a nontrivial nonnegative solution of the problem (1). □

5. **Proof of Theorem 1.6.** First, we consider the following modified problem

$$\begin{cases} -M_{\hat{k}}(\int_\Omega |\nabla u|^p \mathrm{d}x)\Delta_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

where $\hat{k} = \frac{b(r-p)}{a(2p-r)}$, and

$$M_{\hat{k}}(s) = \begin{cases} a\hat{k}^{\frac{2p-q}{p}} s^{\frac{q-p}{p}} + b, & s \leq \hat{k}, \\ M(s), & s > \hat{k} \end{cases}$$

is a modified function of $M(s)$. Then, the corresponding energy functional of the problem (14) is

$$J_{\lambda,M_{\hat{k}}}(u) = \frac{1}{p}\hat{\mathrm{M}}_{\hat{k}}(\|u\|^p) - \frac{\lambda}{q}\int_\Omega f|u|^q \mathrm{d}x - \frac{1}{r}\int_\Omega g|u|^r \mathrm{d}x,$$

where $\hat{\mathrm{M}}_{\hat{k}}(t) = \int_0^t M_{\hat{k}}(s)\mathrm{d}s$, and we have the following lemmas.

Lemma 5.1. *if $r < 2p$ and $0 < \lambda \leq \hat{\Lambda} = a(\frac{b(r-p)}{a(2p-r)})^{\frac{2p-q}{p}}\|f\|^{-1}_\infty S_q^{\frac{q}{p}}$, then*

(i) $N^+_{\lambda,M_{\hat{k}}} = \{u \in N_{\lambda,M_{\hat{k}}} \mid \|u\|^p > \hat{k}\}$;

(ii) $N^-_{\lambda,M_{\hat{k}}} = \{u \in N_{\lambda,M_{\hat{k}}} \mid \|u\|^p \leq \hat{k}\}$;

(iii) $N_{\lambda,M_{\hat{k}}} = N^+_{\lambda,M_{\hat{k}}} \cup N^-_{\lambda,M_{\hat{k}}}$.

Proof. (i) If $u \in N_{\lambda,M_{\hat{k}}}$ with $\|u\|^p > \hat{k}$, it can be deduced to

$$\begin{aligned}K''_{\lambda,M_{\hat{k}}}(1) =& a(2p-r)\|u\|^{2p} - b(r-p)\|u\|^p + \lambda(r-q)\int_\Omega f|u|^q \mathrm{d}x \\ >& [a(2p-r)\hat{k} - b(r-p)]\|u\|^p + \lambda(r-q)\int_\Omega f|u|^q \mathrm{d}x \\ =& \lambda(r-q)\int_\Omega f|u|^q \mathrm{d}x > 0,\end{aligned}$$

then, $N^+_{\lambda,M_{\hat{k}}} \supset \{u \in N_{\lambda,M_{\hat{k}}} \mid \|u\|^p > \hat{k}\}$.

Next, we prove $N^+_{\lambda,M_{\hat{k}}} \subset \{u \in N_{\lambda,M_{\hat{k}}} \mid \|u\|^p > \hat{k}\}$. Assuming that there exists a u such that $u \in N^+_{\lambda,M_{\hat{k}}}$ with $\|u\|^p \leq \hat{k}$, we have

$$\begin{aligned} K''_{\lambda,M_{\hat{k}}}(1) =& -a(r-q)\hat{k}^{\frac{2p-q}{p}}\|u\|^q - b(r-p)\|u\|^p + \lambda(r-q)\int_\Omega f|u|^q \mathrm{d}x \\ \leq& -a(r-q)\hat{k}^{\frac{2p-q}{p}}\|u\|^q - b(r-p)\|u\|^p + \lambda(r-q)\|f\|_\infty S_q^{-\frac{q}{p}}\|u\|^q \\ =&(r-q)(\lambda\|f\|_\infty S_q^{-\frac{q}{p}} - a\hat{k}^{\frac{2p-q}{p}})\|u\|^q - b(r-p)\|u\|^p < 0 \end{aligned}$$

for $0 < \lambda \leq a\hat{k}^{\frac{2p-q}{p}}\|f\|_\infty^{-1}S_q^{\frac{q}{p}} = a(\frac{b(r-p)}{a(2p-r)})^{\frac{2p-q}{p}}\|f\|_\infty^{-1}S_q^{\frac{q}{p}}$, which is a contradiction. Thus, $N^+_{\lambda,M_{\hat{k}}} = \{u \in N_{\lambda,M_{\hat{k}}} \mid \|u\|^p > \hat{k}\}$.

(ii) Similar to the proof of (i), we have $N^-_{\lambda,M_{\hat{k}}} = \{u \in N_{\lambda,M_{\hat{k}}} \mid \|u\|^p \leq \hat{k}\}$.

(iii) Combining (i) and (ii), we have $N_{\lambda,M_{\hat{k}}} = N^+_{\lambda,M_{\hat{k}}} \cup N^-_{\lambda,M_{\hat{k}}}$ if $0 < \lambda \leq a(\frac{b(r-p)}{a(2p-r)})^{\frac{2p-q}{p}}\|f\|_\infty^{-1}S_q^{\frac{q}{p}}$. The proof of Lemma 5.1 is complete. □

Define $I(u) = \frac{1}{p}\|u\|^p - \frac{1}{r}\int_\Omega g|u|^r \mathrm{d}x$, $M = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} \mid \|u\|^p = \int_\Omega g|u|^r \mathrm{d}x\}$. It is easy to know that there exists a $u_0 \in M$ such that $S = \inf_{u \in M} I(u) = I(u_0)$. Let $v_0 = \frac{\hat{k}^{\frac{1}{p}}u_0}{\|u_0\|}$, then $\|v_0\|^p = \hat{k}$ and

$$\int_\Omega g|v_0|^r \mathrm{d}x = \hat{k}^{\frac{r}{p}}\|u_0\|^{p-r} = \hat{k}^{\frac{r}{p}}\big(\frac{r-p}{prS}\big)^{\frac{r-p}{p}} > \frac{pb^2(r-p)}{a(2p-r)^2}$$

provided that $a < A_* = \frac{p^{\frac{r}{p-r}}(r-p)^2}{Sr}(\frac{2p-r}{b})^{\frac{2p-r}{r-p}}$.

Lemma 5.2. *For each $a < A_*$ and $r < 2p$, there exists $0 < \hat{\lambda_*} \leq \hat{\Lambda}$ such that for $\lambda < \hat{\lambda_*}$, there exists $\hat{t_\lambda} > 1$ such that $\hat{t_\lambda}v_0 \in N^+_{\lambda,M_{\hat{k}}}$.*

Proof. Let

$$\begin{aligned} \overline{m}(\lambda,t) =& at^{2p-r}\|v_0\|^{2p} + bt^{p-r}\|v_0\|^p - \lambda t^{q-r}\int_\Omega f|v_0|^q \mathrm{d}x \\ =& at^{2p-r}\hat{k}^2 + bt^{p-r}\hat{k} - \lambda t^{q-r}\int_\Omega f|v_0|^q \mathrm{d}x \quad \text{ for } t > 0. \end{aligned}$$

Clearly, $\lim_{t\to 0^+}\overline{m}(\lambda,t) = -\infty$, $\lim_{t\to+\infty}\overline{m}(\lambda,t) = +\infty$.

Since $\overline{m}'(0,t) = b(r-p)\hat{k}t^{p-r-1}(t^p-1)$, then $\overline{m}(0,t)$ achieves its minimum at $t = 1$, decreasing for $t \in (0,1)$, increasing for $t \in (1,+\infty)$ and

$$\min_{t>0}\overline{m}(0,t) = \overline{m}(0,1) = \frac{pb^2(r-p)}{a(2p-r)^2} < \int_\Omega g|v_0|^r \mathrm{d}x.$$

Hence, there exists a $\overline{t}_0 > 1$ such that $\overline{m}(0,\overline{t}_0) = \int_\Omega g|v_0|^r \mathrm{d}x$ and $\overline{m}'(0,\overline{t}_0) > 0$. Moreover, by the implicit function theorem, we know that there is a positive number $\hat{\lambda_*} \leq a(\frac{b(r-p)}{a(2p-r)})^{\frac{2p-q}{p}}\|f\|_\infty^{-1}S_q^{\frac{q}{p}}$ such that for $\lambda < \hat{\lambda_*}$, there exists a $\hat{t_\lambda} > 1$ such that $\overline{m}(\lambda,\hat{t_\lambda}) = \int_\Omega g|v_0|^r \mathrm{d}x$.

On the other hand, since

$$\begin{aligned}\left\langle J'_{\lambda,M}(\hat{t_\lambda}v_0,),\hat{t_\lambda}v_0\right\rangle =&a\hat{t_\lambda}^{2p}\|v_0\|^{2p}+b\hat{t_\lambda}^{p}\|v_0\|^p-\lambda\hat{t_\lambda}^{q}\int_\Omega f|v_0|^q\mathrm{d}x\\&-\hat{t_\lambda}^{r}\int_\Omega g|v_0|^r\mathrm{d}x\\=&\hat{t_\lambda}^{r}[\overline{m}(\lambda,\hat{t_\lambda})-\int_\Omega g|v_0|^r\mathrm{d}x]\\=&0,\end{aligned}$$

and $\|\hat{t_\lambda}v_0\|^p=\hat{t_\lambda}^{p}\hat{k}>\hat{k}$, thus, $\hat{t_\lambda}v_0\in N_{\lambda,M}$ and $\hat{t_\lambda}v_0\in N^+_{\lambda,M_{\hat{k}}}$ by Lemma 5.1. □

Theorem 5.3. *For each $a<A_*$ and $r<2p$, there exists $0<\hat{\lambda_*}\le\hat{\Lambda}$ such that for $0<\lambda<\hat{\lambda_*}$, the problem (1) has at least one positive solution $u^{(2)}_{\lambda,M}$ with $\|u^{(2)}_{\lambda,M}\|^p>\hat{k}$.*

Proof. By Lemma 5.1 and Lemma 5.2, we know that $N^+_{\lambda,M_{\hat{k}}}\neq\emptyset$. On the other hand, using a similar argument to Lemma 2.2 (i), we know that the energy functional $J_{\lambda,M_{\hat{k}}}(u)$ is coercive and bounded in $N^+_{\lambda,M_{\hat{k}}}$. Define

$$\theta^+_{\lambda,M_{\hat{k}}}=\inf_{u\in N^+_{\lambda,M_{\hat{k}}}}J_{\lambda,M_{\hat{k}}}(u).$$

Applying the Ekeland variational principle [13], there exists a minimizing sequence $\{u_n\}$ for $J_{\lambda,M_{\hat{k}}}(u)$ on $N^+_{\lambda,M_{\hat{k}}}$ such that

$$J_{\lambda,M_{\hat{k}}}(u_n)=\theta^+_{\lambda,M_{\hat{k}}}+o_n(1),\quad J'_{\lambda,M_{\hat{k}}}(u_n)=o_n(1).$$

Then by Lemma 2.3, we know that there exists a subsequence still denoted by $\{u_n\}\subset N^+_{\lambda,M_{\hat{k}}}$ and $u^{(2)}_{\lambda,M}\in W^{1,p}_0(\Omega)$ such that

$$u_n\to u^{(2)}_{\lambda,M}\text{ strongly in }\ W^{1,p}_0(\Omega),$$

so $u^{(2)}_{\lambda,M}\in N^+_{\lambda,M_{\hat{k}}}$ is a nonzero solution of the problem (14) and $J_{\lambda,M_{\hat{k}}}(u^{(2)}_{\lambda,M})=\theta^+_{\lambda,M_{\hat{k}}}$. Due to $J_{\lambda,M_{\hat{k}}}(u^{(2)}_{\lambda,M})=J_{\lambda,M_{\hat{k}}}(|u^{(2)}_{\lambda,M}|)$, $|u^{(2)}_{\lambda,M}|\in N^+_{\lambda,M}$ and $\|u^{(2)}_{\lambda,M}\|^p>\hat{k}$, then we can get that $u^{(2)}_{\lambda,M}\in N^+_{\lambda,M_{\hat{k}}}$ is a nontrivial nonnegative solution of the problem (1). □

Proof of Theorem 1.6. Applying Theorem 1.5 and Theorem 5.3, we see that for each $\vartheta>0$ and $0<a<\min\{\frac{b(p-2)}{pA_0L(\vartheta),},A_*\}$, there exists a positive number $\tilde{\lambda^*}\le\min\{\vartheta,\hat{\Lambda},\lambda_*\}$ such that for $0<\lambda<\tilde{\lambda^*}$, the problem (1) has at least two nontrivial nonnegative solutions $u^{(1)}_{\lambda,M}\in N^+_{\lambda,M}$, $u^{(2)}_{\lambda,M}\in N^+_{\lambda,M}$ and

$$\|u^{(1)}_{\lambda,M}\|^p<\frac{b(r-p)}{pa}<\|u^{(2)}_{\lambda,M}\|^p.$$

This completes the proof of Theorem 1.6. □

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