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Existence and Stability of Traveling Waves for Degenerate Reaction–Diffusion Equation with Time Delay

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Abstract This paper deals with the existence and stability of traveling wave solutions for a degenerate reaction–diffusion equation with time delay. The degeneracy of spatial diffusion together with the effect of time delay causes us the essential difficulty for the existence of the traveling waves and their stabilities. In order to treat this case, we first show the existence of smooth- and sharp-type traveling wave solutions in the case of $c \geq c^*$ for the degenerate reaction–diffusion equation without delay, where $c^* > 0$ is the critical wave speed of smooth traveling waves. Then, as a small perturbation, we obtain the existence of the smooth non-critical traveling waves for the degenerate diffusion equation with small time delay $\tau > 0$. Furthermore, we prove the global existence and uniqueness of $C^{\alpha,\beta}$-solution to the time-delayed degenerate reaction–diffusion equation via compactness analysis. Finally, by the weighted energy method, we prove that the smooth non-critical traveling wave is globally stable in the weighted L^1-space. The exponential convergence rate is also derived.

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1 Introduction and Main Results

In this paper, we consider the degenerate reaction–diffusion equation with time delay

$$\frac{\partial u}{\partial t} - D\Delta u^m + d(u) = f(u_\tau), \quad t \ge 0, \quad x \in \mathbb{R} \tag{1.1}$$

with initial datum $u_0(s, x)$, $s \in [-\tau, 0]$. This equation models the population of the Australian sheep-blowfly Lucilia cuprina, where $u = u(t, x) \ge 0$ represents the population density of the blowflies at time t and location x, $u_\tau = u(t - \tau, x)$ is the time-delayed population, and $\tau \ge 0$ is the generation time (the matured age); mathematically, we call it time delay. $D\Delta u^m$ with $m > 1$ is the nonlinear diffusion with more ecological sense, that is, the diffusive velocity of population depends on the current population density, and the smaller population, the slower diffusion, in particular, when $u = 0$ (the zero population), the diffusion speed is zero. The function $f(u_\tau)$ is the birth rate function, and $d(u)$ represents the death rate. Without loss of generality, throughout this paper we take $f(u_\tau)$ and $d(u)$ as the followings, captured as Nicholson's birth rate function and death rate function

$$f(u_\tau) = p u_\tau e^{-a u_\tau}, \quad d(u) = du, \tag{1.2}$$

where $p > 0$ is the maximum per capita daily egg production rate, $\frac{1}{a} > 0$ is the size at which the blowfly population reproduces at its maximum rate, d with $d \in (0, p)$ is the per capita daily adult death rate. Besides the degenerate diffusion case ($m > 1$), we also give a brief discussion for the singular diffusion case, that is $0 < m < 1$.

The degenerate equation with time delay (1.1) possesses two constant equilibria

$$u_- = 0 \quad \text{and} \quad u_+ = \frac{1}{a} \ln \frac{p}{d}, \quad \text{and} \quad u_+ > u_- \text{ when } \frac{p}{d} > 1.$$

The initial data $u_0(s, x) \ge 0$ satisfies

$$u_0(s, x) \to u_\pm \text{ as } x \to \pm\infty, \quad \text{uniformly in } s \in [-\tau, 0]. \tag{1.3}$$

In this paper, we are interested in the existence and uniqueness (up to shift) of the traveling wave solutions to Eq. (1.1), as well as their asymptotic stability. By a traveling wave solution, we mean a solution of the form $u(t, x) = \varphi(x + ct)$ connecting the constant states $u_\pm$ with the wave speed $c > 0$:

$$\begin{cases} c\varphi'(\xi) - D(\varphi^m)''(\xi) + \mathrm{d}\varphi = p\varphi_{c\tau}e^{-a\varphi_{c\tau}}, \\ \phi(\pm\infty) = u_{\pm}, \end{cases} \tag{1.4}$$

where $\varphi_{c\tau} = \varphi(\xi - c\tau)$.

In the past years, the existence of traveling wave solutions and their stabilities for the time-delayed diffusion equations have been studied intensively. Speaking of which, Schaaf (1987) firstly investigated the existence of traveling wavefronts for the linear diffusion equation with time delay, more precisely, for the time-delayed Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(t - \tau, x)(1 - u(t, x)),$$

there is a minimal wave velocity $c^*(\tau) > 0$, such that

  (i) if $c < c^*$, there are only trivial waves;
 (ii) if $c > c^*$, there are nontrivial wave solutions determined uniquely.

Thereafter, Wu and Zou (2001) studied the traveling waves for the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(t, x)(1 - u(t - \tau, x)),$$

where the delay occurs in the removal term instead of the first factor, and they showed that for any given $c > 2$, there exists $\tau^*(c) > 0$ such that if $\tau \leq \tau^*(c)$, the above equation has a traveling wave front with wave speed c. But two questions arise: for the existent traveling wave, how small the time delay τ should be, and once τ is big, what should we expect? These are unclear in Wu and Zou (2001). Later, Kwong and Ou (2010) gave the explicit estimate on the size of τ. Namely, for any given $c \geq 2$, there exists a critical value $\tau^*(c) > 0$, which is determined by

$$\tau^*(c) = \sup\{\tau > 0; \lambda^2 + c\lambda - e^{c\lambda\tau} = 0 \ \text{ has a positive root } \lambda\},$$

such that the equation has a monotone traveling wave with wave speed c if $\tau \leq \tau^*$, but no monotone traveling wave if $\tau > \tau^*$. Besides the Fisher-KPP model, there are also some interesting studies related to Nicholson's blowflies model, Eq. (1.1) with $m = 1$. For example, So and Zou (2001) showed that when $1 < \frac{p}{d} \leq e$, for any time delay τ, the monotone traveling waves exist when $c \geq c^*$, where $c^* > 0$ is the critical wave speed; while when $\frac{p}{d} > e$, the problem becomes more challenging because the birth function loses monotonicity, and the solutions may be oscillating as numerically reported. As shown in Faria and Trofimchuk (2006, 2007), Ma (2007), Trofimchuk et al. (2008) and Trofimchuk and Trofimchuk (2008), when $e < \frac{p}{d} \leq e^2$, for any time delay, the traveling waves exist when $c \geq c^*$, but the traveling waves may be non-monotone for some cases; when $\frac{p}{d} > e^2$, the existence is proved only for small time delay $\tau < \tau^*$. Beside these, there are some significant research concerning with the traveling waves for the linear diffusion equations with nonlocal time delay, see, for example Li et al. (2007) and Xu and Xiao (2016).

On the other hand, the topic on the stability of traveling waves for time-delayed reaction–diffusion equations is also one of the hot research spots from both mathematical and physical points of view (Chern et al. 2015; Huang et al. 2012, 2016; Lin et al. 2014; Mei et al. 2009a, b, 2004, 2010; Mei and Wang 2011; Wu et al. 2011). For the non-degenerate case, namely $m = 1$ in Eq. (1.1), Schaaf (1987) proved the linear stability of traveling wavefronts by the spectral method. Since then, the topic was not touched until Mei et al. (2004) showed the nonlinear stability of traveling waves by the weighted energy method when the initial perturbation around the wave is small enough. Furthermore, Mei and his collaborators (Huang et al. 2012; Mei et al. 2009a, b, 2010; Mei and Wang 2011) obtained the global stability for the traveling waves, while the non-critical traveling waves are exponentially stable and the critical traveling waves are algebraically stable. The adopted approach is the combination of the comparison principle, Fourier transform and the weighted energy method. When the birth rate function $f(u_\tau)$ is non-monotone for $u_\tau \in [0, u_+]$ under consideration, the equation losses its monotonicity, and when the time delay τ is bigger, then both the traveling waves and the original solution are oscillating around u_+. In this case, Lin et al. (2014) first proved the exponential stability for the non-critical oscillatory traveling wave, and Chern et al. (2015) showed the stability for the critical oscillatory waves, but no convergence rates are addressed. The approach for proof is the anti-weighted energy method with help of Hanalay's inequality. A similar result for the nonlocal diffusion case was obtained by Huang et al. (2016). Very recently, by using Fourier transform to derive the fundamental solution for the linearized equation with delay, Mei et al. (submitted) further obtained the optimal exponential/algebraic convergence rates for the non-critical/critical oscillatory waves.

However, when the time-delayed equation is degenerate in its diffusion, namely, $m > 1$ in Eq. (1.1), the study on the existence of traveling waves and their stability was not related yet as we know. In fact, even for the degenerate diffusion equations without time delay, there are only limited research works dealing with this subject. Aronson (1980) studied the porous medium equation with Fisher-KPP source

$$u_t = (u^m)_{xx} + u(1 - u), \quad \text{with } m > 1,$$

and confirmed the existence of the sharp-type traveling wave with the critical wave speed c^*. In particular, when $m = 2$, Gilding and Kersner (2005) obtained the exact traveling wave solution corresponding to c^*. De Pablo and Vázquez (1991) studied the equation

$$u_t = (u^m)_{xx} + u^n(1 - u), \quad \text{with } m > 1.$$

From these studies, we see that the most obvious difference between the linear diffusion equations and the degenerate diffusion equations is that there may exist traveling waves of sharp type for the degenerate diffusion equation, that is, the solution is piecewise smooth and decreases to 0 at a finite spatial position, at which the solution is non-differentiable.

The degeneracy for $m > 1$ with the effect of time delay $\tau > 0$ causes us some essential difficulties. To attack such a problem will be the main target in the present paper. Here, we focus on the degenerate equation (1.1) with Nicholson blowflies' birth

rate function and death rate function (1.2), and investigate the existence, the uniqueness and the stability of traveling waves for Eq. (1.1). Note that, $u_- = 0$ is an unstable node and u_+ is the stable node, which makes Eq. (1.1) as the so-called mono-stable equation just like the classical Fisher-KPP equation. For the existence of traveling waves to the regular time-delayed equations without degenerate diffusion, the usual approaches adopted are the monotonic method based on the comparison principle (Wu and Zou 2001; So and Zou 2001), the Leray Schauder fixed point theorem (Ma 2007) and the spectral method (Faria and Trofimchuk 2006, 2007; Trofimchuk et al. 2008; Trofimchuk and Trofimchuk 2008) based on the good regularity of differential operators. Due to the degeneracy of the time-delayed equation, the existing methods for treating regular diffusion equations seem not directly applicable to our case. So, we have to try some new techniques. The idea is that, to show the existence of traveling waves for the time-delayed equation, we first solve the problem without time delay by using phase plane analysis and some comparison methods, and then, we use the perturbation method to establish the existence of traveling waves for the equation with a small time delay. Unfortunately, this technique does not work for the case with large time delay. Furthermore, by using the technical compactness analysis, we prove the global existence and uniqueness of the original solution to the time-delayed reaction–diffusion equation with degeneracy, and the $C^{\frac{1}{4m}, \frac{1}{2m}}$-regularity. Finally, by using the weighted energy method, we prove that the smooth traveling waves with $c > c^*$ are L^1-exponentially stable. Note that the L^1-stability was obtained by Mei et al. (2010) for the non-degenerate time-delayed Nicholson's equation, where the comparison principle guarantees that $\int |v| dx = \int v dx$ once the initial perturbations are positive. In the present paper, the initial perturbations are allowed to change sign, by introducing a smooth approximation of sign functions, and taking compactness analysis, we get the L^1-stability for the case with time delay and degenerate diffusion.

Our main results are as follows.

Theorem 1.1 (Smooth and sharp traveling waves without time delay) *For the degenerate reaction–diffusion equation* (1.1) *without time delay (i.e., $\tau = 0$),*

1. *when $m > 1$, there exists a critical wave speed $c^* > 0$, such that for any $c \geq c^*$, the non-delayed equation* (1.1) *admits a unique (up to shift) monotone traveling wavefront $\varphi(x + ct)$ connecting $u_\pm$; and no monotone traveling wavefront exists for $0 < c < c^*$. Here,*

$$c^* = \sup_{g \in \mathcal{D}_1} \left\{ 2\sqrt{D} \int_0^{u_+} \sqrt{-mg(s)s^m(pe^{-as} - d)g'(s)} ds \right\}, \qquad (1.5)$$

where

$$\mathcal{D}_1 = \Big\{ g \in C^1(0, u_+); \ g(u_+) = 0, \ \int_0^{u_+} g(s) ds = 1,$$
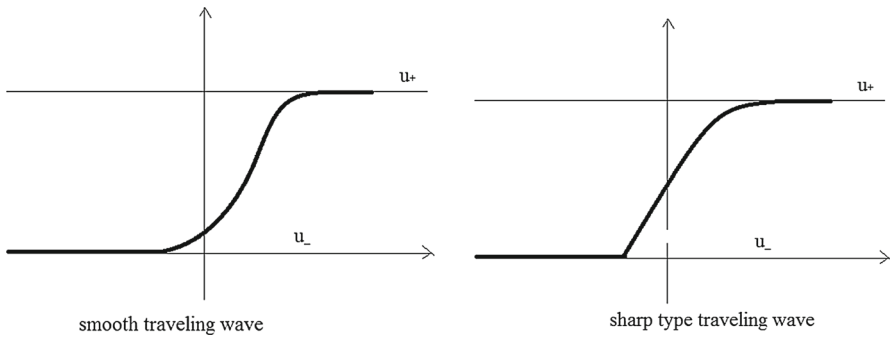$$- g'(\varphi) > 0 \quad \text{for any } \varphi \in (0, u_+) \Big\}.$$

Fig. 1 Smooth traveling waves and sharp-type traveling waves

In particular,

- *when $c > c^*$, the traveling wave is smooth: $\varphi(\xi) \in C^2(\mathbb{R})$;*
- *when $c = c^*$, the traveling wave is a semi-finite traveling wave, that is, there exists ξ_0 such that $\varphi(\xi) = 0$ for $\xi \leq \xi_0$ and $0 < \varphi(\xi) < u_+$ for $\xi > \xi_0$, and $\varphi \in C^2(\xi_0, +\infty)$. In this case, when $1 < m < 2$, the traveling wave is smooth; and when $m \geq 2$, the traveling wave is sharp type.*

Furthermore, the traveling waves satisfy

$$\varphi(\xi) \sim e^{\frac{p-d}{c}\xi}, \quad as \ \xi \to -\infty, \quad for \ any \ c > c^*,$$

and

$$u_+ - \varphi(\xi) \sim e^{\lambda_1^+ \xi}, \quad as \ \xi \to +\infty, \quad for \ any \ c \geq c^*,$$

where

$$\lambda_1^+ = \frac{cu_+^{1-m} - \sqrt{c^2 u_+^{2-2m} + 4m D a d u_+^{2-m}}}{2mD};$$

2. *when $0 < m < 1$, then no nonnegative traveling wave solution connecting $u_\pm$ exits for Eq. (1.1) with $\tau = 0$.*

Remark 1.1 The types of traveling waves for the degenerate diffusion equation described in Theorem 1.1 can be sketched as follows: when $1 < m < 2$, all traveling waves $\phi(x + ct)$ with $c \geq c^*$ are smooth (see the first part of Fig. 1), and when $m \geq 2$ and $c = c^*$, the traveling wave $\phi(x + c^*t)$ is non-differentiable (see the second part of Fig. 1 for the sharp-type traveling wave). This is essentially different from the non-degenerate diffusion equations, where the traveling waves all are smooth.

Theorem 1.2 (Smooth traveling waves with small time delay) *Assume $\tau > 0$ and $m > 1$ in the degenerate reaction–diffusion equation with time delay (1.1). Then, for any $c_0 > c^*$ (c^* is defined in (1.5)), there exists a sufficiently small constant $\tau_0 > 0$, and a continuously differentiable function $c = c(\tau) > 0$, $\tau \in [0, \tau_0)$ with $c(0) = c_0$, such that the problem (1.4) admits a traveling wave front $\varphi(x + ct) = \varphi(\xi) \in C^2(\mathbb{R})$ corresponding to τ and $c(\tau)$. Furthermore, it holds*

$$\varphi(\xi) \sim e^{\lambda_*(\tau)\xi}, \quad as \ \xi \to -\infty,$$

and

$$u_+ - \varphi(\xi) \sim e^{\lambda_-(\tau)\xi}, \quad as \ \xi \to +\infty,$$

where $\lambda_*(\tau)$ is the eigenvalue of the characteristic equation

$$c\lambda + d = pe^{-\lambda c \tau},$$

and $\lambda_-(\tau)$ is the negative root of

$$-Dmu_+^{m-1}\lambda^2 + c\lambda + d - d(1 - au_+)e^{-\lambda c \tau} = 0.$$

Remark 1.2 Different from the case without time delay in Theorem 1.1, the time delay, no matter how small it is, causes us a particular difficulty for the proof of uniqueness of the smooth traveling wave obtained in Theorem 1.2. The existing approach used in Aguerrea et al. (2012) and Yu and Mei (2016) for time-delayed equations seems not able to be applied to the degenerate case $m > 1$, because the maximum principle does not hold for the time-delayed equation with degenerate diffusion. So the uniqueness of smooth traveling waves in the case with time delay still remains open. In this sense, we can also see that the dual effects made by the time delay and the degeneracy are essential.

Theorem 1.3 (Global existence and $C^{\alpha,\beta}$-regularity of original solutions) *For the degenerate diffusion equation* (1.1) *with time delay (i.e., $m > 1$ and $\tau > 0$), let the initial data be $0 \le u_0 \in L^\infty([-\tau, 0] \times \mathbb{R})$, and*

$$\lim_{x \to -\infty} u_0(s, x) = 0, \quad \lim_{x \to +\infty} u_0(s, x) = u_+, \quad uniformly \ in \ s \in [-\tau, 0].$$

Then the cauchy problem of Eq. (1.1) *admits a uniquely global solution $u \in \mathcal{D}_2$, where*

$$\mathcal{D}_2 = \{u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap C^{\alpha,\beta}(\mathbb{R}_+ \times \mathbb{R}); (u^m)_x \in L^\infty(\mathbb{R}_+; L^2_{loc}(\mathbb{R})),$$
$$(u^m)_t \in L^2_{Loc}(\mathbb{R}_+ \times \mathbb{R})\}$$

with $\alpha = \frac{1}{4m}$ and $\beta = \frac{1}{2m}$, and

$$0 \le \|u\|_{L^\infty} \le \max \left\{ \|u_0\|_{L^\infty}, \frac{p}{aed} \right\}.$$

Theorem 1.4 (L^1-global stability of smooth traveling waves) *Let $m > 1$, and $\tau \in [0, \tau_0)$, and $\phi(x + ct)$ be a smooth traveling wave with $c > c^*(\tau)$, and $w(x + ct) = w(\xi) = e^{-\lambda\xi}$ with $\xi = x + ct$ be the weight function, where $\lambda \in (\lambda_1, \lambda_2)$, and $\lambda_1 > 0$ and $\lambda_2 > 0$ are the roots of*

$$J_\tau(c, \lambda) = d + c\lambda - Dm\kappa^{m-1}\lambda^2 - pe^{-\lambda c \tau} = 0$$

with $\kappa = \max\{\|u_0\|_{L^\infty}, \frac{p}{aed}\}$, $c^*(\tau)$ satisfies that $J_\tau(c^*, \lambda) = 0$ admits a unique positive solution λ^*, see Fig. 5. When the initial perturbation around the smooth traveling wave satisfies

$$v_0(s, \xi) := u_0(s, x) - \phi(x + cs) \in C(-\tau, 0; L_w^1(\mathbb{R})),$$

where $L_w^1(\mathbb{R})$ is the weighted L^1-space with the weight $w(\xi)$, then

$$v(t, \xi) := u(t, x) - \phi(x + ct) \in C(-\tau, 0; L_w^1(\mathbb{R}))$$

and

$$\|(u - \phi)(t)\|_{L_w^1(\mathbb{R})} \leq Ce^{-\mu t}(\|v_0\|_{L^1(-\tau,0;L_w^1(\mathbb{R}))} + \|v_0(0)\|_{L_w^1(\mathbb{R})})$$

for some positive constant $\mu > 0$. Particularly, when $\tau = 0$, the above stability also holds.

The paper is organized as follows. In Sect. 2, we prove Theorem 1.1, namely the existence and asymptotic behavior of uniquely monotonic traveling waves for Eq. (1.1) without time delay, and in Sect. 3, we prove the existence and asymptotic behavior of smooth traveling waves with small time delay (Theorem 1.2). Then, we prove Theorem 1.3 in Sect. 4, that is, the existence, uniqueness and regularity of solutions for the cauchy problem of (1.1). Finally, we prove the global L^1-stability of smooth traveling waves (Theorem 1.4) in Sect. 5.

2 Existence of Traveling Waves for the Equation Without Time Delay

In this section, we first consider the existence of traveling wave front connecting two equilibrium points $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{p}{d} > 0$ for the equation without time delay, that is

$$\begin{cases} c\varphi'(\xi) - D(\varphi^m)''(\xi) + d\varphi = p\varphi e^{-a\varphi}, \\ \varphi(\pm\infty) = u_\pm. \end{cases} \tag{2.1}$$

In what follows, we will discuss the existence of smooth traveling waves. Here, the smooth traveling waves means that the solution $\varphi \in C^2(\mathbb{R})$, which satisfies (2.1) in the classical sense.

Let $\psi(\xi) = (\varphi^m)'(\xi)$. Then, Eq. (2.1) is transformed into

$$\begin{cases} \varphi'(\xi) = \dfrac{1}{m}\varphi^{1-m}(\xi)\psi(\xi) \\ \psi'(\xi) = \dfrac{c}{mD}\varphi^{1-m}\psi + \dfrac{d}{D}\varphi - \dfrac{p}{D}\varphi e^{-a\varphi}. \end{cases} \tag{2.2}$$

The equilibrium point $(u_+, 0)$ is a saddle point. Its two real eigenvalues are

$$\lambda_+ = \frac{cu_+^{1-m} + \sqrt{c^2u_+^{2-2m} + 4mDadu_+^{2-m}}}{2mD},$$

and

$$\lambda_- = \frac{cu_+^{1-m} - \sqrt{c^2 u_+^{2-2m} + 4m \, Da \, d u_+^{2-m}}}{2m D}.$$

Note that if $\varphi'(\xi) > 0$ for $0 < \varphi < u_+$, then (2.2) is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}\psi}{\mathrm{d}\varphi} = \dfrac{c}{D} - \dfrac{m\varphi^m(pe^{-a\varphi} - d)}{D\psi}, \\ \psi(u_+) = 0. \end{cases} \tag{2.3}$$

Let $\psi_c(\varphi)$ be the solution of the above equation, and we have

$$\psi_c(\varphi) \sim mu_+^{m-1}\lambda_-\left(\varphi - u_+\right), \quad \text{as } \varphi \to u_+. \tag{2.4}$$

Then, the existence of traveling wavefront of (2.1) connecting 0 and u_+ is equivalent to that $\psi_c(0) = 0$.

Now, we are going to prove Theorem 1.1. The proof is divided in 4 steps, and the adopted approach is the phase plane analysis method.

Step 1. The existence of traveling waves for $c \geq c_*$ Inspired by Yin and Jin (2010), by a direct calculation, it is not difficult to see that $\psi_c(\varphi)$ is decreasing on the wave speed c. That is, for any positive constants c_1, c_2 with $c_1 > c_2$, we have

$$\psi_{c_1}(\varphi) < \psi_{c_2}(\varphi) \tag{2.5}$$

for $\psi_{c_1}(\varphi) \geq 0$. In fact, by (2.4), one sees that (2.5) holds when φ is sufficiently close to 0. Suppose that there exists $\varphi_1 \in (0, u_+)$ such that $\psi_{c_1}(\varphi_1) = \psi_{c_2}(\varphi_1) = \psi^*$, and $\psi_{c_1}(\varphi) < \psi_{c_2}(\varphi)$ for any $\varphi \in (\varphi_1, u_+)$. Then we have $\psi'_{c_1}(\varphi_1) < \psi'_{c_2}(\varphi_1)$. While by (2.3), we see that

$$\psi'_{c_1}(\varphi_1) = \frac{c_1}{D} - \frac{m\varphi_1^m(pe^{-a\varphi_1} - d)}{D\psi^*} > \frac{c_2}{D} - \frac{m\varphi_1^m(pe^{-a\varphi_1} - d)}{D\psi^*} = \psi'_{c_2}(\varphi_1),$$

it is a contradiction.

Denote the curve $\tilde{\psi}(\varphi) = \frac{m\varphi^m(pe^{-a\varphi} - d)}{c}$ by Γ_c, see Fig. 2, which divides the upper half plane into two parts E_1, E_2. From Eq. (2.2), we see that $\psi'_c(\varphi) < 0$ in E_1, and $\psi'_c(\varphi) > 0$ in E_2. By (2.4), it is not difficult to see that $\psi < \tilde{\psi}$ when $\varphi < u_+$ and φ is close to u_+ sufficiently. Therefore, for trajectory $\Gamma(\varphi, \psi)$ of (2.2) starting from point $(u_+, 0)$ must be increasing before intersecting with Γ_c. Also, it is clear that $\Gamma(\varphi, \psi)$ does not intersect with φ-axis when $\varphi \in (0, u_+)$, that is $\psi(0) \geq 0$ for any wave speed c.

Next, we show that when c is appropriately large, $\Gamma(\varphi, \psi)$ will arrive at the point $(0, 0)$. Let $\phi(\varphi) = B\varphi^r$ with $1 \leq r \leq m$. Then,

$$\frac{\mathrm{d}\phi}{\mathrm{d}\varphi} \leq \frac{c}{D} - \frac{m\varphi^m(pe^{-a\varphi} - d)}{D\phi}$$

Fig. 2 The graph of $\tilde{\psi}(\varphi)$
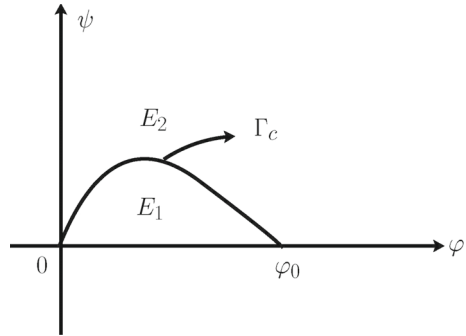
is equivalent to

$$Br\varphi^{r-1} + \frac{m\varphi^{m-r}(pe^{-a\varphi} - d)}{DB} < \frac{c}{D}.$$

When c is appropriately large, the above inequality holds. Note that $\phi(u_+) > 0 = \psi_c(u_+)$, then by comparison, we arrive at

$$0 \leq \psi_c(\varphi) \leq \phi(\varphi), \quad \text{for } \varphi \in [0, u_+], \tag{2.6}$$

which implies that $0 \leq \psi_c(0) \leq \phi(0) = 0$ for the large wave speed c.

On the other hand, if $\psi_c(0) = 0$, then we have

$$\psi_c'(\varphi) \leq \frac{c}{D},$$

which implies $\psi_c(\varphi) \leq \frac{c}{D}\varphi$. Integrating the equation in (2.3) from 0 to u_+, we have

$$\frac{cu_+}{D} = \int_0^{u_+} \frac{ms^m(pe^{-as} - d)}{D\psi(s)} ds \geq \int_0^{u_+} \frac{ms^{m-1}(pe^{-as} - d)}{c} ds,$$

which implies that

$$c \geq \sqrt{\frac{D}{u_+} \int_0^{u_+} ms^m(pe^{-as} - d)ds}. \tag{2.7}$$

Let

$$c^* = \sup\{c \geq 0; \psi_c(\varphi) \quad \text{does not arrive at the point } (0, 0)\}.$$

Then, $c^* > \sqrt{\frac{D}{u_+} \int_0^{u_+} ms^m(pe^{-as} - d)ds}$ is well defined. By the monotonicity of ψ_c on c, it holds that $\psi_c(0) = 0$ for any $c \geq c^*$, and $\psi_c(0) > 0$ for any $0 < c < c^*$.

Step 2. Uniqueness (up to shift) of traveling waves. If there are more than one traveling wave $\phi(x + ct)$ with the same wave speed $c \geq c_*$, we may assume that $\psi_1(\varphi)$, $\psi_2(\varphi)$ are the two solutions of (2.3), then there exists $\varphi_0 \in (0, u_+)$ such that $\psi_1(\varphi_0) \neq \psi_2(\varphi_0)$. Then, we have

$$\frac{d(\psi_1^2 - \psi_2^2)}{d\varphi} = \frac{2c}{D} \frac{\psi_1^2 - \psi_2^2}{\psi_1 + \psi_2},$$

by a direct calculation, we see that

$$(\psi_1^2 - \psi_2^2)(\varphi) = (\psi_1^2 - \psi_2^2)(\varphi_0)e^{\frac{2c}{D} \int_{\varphi_0}^{\varphi} \frac{1}{\psi_1 + \psi_2} ds},$$

then we have $(\psi_1^2 - \psi_2^2)(u_+) \neq 0$. This is a contradiction, and the uniqueness of solutions for the problem (2.3) is proved. It means that the corresponding equation (2.1) admits a unique solution (up to shift). In fact, if φ_1 φ_2 are the two solutions of (2.1), then there exists ξ_1, ξ_2 such that $\varphi_1(\xi_1) = \varphi_2(\xi_2) = \frac{u_+}{2}$. Using the property of inverse function, and recalling (2.2), we see that $\xi'(\varphi) = \frac{1}{\varphi'(\xi)} = \frac{m\varphi^{m-1}}{\psi_c}$. Therefore, we have

$$\xi - \xi_1 = \int_{\frac{u_+}{2}}^{\varphi_1(\xi)} \frac{ms^{m-1}}{\psi(s)} ds,$$

$$\xi - \xi_1 = (\xi - \xi_1 + \xi_2) - \xi_2 = \int_{\frac{u_+}{2}}^{\varphi_2(\xi - \xi_1 + \xi_2)} \frac{ms^{m-1}}{\psi(s)} ds,$$

which means that $\varphi_1(\xi) = \varphi_2(\xi - \xi_1 + \xi_2)$.

Step 3. Variational characterization of c^*, and the asymptotic decay rate of the traveling wavefronts at far fields

Let $\psi_c(\varphi)$ be the solution of (2.3) connecting the equilibrium points $(0, 0)$ and $(u_+, 0)$, and then we have that

$$\psi_c(\varphi) \sim \frac{m(p - d)}{c} \varphi^m \tag{2.8}$$

as $\varphi \to 0^+$ for any $c > c^*$.

In fact, from (2.3), we see that $\psi_c(\varphi) \sim \frac{m(p-d)}{c}\varphi^m$ or $\psi_c(\varphi) \sim \frac{c}{D}\varphi$. However, from (2.6), we see that for large c, $\psi_c(\varphi) \leq \phi(\varphi) = B\varphi^m$ for any $\varphi \in (0, u_+)$ with some positive constant B. Thus, (2.8) holds for large wave speed, while for the other wave speeds, this result can be obtained by the monotonicity of $\psi_c(\varphi)$ on the wave speed c, that is $\psi_c(\varphi)$ is decreasing on the wave speed c. Suppose to the contrary, there exists a wave speed $c_0 > c^*$, such that $\psi_{c_0}(\varphi) \sim \frac{c_0}{D}\varphi$, then $\psi_c(\varphi) \sim \frac{c}{D}\varphi$ for $c^* \leq c < c_0$, which means that $\psi_{c_0} > \psi_c(\varphi)$. This is a contradiction.

Next, we consider the case for the critical wave speed. We show that

$$\psi_{c^*}(\varphi) \sim \frac{c^*}{D}\varphi \tag{2.9}$$

as $\varphi \to 0^+$. Let $\psi_c(\varphi)$ be the solution of (2.3), and then for any $c < c^*$, we have $\psi_c(0) > 0$. Consider the following problem

$$\begin{cases} \dfrac{d\psi}{d\varphi} = \dfrac{c}{D} - \dfrac{m\varphi^m(pe^{-a\varphi} - d)}{D\psi}, \\ \psi(0) = 0. \end{cases} \qquad (2.10)$$

Denote $\tilde{\psi}_c(\varphi)$ be the maximal solution of (2.10), that is take $\psi(0) = \varepsilon > 0$ in (2.10), and the maximal solution is obtained as $\varepsilon \to 0$. Then, $\tilde{\psi}'_c(0) = \frac{c}{D}$. By the comparison principle for solutions of the problem (2.10), we obtain

$$\tilde{\psi}_c(\varphi) \le \psi_c(\varphi), \quad \text{for any } c < c^*,$$

which means that $\tilde{\psi}_c$ goes to zero at some point φ_c with $\varphi_c < u_+$ for $c < c^*$, and $\tilde{\psi}_c(\varphi)(\varphi) > 0$ for any $\varphi \in (0, \varphi_c)$. By a direct calculation, it is also easy to obtain if $c_1 < c_2$,

$$\tilde{\psi}_{c_1}(\varphi) < \tilde{\psi}_{c_2}(\varphi), \quad \text{for any } \varphi \in (0, \min\{\varphi_{c_1}, u_+\}).$$

Note that when $c > c^*$,

$$\tilde{\psi}_c(\varphi) > \psi_c(\varphi), \quad \text{for any } \varphi \in (0, u_+).$$

Then, we have $\tilde{\psi}_{c^*}(u_+) = 0$. Note that $\tilde{\psi}_{c^*}(\varphi) \ge \psi_{c^*}(\varphi)$, by the first equation of (2.3) or (2.10), we have $\tilde{\psi}'_{c^*}(\varphi) \ge \psi'_{c^*}(\varphi)$ for any $\varphi \in (0, u_+)$, which implies that $\tilde{\psi}_{c^*}(\varphi) \equiv \psi_{c^*}(\varphi)$ for any $\varphi \in [0, u_+]$ since $\tilde{\psi}_{c^*}(0) = \psi_{c^*}(0)$ and $\tilde{\psi}_{c^*}(u_+) = \psi_{c^*}(u_+)$. Then (2.9) is obtained.

Inspired by Benguria and Depassier (1996), we give the variational characterization of c^*. Let $g \in c^1(0, u_+)$ be a nonnegative function such that $-g' > 0$ and $\int_0^{u_+} g(s)ds < +\infty$. Multiplying the first equation of (2.3) by g and integrating it from 0 to u_+, we obtain

$$\begin{aligned} \frac{c^*}{D} \int_0^{u_+} g(s)ds &= \int_0^{u_+} \left(\frac{mg(s)s^m(pe^{-as} - d)}{D\psi_{c^*}(s)} - \psi_{c^*}(s)g'(s) \right) ds \\ &\ge \frac{2}{\sqrt{D}} \int_0^{u_+} \sqrt{-mg(s)s^m(pe^{-as} - d)g'(s)}ds \end{aligned}$$

since $\psi_{c^*}, -g', pe^{-as} - d \ge 0$. The above equality holds if $g = \hat{g}$, where $\hat{g}$ satisfies

$$\frac{m\hat{g}(s)s^m(pe^{-as} - d)}{D\psi_{c^*}(s)} = -\psi_{c^*}(s)\hat{g}'(s).$$

Combining with (2.3), we also have

$$-\frac{\hat{g}'}{\hat{g}} = \frac{c^*}{D\psi_{c^*}} - \frac{\psi'_{c^*}}{\psi_{c^*}},$$

which implies that

$$\hat{g}(\varphi) = \frac{D\psi_{c*}}{c^*} e^{-\int_a^\varphi \frac{c^*}{D\psi_{c*}} ds},$$

where $0 < a < u_+$. By (2.4) and (2.9), we can obtain $\hat{g}(u_+) = 0$, and $0 < \hat{g}(0) < \infty$. Then we can normalize $\hat{g}$ such that $\int_0^{u_+} g(s)ds = 1$. Thus we prove (1.5).

In what follows, we show the asymptotic behavior at infinity. Note that $\varphi(\xi)$ is monotone, then for any fixed $\tilde{\varphi} > 0$, using inverse function derivation rule, and recalling (2.2), we see that $\xi'(\varphi) = \frac{1}{\varphi'(\xi)} = \frac{m\varphi^{m-1}}{\psi_c}$, then we get

$$\xi(\tilde{\varphi}) - \xi(\varphi) = \int_\varphi^{\tilde{\varphi}} \xi'(s)ds = \int_\varphi^{\tilde{\varphi}} \frac{ms^{m-1}}{\psi_c(s)} ds.$$

Combining with (2.8) and (2.9), we see that $\xi \to -\infty$ as $\varphi \to 0^+$ for $c > c^*$; and ξ goes to a finite number, say ξ_0, as $\varphi \to 0^+$ for $c = c^*$. Without loss of generality, let us take $\xi_0 = 0$, namely, $\varphi(0) = \varphi(\xi_0) = 0$. Then $(\varphi^m)'(0) = 0$, and by (2.9), we further have $\varphi'_l(0) = 0$ and

$$\varphi'_r(0) = \begin{cases} 0, & \text{if } m < 2, \\ \frac{c^*}{mD}, & \text{if } m = 2, \\ +\infty, & \text{if } m > 2, \end{cases}$$

where φ'_l and φ'_r denote the left-sided and right-sided derivatives, respectively. It implies that φ is still a smooth traveling wave for any $m < 2$, and φ is a sharp-type traveling wave for any $m \geq 2$. Furthermore, for $c > c^*$, using L'Hospital's rule, we further have

$$\lim_{\varphi \to 0^+} \frac{\xi(\varphi)}{\ln \varphi} = \lim_{\varphi \to 0^+} \frac{m\varphi^m}{\psi_c} = \frac{c}{p-d},$$

that is,

$$\varphi(\xi) \sim e^{\frac{p-d}{c}\xi}, \quad \text{as } \xi \to -\infty.$$

Similarly, for any fixed $\hat{\varphi} \in (0, u_+)$, we have

$$\xi(\varphi) - \xi(\hat{\varphi}) = \int_{\hat{\varphi}}^\varphi \frac{ms^{m-1}}{\psi_c(s)} ds.$$

Recalling (2.4), for any $c \geq c^*$, we see that $\xi \to +\infty$ as $\varphi \to u_+^-$, and

$$\lim_{\varphi \to u_+^-} \frac{\xi(\varphi)}{\ln(u_+ - \varphi)} = \frac{1}{\lambda_1^+},$$

which implies $u_+ - \varphi \sim e^{\lambda_1^+ \xi}$ as $\xi \to +\infty$.

Step 4. No traveling waves exist for $0 < m < 1$ We first show that Eq. (1.1) with $m < 1$ and $\tau = 0$ admits no monotone traveling wave solution. Suppose to the

contrary, say, $\psi(\varphi)$ is the solution of (2.3) connecting $(0, 0)$ and $(u_+, 0)$, then we will have a contradiction. In fact, by integrating Eq. (2.3) from 0 to φ, we have

$$\psi(\varphi) \leq \frac{c}{D}\varphi.$$

For any ε with $0 < \varepsilon < p - d$, we further have

$$\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = \frac{c}{D} - \frac{m\varphi^m(pe^{-a\varphi} - d)}{D\psi} \leq \frac{c}{D} - \frac{m(p - \varepsilon - d)}{c\varphi^{1-m}}$$

when φ is sufficiently small such that $pe^{-a\varphi} > p - \varepsilon$. By integrating the above inequality from 0 to φ, it yields

$$\psi(\varphi) \leq \frac{c}{D}\varphi - \frac{p - \varepsilon - d}{c}\varphi^m,$$

which means that $\psi(\varphi) < 0$ for small φ. This is a contradiction.

Next, we show that Eq. (1.1) does not admit any nonnegative and smooth traveling wave solution. If the argument is false, then there exists at least a solution $\varphi(\xi)$ for (2.1). From the above proof, φ is not monotonous. Note that $\varphi(\pm\infty) = u_\pm$, then there exists a local minimum point $\xi_0 \in \mathbb{R}$ such that $\varphi'(\xi_0) = 0$, $-(\varphi^m)'' \leq 0$, by the first equation of (2.1), we see that

$$0 \geq c\varphi'(\xi_0) - D(\varphi^m)''(\xi_0) = \varphi(pe^{-a\varphi} - d),$$

which means that $\varphi(\xi_0) = 0$ or $\varphi(\xi_0) \geq u_+$. If $\varphi(\xi_0) = 0$, then there exist $\xi_1 < \xi_0 < \xi_2$ such that $\varphi(\xi_1) = \varphi(\xi_2) > 0$, $\varphi'(\xi_1) \leq 0$, $\varphi'(\xi_2) \geq 0$ and $0 \leq \varphi(\xi) < u_+$ for $\xi \in (\xi_1, \xi_2)$. Then, integrating the first equation in (2.1) from ξ_1 to ξ_2 yields

$$0 \geq D(\varphi^m)'(\xi_1) - D(\varphi^m)'(\xi_2) = \int_{\xi_1}^{\xi_2} \varphi(s)(pe^{-a\varphi(s)} - d)\mathrm{d}s > 0.$$

It is a contradiction. While if $\varphi(\xi_0) \geq u_+$, then there exists $\xi^* < \xi_0$, such that ξ^* is a local maximum with $\varphi(\xi^*) > u_+$, then there exist ξ_3, ξ_4 with $\xi_3 < \xi^* < \xi_4$ such that $u_+ < \varphi(\xi_3) = \varphi(\xi_4) < \varphi(\xi^*)$, $\varphi'(\xi_3) \geq 0$, $\varphi'(\xi_4) \leq 0$, and $\varphi(\xi) > u_+$ for $\xi \in (\xi_1, \xi_2)$. Then integrating the first equation in (2.1) from ξ_3 to ξ_4 yields

$$0 \leq D(\varphi^m)'(\xi_3) - D(\varphi^m)'(\xi_4) = \int_{\xi_3}^{\xi_4} \varphi(s)(pe^{-a\varphi(s)} - d)\mathrm{d}s < 0.$$

It is a contradiction. It implies that φ is not a solution.

Finally, combining Steps 1–4, we immediately prove Theorem 1.1.

3 Existence of Traveling Waves for Equation with Small Time Delay

In this section, we are going to prove Theorem 1.2. Based on Theorem 1.1, by the perturbation method and the implicit function theory, we can obtain the existence of smooth traveling waves $\phi(x + ct)$ for the time-delayed degenerate diffusion equation (1.1), where the wave speed c is selected as $c = c(\tau) > c_0 \geq c^*$ with $\tau \in (0, \tau_0]$, and $\tau_0 > 0$ is a specified number. At the end of this section, we will give one remark on the uniqueness of traveling waves and one proposition on the case of sharp-type traveling waves.

Proof of Theorem 1.2 Firstly, we reduce Eq. (2.1) into the following system

$$
\begin{cases}
\varphi'(\xi) = \dfrac{1}{m}\varphi^{1-m}(\xi)\psi(\xi), \\[2mm]
\psi'(\xi) = \dfrac{c}{mD}\varphi^{1-m}\psi + \dfrac{d}{D}\varphi - \dfrac{p}{D}\varphi e^{-a\varphi} + \dfrac{p}{D}\left(\varphi e^{-a\varphi} - \varphi_{c\tau} e^{-a\varphi_{c\tau}}\right).
\end{cases}
\tag{3.1}
$$

The existence of smooth monotone traveling waves connecting the two equilibrium points 0 and u_+ is equivalent to the existence of positive solutions for the following problem

$$
\frac{d\psi}{d\varphi} = \frac{c}{D} - \frac{m\varphi^{m-1}(p\varphi_{c\tau}e^{-a\varphi_{c\tau}} - d\varphi)}{D\psi},
$$
$$
\psi(0) = 0, \quad \psi(u_+) = 0,
\tag{3.2}
$$

where $\varphi_{c\tau}$ is defined as

$$
\varphi_{c\tau} = \inf\left\{r; \varphi \geq r \geq 0, \text{ and } \int_r^\varphi \frac{ms^{m-1}}{\psi(s)}ds \leq c\tau\right\}.
\tag{3.3}
$$

When $\tau = 0$, by using Theorem 1.1, we see that (3.2) admits a solution $\psi = F(\varphi, c)$ connecting the points $(0, 0)$ and $(u_+, 0)$ for $c > c^*$. We consider Eq. (3.2) with the initial data $\psi(u_+) = 0$, by (3.3), we see that $\varphi_{c\tau} \to u_+$ as $\varphi \to u_+$, we consider the linearized equation near $(\varphi, \psi) = (u_+, 0)$, that is

$$
-Dmu_+^{m-1}\varphi'' + c\varphi' + d(\varphi - u_+) - p(1 - au_+)e^{-au_+}(\varphi_{c\tau} - u_+) = G_1,
\tag{3.4}
$$

where

$$
\begin{aligned}
G_1 = {}& Dm(\varphi^{m-1} - u_+^{m-1})\varphi'' + Dm(m-1)u_+^{m-2}|\varphi'|^2 \\
& + p(\varphi_{c\tau} - u_+)(e^{-a\varphi_{c\tau}} - e^{-au_+}) \\
& + pu_+(\varphi_{c\tau} - u_+)\left(ae^{-au_+} + \frac{e^{-a\varphi_{c\tau}} - e^{-au_+}}{\varphi_{c\tau} - u_+}\right).
\end{aligned}
$$

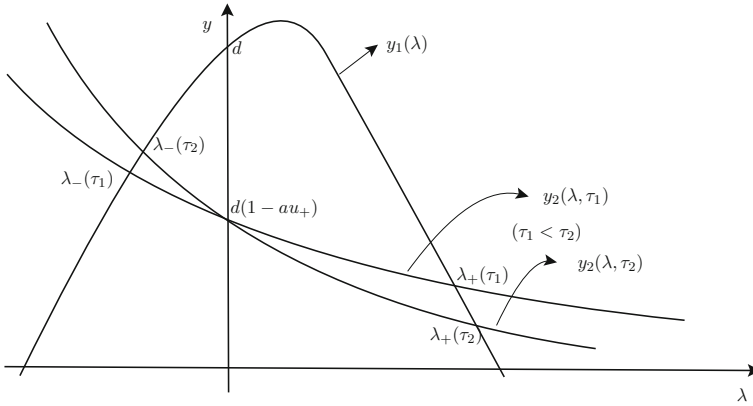**Fig. 3** The graphs of y_1, y_2

Note that $pe^{-au_+} = d$, then the characteristic equation of (3.4) is

$$\Delta(c, \lambda) = -Dmu_+^{m-1}\lambda^2 + c\lambda + d - d(1 - au_+)e^{-\lambda c\tau} = 0. \quad (3.5)$$

Denote $y_1(\lambda) = -Dmu_+^{m-1}\lambda^2 + c\lambda + d$, $y_2(\lambda, \tau) = d(1 - au_+)e^{-\lambda c\tau}$.

Clearly, from Fig. 3, the above equation admits two real roots λ_- and λ_+ with $\lambda_-(\tau) < 0 < \lambda_+(\tau)$, where $\lambda_-(\tau)$ and $\lambda_+(\tau)$ are increasing on τ. Then, we have

$$\psi \sim mu_+^{m-1}\lambda_-(\tau)(\varphi - u_+), \text{ as } \varphi \to u_+^-, \quad (3.6)$$

which means that $\psi(\varphi) > 0$ when φ is in a small left neighborhood of u_+. We denote the trajectory of (3.2) starting from the point $(u_+, 0)$ by

$$\psi = f_1(\varphi, c, \tau), \quad f_1(u_+, c, \tau) = 0.$$

Notice that $\lambda_-(\tau) > \lambda_-(0)$, then we have

$$f_1(\varphi, c, \tau) < f_1(\varphi, c, 0), \quad (3.7)$$

when φ is in a small left neighborhood of u_+. We denote the trajectory of (3.2) starting from the point $(0, 0)$ by

$$\psi = f_0(\varphi, c, \tau), \quad f_0(0, c, \tau) = 0.$$

Consider the linearized equation near $(\varphi, \psi) = (0, 0)$, that is

$$\begin{aligned}
c\varphi' &+ d\varphi - p\varphi_{c\tau} \\
&= Dm(m-1)\varphi^{m-2}|\varphi'|^2 + Dm\varphi^{m-1}\varphi'' + p\varphi_{c\tau}(e^{-a\varphi_{c\tau}} - 1) \\
&= G_2.
\end{aligned} \quad (3.8)$$

Then, the characteristic equation of (3.8) is

$$\Delta(c, \lambda) = c\lambda + d - pe^{-\lambda c\tau}. \tag{3.9}$$

Clearly, the above equation admits a real root $\lambda^*(\tau) > 0$. Then

$$f_0(\varphi, c, \tau) \sim m\varphi^m \lambda^*(\tau), \quad \text{as } \varphi \to 0^+. \tag{3.10}$$

Note that

$$f_0(\varphi, c, 0) = f_1(\varphi, c, 0) = F(\varphi, c), \quad \text{for any } c \geq c^*, \tag{3.11}$$

by continuous dependence of solution trajectories on parameters, the trajectories must cross the line $\varphi = \frac{1}{2}u_+$ when τ appropriately small. For any fixed wave speed $c_0 \geq c^*$, the unperturbed problem corresponds to the solution $\psi = F(\varphi, c_0)$. In what follows, we show that there exists a unique $c = c(\tau)$ near c_0 such that $f_0(\frac{1}{2}u_+, c, \tau) = f_1(\frac{1}{2}u_+, c, \tau)$. Let

$$H(c, \tau) = f_0\left(\frac{1}{2}u_+, c, \tau\right) - f_1\left(\frac{1}{2}u_+, c, \tau\right).$$

Obviously, $H(c_0, 0) = 0$. In what follows, we show that

$$\frac{\partial H}{\partial c}(c_0, 0) \neq 0. \tag{3.12}$$

Let

$$g_0(\varphi) = \frac{\partial f_0}{\partial c}(\varphi, c_0, 0).$$

Then, $g_0(0) = 0$, and

$$\frac{dg_0}{d\varphi} = \frac{\partial}{\partial c}\left(\frac{\partial f_0}{\partial \varphi}(\varphi, c, 0)\right)\Bigg|_{c=c_0} = \frac{1}{D} + \frac{m\varphi^m(pe^{-a\varphi} - d)}{DF^2}g_0,$$

where F is defined in (3.11), and we further have

$$\frac{d}{d\varphi}\left(g_0 e^{\int_\varphi^{\frac{1}{2}u_+} \frac{ms^m(pe^{-as}-d)}{DF^2(s,c)}ds}\right) = \frac{1}{D}e^{\int_\varphi^{\frac{1}{2}u_+} \frac{ms^m(pe^{-as}-d)}{DF^2(s,c)}ds}.$$

Integrating it from 0 to $\frac{1}{2}u_+$ yields

$$g_0\left(\frac{1}{2}u_+\right) = \frac{1}{D}\int_0^{\frac{1}{2}u_+} e^{\int_\varphi^{\frac{1}{2}u_+} \frac{ms^m(pe^{-as}-d)}{DF^2(s,c)}ds}d\varphi. \tag{3.13}$$

Let

$$g_1(\varphi) = \frac{\partial f_1}{\partial c}(\varphi, c_0, 0).$$

Then, $g_1(u_+) = 0$, and

$$\frac{dg_1}{d\varphi} = \frac{\partial}{\partial c}\left(\frac{\partial f_1}{\partial \varphi}(\varphi, c, 0)\right)\Bigg|_{c=c_0} = \frac{1}{D} + \frac{m\varphi^m(pe^{-a\varphi} - d)}{DF^2}g_1,$$

and we further have

$$\frac{d}{d\varphi}\left(g_1 e^{-\int_{\frac{1}{2}u_+}^{\varphi} \frac{ms^m(pe^{-as}-d)}{DF^2(s,c)}ds}\right) = \frac{1}{D}e^{-\int_{\frac{1}{2}u_+}^{\varphi} \frac{ms^m(pe^{-as}-d)}{DF^2(s,c)}ds}.$$

Similarly, integrating the above equation from $\frac{1}{2}u_+$ to u_+, we have

$$g_1\left(\frac{1}{2}u_+\right) = -\frac{1}{D}\int_{\frac{1}{2}u_+}^{u_+} e^{-\int_{\frac{1}{2}u_+}^{\varphi} \frac{ms^m(pe^{-as}-d)}{DF^2(s,c)}ds}d\varphi. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\frac{\partial H}{\partial c}(c_0, 0) = g_0\left(\frac{1}{2}u_+\right) - g_1\left(\frac{1}{2}u_+\right) = \frac{1}{D}\int_0^{u_+} e^{\int_{\varphi}^{\frac{1}{2}u_+} \frac{ms^m(pe^{-as}-d)}{DF^2(s,c)}ds}d\varphi > 0.$$

By using the implicit function theorem, for sufficiently small τ, we have $H(c(\tau), \tau) = 0$. Thus, we have proved Theorem 1.2. $\qquad\square$

Now, we give one remark on the uniqueness of traveling waves and one proposition on the possible sharp-type traveling waves.

Remark 3.1 Different from the case without time delay, the degenerate diffusion equation with time delay loses its comparison principle, and the uniqueness of traveling waves for the case with time delay (even it is small enough) is still open. It seems that the existing techniques for treating uniqueness cannot be directly applied.

Proposition 3.1 *If $\varphi(\xi) \in L^\infty(\mathbb{R})$ is a piecewise smooth monotone traveling wave solution, then $\varphi \in C^2(\mathbb{R})$ if $\varphi(\xi) > 0$ for any $\xi \in \mathbb{R}$. However, when $\varphi(\xi) = 0$ at some point ξ, then a sharp-front type traveling wave is possible to exist.*

Proof Let $u(t, x) \in L^\infty(\mathbb{R} \times \mathbb{R})$ be a piecewise smooth solution of (1.1). Assume that $u(t, x)$ is not smooth or not continuous on the curve $\Gamma : x = x(t) \in Q$, and $x = x(t)$ divides the domain Q into two parts Q_1 and Q_2, where $Q \subset \mathbb{R}^2$ is a bounded domain (Fig. 4). Then, for any $\Phi(t, x) \in C_0^\infty(Q)$, we have

$$-\iint_Q u\Phi_t dxdt - D\iint_Q u^m \Phi_{xx}dxdt + d\iint_Q u\Phi dxdt = \iint_Q f(u_\tau)\Phi dxdt. \tag{3.15}$$

Let u_l, u_r be the left/right-sided limits, and $(u^m)_{xl}$, $(u^m)_{xr}$ denote the left-sided and right-sided derivatives, respectively. By a direct calculation, we see that

Fig. 4 Piecewise smooth solution
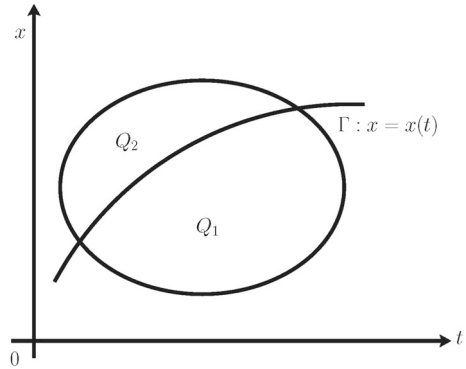
$$-\iint_Q u\Phi_t \mathrm{d}x\mathrm{d}t = -\iint_{Q_1} u\Phi_t \mathrm{d}x\mathrm{d}t - \iint_{Q_2} u\Phi_t \mathrm{d}x\mathrm{d}t$$

$$= \int_\Gamma u_r \Phi x'(t)\mathrm{d}t - \int_\Gamma u_l \Phi x'(t)\mathrm{d}t + \iint_{Q_1} u_t \Phi \mathrm{d}x\mathrm{d}t + \iint_{Q_2} u_t \Phi \mathrm{d}x\mathrm{d}t, \quad (3.16)$$

and

$$-D\iint_Q u^m \Phi_{xx}\mathrm{d}x\mathrm{d}t$$

$$= -D\iint_{Q_1} u^m \Phi_{xx}\mathrm{d}x\mathrm{d}t - D\iint_{Q_2} u^m \Phi_{xx}\mathrm{d}x\mathrm{d}t$$

$$= -D\int_\Gamma u_r^m \Phi_x \mathrm{d}t + D\int_\Gamma u_l^m \Phi_x \mathrm{d}t + D\iint_{Q_1} (u^m)_x \Phi_x \mathrm{d}x\mathrm{d}t$$

$$+ D\iint_{Q_2} (u^m)_x \Phi_x \mathrm{d}x\mathrm{d}t$$

$$= -D\int_\Gamma (u_r^m - u_l^m)\Phi_x \mathrm{d}t + D\int_\Gamma ((u^m)_{rx} - (u^m)_{lx})\Phi \mathrm{d}t$$

$$- D\iint_{Q_1} (u^m)_{xx}\Phi \mathrm{d}x\mathrm{d}t - D\iint_{Q_2} (u^m)_{xx}\Phi \mathrm{d}x\mathrm{d}t. \quad (3.17)$$

Substituting (3.16) and (3.17) into (3.15), and we obtain

$$\int_\Gamma \Big((u_r - u_l)x'(t) + D((u^m)_{rx} - (u^m)_{lx})\Big)\Phi \mathrm{d}t - D\int_\Gamma (u_r^m - u_l^m)\Phi_x \mathrm{d}t$$

$$+ \iint_{Q_1} \Big(u_t - D(u^m)_{xx} + du - f(u_\tau)\Big)\Phi \mathrm{d}x\mathrm{d}t$$

$$+ \iint_{Q_2} \Big(u_t - D(u^m)_{xx} + du - f(u_\tau)\Big)\Phi \mathrm{d}x\mathrm{d}t$$

$$= 0.$$

By the arbitrariness of Φ, we have

$$(u_r - u_l)x'(t) + D((u^m)_{rx} - (u^m)_{lx}) = 0, \quad \text{and } u_r^m - u_l^m = 0,$$

which implies that

$$u_r = u_l, \quad (u^m)_{rx} = (u^m)_{lx}.$$

Note that $\varphi(\xi) = \varphi(x + ct) = u(t, x)$, then $\varphi_r(\xi) = \varphi_l(\xi)$, and $\varphi_r^{m'}(\xi) = \varphi_l^{m'}(\xi)$. If $\varphi_r(\xi) = \varphi_l(\xi) \neq 0$, $\varphi_r' = \varphi_l'$. Recalling (1.4), we further have $\varphi_r'' = \varphi_l''$. That is, $\varphi \in C^2(\mathbb{R})$. While if $\varphi_r(\xi) = \varphi_l(\xi) = 0$, then it is possible for $\varphi_r'(\xi) \neq \varphi_l'(\xi)$. For this cases, so $\varphi(\xi)$ is a sharp traveling wave, which is not smooth only at the point $\varphi = 0$. The proof is complete. □

4 Global Existence and Regularity of Time-Delayed Degenerate Solution

Recalling (1.1), we see that $\tilde{u}(t, \xi) = u(t, \xi - ct) = u(t, x)$ with $\xi = x + ct$ and $c > c^* > 0$ satisfies the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + c \dfrac{\partial u}{\partial \xi} - D\Delta u^m + du = f(u_\tau), \\ \lim_{\xi \to -\infty} u(t, \xi) = 0, \quad \lim_{\xi \to +\infty} u(t, \xi) = u_+, \\ u(s, \xi) = \tilde{u}_0(s, \xi) = u_0(s, \xi - cs), \quad s \in [-\tau, 0], \end{cases} \quad (4.1)$$

where $f(s) = pse^{-as}$. For simplicity, we still denote the solution by $u(t, \xi)$. We are going to prove Theorem 1.3 in the equivalent form to Eq. (4.1).

Let $u_0^l(s, \xi)$ with $0 \leq u_0^l(s, \xi) \leq \tilde{u}_0(s, \xi)$ be a sufficiently smooth function with $u_0^l(s, \xi) \to \tilde{u}_0(s, \xi)$ as $l \to \infty$, uniformly in ξ and s. Consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + c \dfrac{\partial u}{\partial \xi} - D\Delta u^m + du = f(u_\tau), \quad \xi \in (-l, l), \quad t > 0 \\ u(-l, t) = \dfrac{1}{l}, \quad u(l, t) = \dfrac{1}{l}e^{-dt} + u_+, \quad t > 0 \\ u(s, \xi) = u_0^l(s, \xi) + \dfrac{1}{l}, \quad s \in [-\tau, 0]. \end{cases} \quad (4.2)$$

It is easy to show that $f(u_\tau) = pu_\tau e^{-au_\tau} \leq \frac{p}{ae}$. By a direct calculation, $\frac{1}{l}e^{-dt}$ is a lower solution of (4.2), and $\max\{u_+ + \frac{1}{l}, \|\tilde{u}_0\|_{L^\infty} + 1, \frac{p}{aed}\}$ is an upper solution of (4.2). Then, by the comparison principle, we obtain

$$\frac{1}{l}e^{-dt} \leq u(t, \xi) \leq \max\left\{u_+ + \frac{1}{l}, \|\tilde{u}_0\|_{L^\infty} + 1, \frac{p}{aed}\right\}, \quad (4.3)$$

and the problem (4.2) admits a smooth solution $u_l(t, \xi)$. Next, we show the $C^{\alpha,\beta}$-estimate of u_l. For any $a \in (-l+2, l-2)$, multiplying Eq. (4.2) by $u^r \eta^2$ ($r > 0$) with

$\eta(\xi) \in C_0^\infty(a-2, a+2)$, and $0 \leq \eta(\xi) \leq 1$, $|\eta'(\xi)| \leq 1$, $\eta = 1$ for $\xi \in (a-1, a+1)$, we see that

$$\frac{1}{r+1} \frac{d}{dt} \int u^{r+1} \eta^2 d\xi + Dmr \int u^{m+r-2} |u_\xi|^2 \eta^2 d\xi + d \int u^{r+1} \eta^2 d\xi$$

$$= -Dm \int u^{m+r-1} u_\xi \eta \eta_\xi d\xi + \frac{2c}{r+1} \int u^{r+1} \eta \eta_\xi d\xi + \int f(u_\tau) u^r \eta^2 d\xi$$

$$\leq \frac{Dmr}{2} \int u^{m+r-2} |u_\xi|^2 \eta^2 d\xi + C.$$

Then, for any given $\sigma > 0$, and for any $t > 0$, we have

$$\int_t^{t+\sigma} \int u^{m+r-2} |u_\xi|^2 \eta^2 d\xi ds \leq C, \tag{4.4}$$

where C only depends on r and σ. By using the mean value theorem of integrals, we further prove that there exists $t_0 \in (t, t+\sigma)$ such that

$$\int u^{m+r-2}(t_0) |u_\xi(t_0, \xi)|^2 \eta^2(\xi) d\xi \leq C. \tag{4.5}$$

Multiplying Eq. (4.2) by $(u^m)_t \eta^4$, we have

$$m \int u^{m-1} u_t^2 \eta^4 d\xi + \frac{D}{2} \frac{d}{dt} \int |(u^m)_\xi|^2 \eta^4 d\xi + \frac{dm}{m+1} \int u^{m+1} \eta^4 d\xi$$

$$= -m^2 \int u^{2m-2} u_\xi u_t \eta^3 \eta_\xi d\xi - cm \int u^{m-1} u_t u_\xi \eta^4 d\xi + \int f(u_\tau) u^{m-1} u_t \eta^4 d\xi$$

$$\leq \frac{m}{2} \int u^{m-1} u_t^2 \eta^4 d\xi + C \int u^{3(m-1)} |u_\xi|^2 \eta^2 d\xi$$

$$+ C \int u^{m-1} |u_\xi|^2 \eta^4 d\xi + C \int f^2(u_\tau) u^{m-1} \eta^4 d\xi.$$

Combining with (4.4) and (4.5), we further have

$$\int_t^{t+\sigma} \int u^{m-1} u_t^2 \eta^4 d\xi + \sup_t \int |(u^m)_\xi|^2 \eta^4 d\xi \leq C, \tag{4.6}$$

where C is independent of l. Let us denote the solution of (4.2) by u_l and its weak limit by u as $l \to \infty$. Letting $l \to \infty$, we obtain the existence of the generalized solutions for the problem (4.1) satisfying (4.4), (4.6) and

$$0 \leq u \leq \max \left\{ \|\tilde{u}_0\|_{L^\infty}, \frac{p}{aed} \right\}.$$

By the boundedness of u, we also have

$$\int_t^{t+\sigma} \int |(u^m)_t|^2 \eta^4 d\xi + \sup_t \int |(u^m)_\xi|^2 \eta^4 d\xi \le C. \tag{4.7}$$

Then, by Sobole embedding inequality, we have $u^m \in L^\infty((0, \infty), C^{1/2}(\mathbb{R}))$, because of $u^m \in L^\infty((0, \infty), H^1_{loc}(\mathbb{R}))$. We further obtain $u^m(t, \xi) \in C^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R})$ due to $(u^m)_t \in L^2((t, t+\sigma) \times (a-1, a+1))$. In fact, for any $t_1, t_2 \in \mathbb{R}^+$, $x \in \mathbb{R}$, (without loss of generality, we assume that $t_2 \le t_1$.) take a ball B_r of radius r centered at x, with $r = |t_1 - t_2|^{\frac{1}{2}}$. Recalling (4.7) and using Poincáre inequality, we have

$$\int_{B_r} |u^m(t_1, y) - u^m(t_2, y)| dy = \int_{B_r} \left| \int_{t_2}^{t_1} \frac{\partial u^m(t, y)}{\partial t} dt \right| dy$$

$$\le C \left(\int_{t_2}^{t_1} \int_{B_r} \left| \frac{\partial u^m(t, y)}{\partial t} \right|^2 dy dt \right)^{1/2} |t_1 - t_2|^{1/2} r^{1/2}$$

$$\le C |t_1 - t_2|^{1/2} r^{1/2}.$$

By the mean value theorem, there exists $x^* \in B_r$ such that

$$|u^m(t_1, x^*) - u^m(t_2, x^*)| \le C|t_1 - t_2|^{1/2} r^{-1/2} = C|t_1 - t_2|^{1/4}.$$

Then, we have

$$|u^m(t_1, x) - u^m(t_2, x)|$$
$$\le |u^m(t_1, x) - u^m(t_1, x^*)| + |u^m(t_1, x^*) - u^m(t_2, x^*)| + |u^m(t_2, x^*) - u^m(t_2, x)|$$
$$\le C(|x - x^*|^{\frac{1}{2}} + |t_1 - t_2|^{\frac{1}{4}})$$
$$\le C|t_1 - t_2|^{\frac{1}{4}},$$

which implies that $u^m(t, \xi) \in C^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}_+ \times \mathbb{R})$. It means that $u(t, \xi) \in C^{\frac{1}{4m}, \frac{1}{2m}}(\mathbb{R}_+ \times \mathbb{R})$.

Next, we show the uniqueness. Let $u_1, u_2 \in \mathcal{D}_2$ be the solutions of (4.1), and assume $u = u_1 - u_2$. For simplicity, we replace $u(t, \xi)$ by $u(t, x)$ with $\xi = x + ct$, and note that when $t \in (0, \tau)$, $t - \tau \in (-\tau, 0)$, then $f(u_{1,\tau}) = f(u_{2,\tau}) = f(\tilde{u}_0(t - \tau, x))$. Thus, we have

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta(u_1^m - u_2^m) + du = 0, & (t, x) \in \mathbb{R} \times (0, \tau), \\ \lim\limits_{x \to -\infty} u(t, x) = 0, \quad \lim\limits_{x \to +\infty} u(t, x) = 0, & t \in (0, \tau), \\ u(s, x) = 0, \quad s \in [-\tau, 0]. \end{cases} \tag{4.8}$$

Take the test function $\psi_n(t, x) = \alpha_n(x) \int_\tau^t (u_1^m - u_2^m) \mathrm{d}s$, where $\alpha_n(x) \in C_0^\infty(\mathbb{R})$, which is a concave function with $0 \le \alpha_n(x) \le 1$, $|\alpha_n'(x)| \le 1$, and

$$\alpha_n(x) = \begin{cases} 1, & |x| \le n; \\ 0, & |x| \ge n+1. \end{cases} \tag{4.9}$$

Then, $\psi_n(t, x) \in \mathring{H}^1(\mathbb{R} \times (0, \tau))$. Note that $u(0, x) = \psi_n(\tau, x) \equiv 0$, we obtain

$$\iint_{Q_\tau} u e^{\mathrm{d}t} \alpha_n(x)(u_1^m - u_2^m) \mathrm{d}x \mathrm{d}t$$

$$= D \iint_{Q_\tau} e^{\mathrm{d}t} \alpha_n(x)(u_1^m - u_2^m)_x \left(\int_\tau^t (u_1^m - u_2^m)_x \mathrm{d}s \right) \mathrm{d}x \mathrm{d}t$$

$$+ D \iint_{Q_\tau} e^{\mathrm{d}t} \alpha_n'(x)(u_1^m - u_2^m)_x \left(\int_\tau^t (u_1^m - u_2^m) \mathrm{d}s \right) \mathrm{d}x \mathrm{d}t$$

$$= \frac{D}{2} \iint_{Q_\tau} e^{\mathrm{d}t} \alpha_n(x) \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\tau^t (u_1^m - u_2^m)_x \mathrm{d}s \right)^2 \mathrm{d}x \mathrm{d}t$$

$$+ D \iint_{Q_\tau} e^{\mathrm{d}t} \alpha_n'(u_1^m - u_2^m)_x \left(\int_\tau^t (u_1^m - u_2^m) \mathrm{d}s \right) \mathrm{d}x \mathrm{d}t$$

$$= -\frac{D}{2} \iint_{Q_\tau} \alpha_n(x) \left(\int_\tau^0 (u_1^m - u_2^m)_x \mathrm{d}s \right)^2 \mathrm{d}x \mathrm{d}t$$

$$- \frac{\mathrm{d}D}{2} \iint_{Q_\tau} e^{\mathrm{d}t} \alpha_n(x) \left(\int_\tau^t (u_1^m - u_2^m)_x \mathrm{d}s \right)^2 \mathrm{d}x \mathrm{d}t$$

$$+ D \iint_{Q_\tau} e^{\mathrm{d}t} \alpha_n'(x)(u_1^m - u_2^m)_x \left(\int_\tau^t (u_1^m - u_2^m) \mathrm{d}s \right) \mathrm{d}x \mathrm{d}t$$

$$\le D \iint_{Q_\tau} e^{\mathrm{d}t} \alpha_n'(x)(u_1^m - u_2^m)_x \left(\int_\tau^t (u_1^m - u_2^m) \mathrm{d}s \right) \mathrm{d}x \mathrm{d}t,$$

where $Q_\tau = (0, \tau) \times (-\infty, +\infty)$. Note that $\alpha_n'(x) = 0$ for $|x| < n$ and $|x| > n+1$, by using (4.6), we further have

$$\iint_{Q_\tau} u e^{\mathrm{d}t} \alpha_n(x)(u_1^m - u_2^m) \mathrm{d}x \mathrm{d}t$$

$$\le D\sqrt{\tau} e^{\mathrm{d}\tau} \left(\iint_{Q_\tau} |\alpha_n'(x)||(u_1^m - u_2^m)_x|^2 \mathrm{d}x \mathrm{d}t \right)^{1/2}$$

$$\times \left(\iint_{Q_\tau} |\alpha_n'(x)|(u_1^m - u_2^m)^2 \mathrm{d}x \mathrm{d}t \right)^{1/2}$$

$$\le C \left(\iint_{Q_\tau} |\alpha_n'(x)|(u_1^m - u_2^m)(u_1 - u_2) \mathrm{d}x \mathrm{d}t \right)^{1/2}. \tag{4.10}$$

Note that $\alpha'_n(x) = 0$ for $|x| < n$ and $|x| > n + 1$, from the above inequality, we have

$$\iint_{Q_\tau} u e^{dt} \alpha_n(x)(u_1^m - u_2^m) dx dt$$

is bounded uniformly. It means that

$$\iint_{Q_\tau} |\alpha'_n(x)|(u_1^m - u_2^m)(u_1 - u_2) dx dt \to 0, \quad \text{as } n \to \infty.$$

Let $n \to \infty$, we have

$$\iint_{Q_\tau} e^{dt} u(u_1^m - u_2^m) dx dt = 0,$$

which implies that $u_1 = u_2$ in Q_τ. Repeating the above procedure on $(\tau, 2\tau)$, $(2\tau, 3\tau), \ldots$, we then prove

$$u_1(t, x) = u_2(t, x), \quad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

The uniqueness is proved.

5 Global Stability of Smooth Traveling Waves

In this section, we study the global stability of smooth traveling waves.

Let $u(t, x)$ be the solution of Eq. (1.1) with initial datum $u_0(s, x)$, $s \in [-\tau, 0]$, $\varphi(x + ct) = \varphi(\xi)$ be a given smooth traveling wave connecting 0 and u_+ with $c > c^*$, and let $v(t, \xi) := u(t, x) - \varphi(\xi)$, $v_0(s, \xi) := u_0(s, \xi - cs) - \varphi(\xi)$. Then, $v(t, \xi)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} + c \dfrac{\partial v}{\partial \xi} - D\Delta(u^m - \varphi^m) + dv = f(u_\tau) - f(\varphi_{c\tau}), \\ v(s, \xi) = v_0(s, \xi), \quad s \in [-\tau, 0], \end{cases} \quad (5.1)$$

where $f(s) = pse^{-as}$.

Let $w(\xi) > 0$ be a weight function, and then we define the weighted $L_w^1(\mathbb{R})$ by

$$v \in L_w^1(\mathbb{R}) \Leftrightarrow \|v\|_{L_w^1(\mathbb{R})} = \int_\mathbb{R} |v(\xi)| w(\xi) d\xi \leq C, \quad (5.2)$$

and for any $T > 0$, $L^\infty([0, T]; L_w^1(\mathbb{R}))$ by

$$\|v\|_{L^\infty([0,T];L_w^1(\mathbb{R}))} = \sup_{t \in [0,T]} \|v(t)\|_{L_w^1(\mathbb{R})}.$$

We are going to prove Theorem 1.4. The proof is divided into 3 steps.

Step 1. Weighted L_w^1-regularity for the perturbed solution: $v \in L^\infty(\mathbb{R}_+; L_w^1(\mathbb{R}))$
Let $J_\varepsilon(s) \in C^1(R)$ be the approximation of the sign function satisfying $J'_\varepsilon(s) \geq 0$,

and $J_\varepsilon(s) \to \text{sgn}(s)$, $J_\varepsilon'(s) \to \delta(s)$, as $\varepsilon \to 0$, where $\delta(s)$ is the Delta function. Multiplying the first equation of (5.1) by $J_\varepsilon(\hat{v})w(\xi)\alpha_n(\xi)$, where $w(\xi) = e^{-\lambda\xi}$ is the weight function, $\alpha_n(\xi)$ is defined in (4.9), and $\hat{v} = u^m - \varphi^m$, and integrating the resultant equation on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} \partial_t(\alpha_n v w) J_\varepsilon(\hat{v})d\xi + D \int_{\mathbb{R}} \alpha_n J_\varepsilon'(\hat{v})w|\partial_\xi \hat{v}|^2 d\xi + D \int_{\mathbb{R}} \alpha_n' J_\varepsilon(\hat{v})w\partial_\xi \hat{v}d\xi$$
$$+ D \int_{\mathbb{R}} w'(\xi)\alpha_n J_\varepsilon(\hat{v})\partial_\xi \hat{v}d\xi + d \int_{\mathbb{R}} \alpha_n v w J_\varepsilon(\hat{v})d\xi$$
$$= \int_{\mathbb{R}} \alpha_n (f(u_\tau) - f(\varphi_{c\tau})) w J_\varepsilon(\hat{v})d\xi.$$

Let $h_\varepsilon(\hat{v}) := \int_0^{\hat{v}} J_\varepsilon(s)ds$, and then we further have

$$\int_{\mathbb{R}} \partial_t(\alpha_n v w) J_\varepsilon(\hat{v})d\xi + D \int_{\mathbb{R}} \alpha_n J_\varepsilon'(\hat{v})w|\partial_\xi \hat{v}|^2 d\xi$$
$$- D \int_{\mathbb{R}} (\lambda^2 w(\xi)\alpha_n - 2\lambda w(\xi)\alpha_n' + w\alpha_n'')h_\varepsilon(\hat{v})d\xi$$
$$+ d \int_{\mathbb{R}} \alpha_n v w J_\varepsilon(\hat{v})d\xi$$
$$= \int_{\mathbb{R}} \alpha_n (f(u_\tau) - f(\varphi_{c\tau})) w J_\varepsilon(\hat{v})d\xi.$$

Letting $\varepsilon \to 0$, and noting that $h_\varepsilon(\hat{v}) \to |\hat{v}|$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \alpha_n |v|wd\xi + D \int_{\mathbb{R}} \alpha_n \delta(\hat{v})w|\partial_\xi \hat{v}|^2 d\xi - D \int_{\mathbb{R}} w\alpha_n''|\hat{v}|d\xi + d \int_{\mathbb{R}} \alpha_n |v|wd\xi$$
$$= D\lambda^2 \int_{\mathbb{R}} \alpha_n |\hat{v}|wd\xi - 2\lambda D \int_{\mathbb{R}} w(\xi)\alpha_n'|\hat{v}|d\xi + \int_{\mathbb{R}} \alpha_n (f(u_\tau) - f(\varphi_{c\tau})) w \, \text{sgn}(\hat{v})d\xi$$
$$\leq M \int_{\mathbb{R}} w(\xi)\alpha_n |v|d\xi - 2\lambda D \int_{\mathbb{R}} w(\xi)\alpha_n'|\hat{v}|d\xi + p \int_{\mathbb{R}} \alpha_n |u_\tau - \varphi_{c\tau}|wd\xi.$$

Note that $\alpha_n'' \leq 0$, $\alpha_n'(\xi) \geq 0$ for $-n-1 \leq \xi \leq -n$, and $\alpha_n'(\xi) \leq 0$ for $n \leq \xi \leq n+1$, and then we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \alpha_n |v|wd\xi \leq (M - d) \int_{\mathbb{R}} \alpha_n |v|wd\xi + 2\lambda D \int_n^{n+1} e^{-\lambda\xi}|\hat{v}|d\xi$$
$$+ p \int_{\mathbb{R}} \alpha_n |u_\tau - \varphi_{c\tau}|wd\xi,$$

which implies

$$\frac{d}{dt} \left(e^{-(M-d)t} \int_{\mathbb{R}} \alpha_n |v|wd\xi \right) \leq Ce^{-(M-d)t}, \quad \text{for } 0 \leq t \leq \tau.$$

Letting $n \to \infty$, we obtain

$$\sup_{t \in (0,\tau)} \int_{\mathbb{R}} |v| w \mathrm{d}\xi \leq C.$$

Repeating the same procedure on $(\tau, 2\tau)$, $(2\tau, 3\tau)$, $\cdots$, then we prove

$$\sup_{t \in \mathbb{R}_+} \int_{\mathbb{R}} |v(t,\xi)| w(\xi) \mathrm{d}\xi \leq C.$$

The proof is complete.

Similarly to Step 1, we also have the following L^1-regularity of the solution $v(t, \xi)$ without weight function.

Step 2. L^1-regularity of perturbed solution: $v \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}))$ For simplicity, we do not take the approximation of the sign function. Multiplying the equation by $\alpha_n(x)\mathbf{sgn}(\hat{v})$, and integrating it over $\mathbb{R}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \alpha_n |v| \mathrm{d}\xi + D \int_{\mathbb{R}} \alpha_n \delta(\hat{v}) |\partial_\xi \hat{v}|^2 \mathrm{d}\xi - D \int_{\mathbb{R}} \alpha_n'' |\hat{v}| \mathrm{d}\xi + \mathrm{d} \int_{\mathbb{R}} \alpha_n |v| \mathrm{d}\xi$$

$$= \int_{\mathbb{R}} \alpha_n \left(f(u_\tau) - f(\varphi_{c\tau}) \right) \mathbf{sgn}(\hat{v}) \mathrm{d}\xi + c \int_{\mathbb{R}} \alpha_n' |v| \mathrm{d}\xi$$

$$\leq C \int_{\mathbb{R}} |u_\tau - \varphi_{c\tau}| \mathrm{d}\xi + C,$$

where α_n is defined in (4.9). Then, for any given $t > 0$, we obtain

$$\int_{\mathbb{R}} \alpha_n |v| \mathrm{d}\xi \leq C,$$

where C is independent of n. Letting $n \to \infty$, we prove

$$\int_{\mathbb{R}} |v(t,\xi)| \mathrm{d}\xi \leq C \quad \text{for } t \in \mathbb{R}_+.$$

The proof is complete.

Step 3. Global stability of traveling waves: $\|v(t)\|_{L^1_w(\mathbb{R})} \leq Ce^{-\mu t}$ $(\|v_0\|_{L^1(-\tau,0;L^1_w(R))} + \|v_0(0)\|_{L^1_w})$ Next, we are going to show the global stability of smooth traveling waves in the weighted L^1-space. By Theorem 1.3, we see that the solution of (5.1) satisfies

$$\|u\|_{L^\infty} \leq \max \left\{ \|u_0\|_{L^\infty}, \frac{p}{aed} \right\} = \kappa,$$

which means $\frac{\hat{v}}{v} < m\kappa^{m-1}$, where $v = u - \phi$ is the solution of (5.1), and $\hat{v} = u^m - \phi^m$ as defined before. Define

$$J_\tau(c, \lambda) := d + c\lambda - Dm\kappa^{m-1}\lambda^2 - pe^{-\lambda c\tau}. \tag{5.3}$$
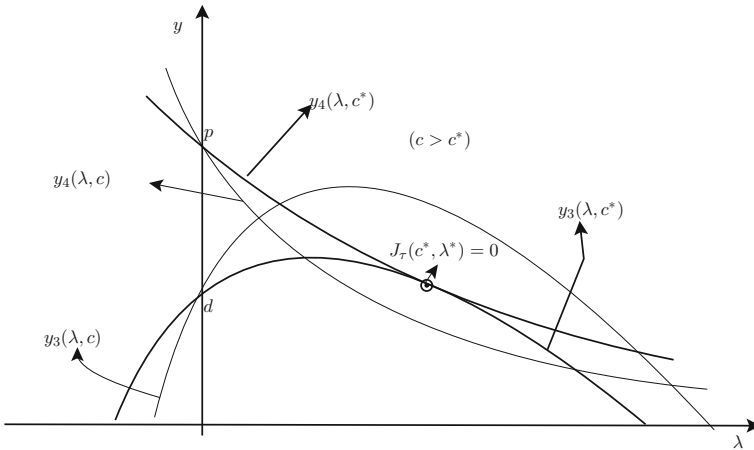
Fig. 5 The graphs of y_3 and y_4

Denote $y_3(\lambda, c) = d + c\lambda - Dm\kappa^{m-1}\lambda^2$, $y_4(\lambda, c) = pe^{-\lambda c\tau}$. From Fig. 5, we see that, there exists $c^*(\tau) > 0$ such that $J_\tau(c^*, \lambda) = 0$ admits only one real root λ^*, especially, $c^*(0) = 2\sqrt{Dm\kappa^{m-1}(p-d)}$, and for any $c > c^*$, $J_\tau(c, \lambda) = 0$ has two positive real roots λ_1, λ_2 with $0 < \lambda_1 < \lambda^* < \lambda_2$.

Firstly, similarly to the proof of above Step 1, we multiply the first equation of (5.1) by $\alpha_n J_\varepsilon(\hat{v})w$, where $w = e^{-\lambda\xi}$, $\alpha_n(\xi)$ is defined in (4.9), and $\hat{v} = u^m - \varphi^m$, then integrate the resultant equation on $\mathbb{R}$ to have

$$\int_{\mathbb{R}} \partial_t(\alpha_n v w) J_\varepsilon(\hat{v})\mathrm{d}\xi$$
$$- c\int_{\mathbb{R}} v\Big(\alpha_n'(\xi)w(\xi)J_\varepsilon(\hat{v}) + \alpha_n(\xi)w'(\xi)J_\varepsilon(\hat{v}) + \alpha_n(\xi)w(\xi)J_\varepsilon'(\hat{v})\partial_\xi\hat{v}\Big)\mathrm{d}\xi$$
$$+ D\int_{\mathbb{R}} \alpha_n J_\varepsilon'(\hat{v})w|\partial_\xi\hat{v}|^2\mathrm{d}\xi + D\int_{\mathbb{R}} \alpha_n' J_\varepsilon(\hat{v})w\partial_\xi\hat{v}\,\mathrm{d}\xi$$
$$+ D\int_{\mathbb{R}} \alpha_n J_\varepsilon(\hat{v})w'\partial_\xi\hat{v}\,\mathrm{d}\xi + \mathrm{d}\int_{\mathbb{R}} \alpha_n v w J_\varepsilon(\hat{v})\mathrm{d}\xi$$
$$= \int_{\mathbb{R}} \alpha_n\left(f(u_\tau) - f(\varphi_{c\tau})\right) w J_\varepsilon(\hat{v})\mathrm{d}\xi.$$

Letting $h_\varepsilon(\hat{v}) = \int_0^{\hat{v}} J_\varepsilon(s)\mathrm{d}s$, we further have

$$\int_{\mathbb{R}} \partial_t(\alpha_n v w) J_\varepsilon(\hat{v})\mathrm{d}\xi$$
$$- c\int_{\mathbb{R}} v\Big(\alpha_n'(\xi)w(\xi)J_\varepsilon(\hat{v}) + \alpha_n(\xi)w'(\xi)J_\varepsilon(\hat{v}) + \alpha_n(\xi)w(\xi)J_\varepsilon'(\hat{v})\partial_\xi\hat{v}\Big)\mathrm{d}\xi$$
$$+ D\int_{\mathbb{R}} \alpha_n J_\varepsilon'(\hat{v})w|\partial_\xi\hat{v}|^2\mathrm{d}\xi - D\int_{\mathbb{R}} h_\varepsilon(\hat{v})(\alpha_n''w + 2\alpha_n'w' + \alpha_n w'')\mathrm{d}\xi$$

$$+ \, \mathrm{d} \int_{\mathbb{R}} \alpha_n v w J_\varepsilon(\hat{v}) \mathrm{d}\xi$$

$$= \int_{\mathbb{R}} \alpha_n \left(f(u_\tau) - f(\varphi_{c\tau}) \right) w J_\varepsilon(\hat{v}) \mathrm{d}\xi.$$

By (4.6), we see that $v, \partial_\xi \hat{v} \in L^2_{loc}(\mathbb{R})$, then let $\varepsilon \to 0$, we have

$$\int_{\mathbb{R}} v \alpha_n(\xi) w(\xi) J'_\varepsilon(\hat{v}) \partial_\xi \hat{v} \mathrm{d}\xi \to \int_{\mathbb{R}} v \alpha_n(\xi) w(\xi) \delta(\hat{v}) \partial_\xi \hat{v} \mathrm{d}\xi = 0.$$

Thus, we have

$$\frac{d}{dt} \int_{\mathbb{R}} \alpha_n |v| w \mathrm{d}\xi - c \int_{\mathbb{R}} \left(|v| \alpha'_n(\xi) w(\xi) + |v| \alpha_n(\xi) w'(\xi) \right) \mathrm{d}\xi + D \int_{\mathbb{R}} \alpha_n \delta(\hat{v}) w |\partial_\xi \hat{v}|^2 \mathrm{d}\xi$$

$$- D \int_{\mathbb{R}} |\hat{v}| (\alpha''_n w + 2\alpha'_n w' + \alpha_n w'') \mathrm{d}\xi + \mathrm{d} \int_{\mathbb{R}} \alpha_n v w J_\varepsilon(\hat{v}) \mathrm{d}\xi$$

$$= \int_{\mathbb{R}} \alpha_n \left(f(u_\tau) - f(\varphi_{c\tau}) \right) w J_\varepsilon(\hat{v}) \mathrm{d}\xi.$$

By the weighted L^1_w-regularity for the perturbed solution showed in the above Step 1, we see that for any $t > 0$, $|v| w \in L^1(\mathbb{R})$. By taking $n \to \infty$, we then have

$$\frac{d}{dt} \int_{\mathbb{R}} |v| w \mathrm{d}\xi - c \int_{\mathbb{R}} |v| w'(\xi) \mathrm{d}\xi + D \int_{\mathbb{R}} \delta(\hat{v}) w |\partial_\xi \hat{v}|^2 \mathrm{d}\xi$$

$$- D \int_{\mathbb{R}} w''(\xi) |\hat{v}| \mathrm{d}\xi + \mathrm{d} \int_{\mathbb{R}} |v| w \mathrm{d}\xi$$

$$= \int_{\mathbb{R}} f'(\varphi_{c\tau} + \theta v(t - \tau, \xi - c\tau)) v(t - \tau, \xi - c\tau) w \, \mathrm{sgn}(\hat{v}) \mathrm{d}\xi.$$

Multiplying the above equation by $e^{\mu t}$, and integrating it from 0 to t, we then have

$$\int_{\mathbb{R}} e^{\mu t} |v(t, \xi)| w \mathrm{d}\xi + \int_0^t \int_{\mathbb{R}} \left(d - c \frac{w'}{w} - D \frac{w''}{w} \frac{\hat{v}}{v} - \mu \right) e^{\mu s} |v| w \mathrm{d}\xi \mathrm{d}s$$

$$+ D \int_0^t \int_{\mathbb{R}} e^{\mu s} \delta(\hat{v}) w |\partial_\xi \hat{v}|^2 \mathrm{d}\xi \mathrm{d}s$$

$$= \int_0^t \int_{\mathbb{R}} e^{\mu s} f'(\varphi_{c\tau} + \theta v(s - \tau, \xi - c\tau)) v(s - \tau, \xi - c\tau) w(\xi) \, \mathrm{sgn}(\hat{v}) \mathrm{d}\xi \mathrm{d}s$$

$$+ \int_{\mathbb{R}} |v(0, \xi)| w(\xi) \mathrm{d}\xi$$

$$=: I_1(t) + \int_{\mathbb{R}} |v(0, \xi)| w(\xi) \mathrm{d}\xi. \tag{5.4}$$

Next, we estimate I_1. Clearly, we have $u(t, x) \geq 0$ since $u_0(s, x) \geq 0$, and note that $f'(s) = p(1 - as)e^{-as}$, then $|f'(s)| \leq p$ for any $s \geq 0$, therefore, we have

$$
\begin{aligned}
|I_1| &\leq p \int_0^t \int_{\mathbb{R}} e^{\mu s} |v(s - \tau, \xi)| w(\xi + c\tau) d\xi ds \\
&= pe^{\mu \tau} \int_{-\tau}^{t-\tau} \int_{\mathbb{R}} e^{\mu s} |v(s, \xi)| w(\xi + c\tau) d\xi ds \\
&\leq pe^{\mu \tau} \int_{-\tau}^0 \int_{\mathbb{R}} e^{\mu s} |v_0(s, \xi)| w(\xi + c\tau) d\xi ds \\
&\quad + pe^{\mu \tau} \int_0^t \int_{\mathbb{R}} e^{\mu s} |v(s, \xi)| w(\xi + c\tau) d\xi ds.
\end{aligned}
$$

Substituting it into (5.4), we arrive at

$$
e^{\mu t} \int_{\mathbb{R}} |v(t, \xi)| w d\xi + \int_0^t \int_{\mathbb{R}} e^{\mu s} \left(d + c\lambda - D \frac{\hat{v}}{v} \lambda^2 - \mu - pe^{-\lambda c\tau} e^{\mu \tau} \right) |v| w d\xi ds
$$
$$
+ D \int_0^t \int_{\mathbb{R}} e^{\mu s} \delta(\hat{v}) w |\partial_\xi \hat{v}|^2 d\xi ds
$$
$$
\leq pe^{\mu \tau} \int_{-\tau}^0 \int_{\mathbb{R}} e^{\mu s} |v_0(s, \xi)| w(\xi + c\tau) d\xi ds + \int_{\mathbb{R}} |v(0, \xi)| w(\xi) d\xi.
$$

For any fixed $\lambda \in (\lambda_1, \lambda_2)$, then there exists a constant $m_1 > 0$, such that $J_\tau(c, \lambda) > 2m_1$, where $J_\tau(c, \lambda)$ is defined in (5.3), and take $\mu > 0$ appropriately small such that

$$
d + c\lambda - D \frac{\hat{v}}{v} \lambda^2 - \mu - pe^{-\lambda c\tau} e^{\mu \tau} > m_1.
$$

Then, we obtain

$$
\int_{\mathbb{R}} |v(t, \xi)| w d\xi + m_1 \int_0^t \int_{\mathbb{R}} e^{\mu(s-t)} |v| w d\xi ds
$$
$$
\leq pe^{-\mu t} e^{\mu \tau} \int_{-\tau}^0 \int_{\mathbb{R}} e^{\mu s} |v_0(s, \xi)| w(\xi + c\tau) d\xi ds + e^{-\mu t} \int_{\mathbb{R}} |v(0, \xi)| w(\xi) d\xi,
$$

which implies the weighted L_w^1-stability:

$$
\|(u - \phi)(t)\|_{L_w^1(\mathbb{R})} \leq Ce^{-\mu t} (\|v_0\|_{L^1(-\tau, 0; L_w^1(\mathbb{R}))} + \|v_0(0)\|_{L_w^1(\mathbb{R})}).
$$

When $\tau = 0$, similar to (5.4), we have

$$\int_{\mathbb{R}} e^{\tilde{\mu}t}|v(t,\xi)|\tilde{w}\mathrm{d}\xi + \int_0^t \int_{\mathbb{R}} \left(d - c\frac{\tilde{w}'}{\tilde{w}} - D\frac{\hat{v}}{v}\frac{\tilde{w}''}{\tilde{w}} - \tilde{\mu}\right) e^{\tilde{\mu}s}|v|\tilde{w}\mathrm{d}\xi\mathrm{d}s$$

$$+ D\int_0^t \int_{\mathbb{R}} e^{\tilde{\mu}s}\delta(\hat{v})\tilde{w}|\partial_\xi\hat{v}|^2\mathrm{d}\xi\mathrm{d}s$$

$$= \int_0^t \int_{\mathbb{R}} e^{\tilde{\mu}s}f'(\varphi + \theta v(s,\xi))|v(s,\xi)|\tilde{w}(\xi)\mathrm{d}\xi\mathrm{d}s + \int_{\mathbb{R}}|v(0,\xi)|\tilde{w}(\xi)\mathrm{d}\xi$$

$$\leq p\int_0^t \int_{\mathbb{R}} e^{\tilde{\mu}s}|v(s,\xi)|\tilde{w}(\xi)\mathrm{d}\xi\mathrm{d}s + \int_{\mathbb{R}}|v(0,\xi)|\tilde{w}(\xi)\mathrm{d}\xi.$$

Taking $\tilde{w}(\xi) = e^{-\tilde{\lambda}\xi}$, then

$$\int_{\mathbb{R}}|v(t,\xi)|\tilde{w}\mathrm{d}\xi + \int_0^t \int_{\mathbb{R}} \left(d + c\tilde{\lambda} - D\frac{\hat{v}}{v}\tilde{\lambda}^2 - \tilde{\mu} - p\right) e^{\tilde{\mu}(s-t)}|v|\tilde{w}\mathrm{d}\xi\mathrm{d}s$$

$$\leq e^{-\tilde{\mu}t}\int_{\mathbb{R}}|v(0,\xi)|\tilde{w}(\xi)\mathrm{d}\xi.$$

For any fixed $\tilde{\lambda} \in (\tilde{\lambda}_1, \tilde{\lambda}_2)$, then there exists a constant $\tilde{m}_1 > 0$, such that $J_0(c, \tilde{\lambda}) > 2\tilde{m}_1$, and take $\tilde{\mu} > 0$ appropriately small such that

$$d + c\tilde{\lambda} - D\frac{\hat{v}}{v}\tilde{\lambda}^2 - \tilde{\mu} - p > \tilde{m}_1.$$

Then, we obtain

$$\int_{\mathbb{R}}|v(t,\xi)|\tilde{w}\mathrm{d}\xi + \tilde{m}_1\int_0^t \int_{\mathbb{R}} e^{\tilde{\mu}(s-t)}|v|\tilde{w}\mathrm{d}\xi\mathrm{d}s \leq e^{-\tilde{\mu}t}\int_{\mathbb{R}}|v(0,\xi)|\tilde{w}(\xi)\mathrm{d}\xi.$$

The proof is complete. □

Based on the above Steps 1–3, we immediately prove Theorem 1.4.

Remark 5.1 When $\tau = 0$, namely, the degenerate reaction–diffusion equation (1.1) without time delay, from Theorem 1.4, we see that $v(t,\xi) \in L^1(\mathbb{R})$ for any $t > 0$ if $v_0(0,\xi) \in L^1(\mathbb{R})$. By Theorem 1.3, we also see that $v \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap C^{\alpha,\beta}(\mathbb{R}_+ \times \mathbb{R})$. Then, we always have

$$\lim_{\xi \to \infty} |u(t,\xi) - \varphi(\xi)| = 0.$$

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