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Relaxation limit in bipolar semiconductor hydrodynamic model with non-constant doping profile

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A B S T R A C T

The relaxation limit from bipolar semiconductor hydrodynamic (HD) model to drift–diffusion (DD) model is shown under the non-constant doping profile assumption for both stationary solutions and global-in-time solutions, which satisfy the general form of the Ohmic contact boundary condition. The initial layer phenomenon will be analyzed because the initial data is not necessarily in the momentum equilibrium. Due to the bipolar coupling structure, the analysis is hard and different from the previous literature on unipolar model or bipolar model with zero doping profile restriction. We first construct the non-constant uniform stationary solutions by the operator method for both HD and DD models in a unified procedure. Then we prove the global existence of DD model and uniform global existence of HD model by the elementary energy method but with some new developments. Based on the above existence results, we further calculate the convergence rates in relaxation limits.

1. Introduction

We consider the following bipolar isothermal hydrodynamic (HD) model for semiconductors

$$\begin{cases} n_{it} + j_{ix} = 0, & \text{(a)} \\ j_{it} + \left(j_i^2/n_i + K_i n_i\right)_x = (-1)^{i-1} n_i \phi_x - j_i/\tau, & \text{(b)} \\ \phi_{xx} = n_1 - n_2 - D(x), \quad i = 1, 2, \quad \forall (t, x) \in (0, +\infty) \times \Omega, & \text{(c)} \end{cases} \tag{1.1}$$

where $\Omega := (0, 1)$ is a bounded interval occupied by the semiconductor device. The unknown functions $n_i(t, x)$ and $j_i(t, x)$ stand for the charge density, current distribution for electrons $(i = 1)$ and holes $(i = 2)$

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respectively, and ϕ is the electrostatic potential. The positive constants τ, K_1 and K_2 are the relaxation time, temperature constant of electrons and temperature constant of holes respectively. The given function $D(x)$ means the non-constant doping profile, the density of impurities in semiconductor devices. Mathematically, the system (1.1) takes the form of the compressible fluids coupled with self-consistent Poisson equation, which leads to a hyperbolic–elliptic system.

In the present paper, we are interested in the behavior of solutions of the bipolar HD model (1.1) as the relaxation time $\tau \to 0^+$. Thus, we suppose $\tau \in (0,1]$ and introduce a scaling of time $s = \tau t$ and define

$$n_i^\tau(s,x) = n_i\left(\frac{s}{\tau}, x\right), \quad j_i^\tau(s,x) = \frac{1}{\tau} j_i\left(\frac{s}{\tau}, x\right), \quad \phi^\tau(s,x) = \phi\left(\frac{s}{\tau}, x\right). \tag{1.2}$$

Substituting the scaling transform (1.2) into the original HD model (1.1) and setting again $t = s$, we obtain the scaled HD model

$$\begin{cases} n_{it}^\tau + j_{ix}^\tau = 0, & \text{(a)} \\ \tau^2 j_{it}^\tau + \left(\tau^2 (j_i^\tau)^2/n_i^\tau + K_i n_i^\tau\right)_x = (-1)^{i-1} n_i^\tau \phi_x^\tau - j_i^\tau, & \text{(b)} \\ \phi_{xx}^\tau = n_1^\tau - n_2^\tau - D(x), \quad i = 1,2, \quad \forall (t,x) \in (0,+\infty) \times \Omega. & \text{(c)} \end{cases} \tag{1.3}$$

From now on, we only consider the scaled HD model (1.3) and also call it the HD model. The system (1.3) is complemented by the initial and boundary data

$$(n_i^\tau, j_i^\tau)(0,x) = (n_{i0}, j_{i0})(x), \tag{1.4}$$

and

$$n_i^\tau(t,0) = n_{il} > 0, \qquad n_i^\tau(t,1) = n_{ir} > 0, \tag{1.5a}$$

$$\phi^\tau(t,0) = 0, \qquad \phi^\tau(t,1) = \phi_r > 0, \tag{1.5b}$$

where n_{il}, n_{ir} and ϕ_r are positive constants. The physical boundary condition (1.5) is called the Ohmic contact boundary condition. Since we intend to establish the existence of a classical solution to the initial–boundary value problem (IBVP for abbreviation) (1.3)–(1.5), it is necessary to assume that the initial data (1.4) are compatible with the boundary data (1.5). Namely,

$$n_{i0}(0) = n_{il}, \quad n_{i0}(1) = n_{ir}, \quad j_{i0x}(0) = j_{i0x}(1) = 0. \tag{1.6}$$

Formally substituting $\tau = 0$ into the HD model (1.3) and expressing the solution of the limit system by $(n_1^0, j_1^0, n_2^0, j_2^0, \phi^0)$, we have the bipolar drift–diffusion (DD) model

$$\begin{cases} n_{it}^0 + j_{ix}^0 = 0, & \text{(a)} \\ j_i^0 = (-1)^{i-1} n_i^0 \phi_x^0 - K_i n_{ix}^0, & \text{(b)} \\ \phi_{xx}^0 = n_1^0 - n_2^0 - D(x), \quad i = 1,2, \quad \forall (t,x) \in (0,+\infty) \times \Omega. & \text{(c)} \end{cases} \tag{1.7}$$

The initial and boundary data for the DD model (1.7) are given by

$$n_i^0(0,x) = n_{i0}(x), \tag{1.8}$$

and

$$n_i^0(t,0) = n_{il} > 0, \qquad n_i^0(t,1) = n_{ir} > 0, \tag{1.9a}$$

$$\phi^0(t,0) = 0, \qquad \phi^0(t,1) = \phi_r > 0. \tag{1.9b}$$

To consider the existence of solutions of both HD and DD models, we need to assume the subsonic condition of the electric flow and the positivity of the density. These conditions are written as

$$\inf_{x\in\Omega} n_i^\tau > 0, \quad \inf_{x\in\Omega} S_i[n_i^\tau, j_i^\tau] > 0, \quad \forall \tau \in [0,1], \tag{1.10}$$

where

$$S_i[n_i^\tau, j_i^\tau] := K_i - \frac{(\tau j_i^\tau)^2}{(n_i^\tau)^2}, \quad \forall \tau \in [0,1].$$

Apparently, if we want to construct the solutions in the above physical region (1.10), then the initial data (n_{i0}, j_{i0}) must satisfy the same conditions (1.10).

The stationary boundary value problem (BVP) of the HD-IBVP (1.3)–(1.5) and the stationary BVP of the DD-IBVP (1.7)–(1.9) can be written as a unified form with small parameter $\tau \in [0,1]$, namely,

$$\begin{cases} \tilde{j}_{ix}^\tau = 0, & \text{(a)} \\ S_i[\tilde{n}_i^\tau, \tilde{j}_i^\tau]\tilde{n}_{ix}^\tau = (-1)^{i-1}\tilde{n}_i^\tau \tilde{\phi}_x^\tau - \tilde{j}_i^\tau, & \text{(b)} \\ \tilde{\phi}_{xx}^\tau = \tilde{n}_1^\tau - \tilde{n}_2^\tau - D(x), \quad i = 1,2, \quad \forall x \in \Omega, & \text{(c)} \end{cases} \tag{1.11}$$

and

$$\tilde{n}_i^\tau(0) = n_{il} > 0, \qquad \tilde{n}_i^\tau(1) = n_{ir} > 0, \tag{1.12a}$$

$$\tilde{\phi}^\tau(0) = 0, \qquad \tilde{\phi}^\tau(1) = \phi_r > 0, \tag{1.12b}$$

where

$$\tilde{S}_i^\tau = S_i[\tilde{n}_i^\tau, \tilde{j}_i^\tau] := K_i - \frac{(\tau \tilde{j}_i^\tau)^2}{(\tilde{n}_i^\tau)^2}, \quad \forall \tau \in [0,1].$$

We also assume that the stationary solution $(\tilde{n}_i^\tau, \tilde{j}_i^\tau)$ satisfies the subsonic condition and the positivity of the density, that is,

$$\inf_{x\in\Omega} \tilde{n}_i^\tau > 0, \quad \inf_{x\in\Omega} \tilde{S}_i^\tau > 0, \quad \forall \tau \in [0,1]. \tag{1.13}$$

The HD and DD models are two important mathematical models for semiconductor devices, which were introduced to remedy the high cost in dealing with the basic kinetic transport equations in real applications. These macroscopic fluid models give a good compromise between the physical accuracy and the reduction of computational cost. For more information on the semiconductor device modeling involved, we refer to Roosbroeck [29], Markowich et al. [20], Jüngel [15,16], Bløtekjær [2], Ben Abdallah and Degond [1].

Actually, in the present paper, we only study the isothermal models without loss of generality. But one can also consider the more general pressure law (e.g. $p_i(n_i) = K_i n_i^\gamma$ with $\gamma > 1$) in the models, which are called isentropic models. The main difference between isothermal and isentropic models is that the former contains the linear pressure term but the latter possesses the nonlinear one. For unipolar HD model with the general pressure law, as we all know, the relaxation term together with the electric field term provides strong dissipation effect enough to prevent the formation of singularities for small and smooth initial data [25]. However, for large initial data, one has to consider the global weak solution. In the studies on weak solution [14,11], the isothermal case is more difficult than the isentropic case because the term j^2/n is not Lipschitz continuous near the vacuum due to the infiniteness of the velocity. In the setting of the bipolar HD model for the problems starting with small smooth initial data, we do believe that the same methods used in isothermal case could probably cover the isentropic case. In order to clarify the competition between the bipolar coupling structure and the non-constant doping profile, we ignore the impact of the nonlinear pressure law, instead to consider the isothermal case only for simplicity.

We introduce some known results about both DD and HD models as follows. To our knowledge, Mock [22] first investigated the bipolar DD model without recombination–generation rate on the bounded domain and proved existence theorems for stationary solutions. As for the time-dependent DD model with recombination–generation rate, Mock [23] was the first to prove a global existence and uniqueness result. Moreover, Mock [24] proved that the above global solution decays exponentially into the corresponding thermal equilibrium of which current density is zero. All the results in [22–24] are shown under the isothermal assumption and the insulating boundary conditions. For more general boundary conditions, Gajewski and Gröger [5] established the asymptotic stability of the thermal equilibrium. Lou [18] proved the global existence and the uniqueness of a solution to the DD model with heat conduction under the Dirichlet boundary condition, and also showed the existence, uniqueness and local asymptotic stability of the stationary solution if the domain is sufficiently narrow in one direction. For the HD model, Degond and Markowich [3] first studied the existence and uniqueness of the stationary solution of the unipolar HD model on the bounded interval. Luo, Natalini and Xin [19] first studied the large time behavior of the solutions to the Cauchy problem of the unipolar HD model in the whole real line. In fact, there are many mathematical results on existence, uniqueness, large time asymptotic behavior and stability of stationary solutions. For example, see [17,7,25,26,13] and the references therein for unipolar HD model, and for bipolar one we refer to [6,12,4,21,30,28,9,10] and the references therein.

There are few results on the hierarchy between these two models, but it has been increasingly attracting the interests of researchers. For unipolar model, Nishibata and Suzuki [27] verified the relaxation limit of the global smooth solution of the isothermal HD model with non-flat doping profile on the bounded interval. In several space dimensions, Xu [31] proved the relaxation limit of global classical solution to Cauchy problem of the isothermal HD model with positive constant doping profile in the critical Besov space. Xu and Yong [32] further extended the result in [31] to the non-isentropic case but still use the positive constant doping profile. For bipolar model, there is no relaxation limit result both in the smooth solution regime and in the setting of non-constant doping profile as the existing literature mostly deals with the unipolar model. Therefore, in the present paper, we will give a rigorous proof to this kind of singular limit. It is worth mentioning that an initial layer will occur in the relaxation limit provided the initial data $j_{i0}(x) \neq j_i^0(0,x)$, namely, the initial data of HD model is not in momentum equilibrium.

Before stating our main results, we firstly list the notations and settings used in this paper,

- $\mathcal{B}^l(\overline{\Omega})$: The space of l-times bounded differentiable functions on $\overline{\Omega}$ with the norm $|\cdot|_l := \sum_{m=0}^{l} \sup_{x\in\overline{\Omega}} |\partial_x^m \cdot|$ (integer $l \geq 0$). The stationary solutions will be found in this class of function spaces.
- $H^l(\Omega)$: The usual L^2-Sobolev space over Ω of integer order l with the norm $\|\cdot\|_l$ ($l \geq 0$). In particular, $\|\cdot\|_0 = \|\cdot\|$.
- $C^l([0,T]; H^m(\Omega))$: The space of l-times continuously differentiable functions on time interval $[0,T]$ with values in $H^m(\Omega)$. Similarly, one can define the function spaces $L^2(0,T;H^1(\Omega))$ and $L^2_{loc}(0,T;H^2(\Omega))$. The time-dependent solutions will be constructed in these classes of function spaces. More precisely, the solution spaces used in HD-IBVP (1.3)–(1.5):

$$\mathfrak{X}_l^m([0,T]) := \bigcap_{k=0}^{l} C^k([0,T]; H^{m+l-k}(\Omega)), \quad \mathfrak{X}_l([0,T]) := \mathfrak{X}_l^0([0,T]), \quad l,m = 0,1,2,$$

and the solution space used in DD-IBVP (1.7)–(1.9):

$$\mathfrak{Y}([0,T]) := \Big\{ (n_1^0, j_1^0, n_2^0, j_2^0, \phi^0)(t,x) \;\Big|\; (n_i^0, j_i^0, \phi^0) \in C([0,T]; (H^2 \times H^1 \times H^2)(\Omega)) \\ n_{it}^0 \in C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap L^2_{loc}(0,T; H^2(\Omega)),\ i = 1,2 \Big\}.$$

- The strength parameter of the given data is defined as

$$\delta := \sum_{i=1}^{2} |n_{il} - n_{ir}| + |\phi_r| + \|D - \bar{d}\|_1, \tag{1.14}$$

where $\bar{d} := n_{1l} - n_{2l}$ and the assumption $\delta \ll 1$ will play an important role in what follows.
- C denotes the generic positive constant and N, γ_k, C_k, C_{kl} and C_{kr} $(k = 1,2,\cdots)$ stand for the specific positive constants. It is worth mentioning that all these constants only depend on the state constants n_{1l}, n_{2l}, K_1 and K_2 throughout the paper. This fact allows us to establish the relaxation limits.

Now we can state the main results in the present paper as follows.

Theorem 1.1 *(Existence and uniqueness of stationary wave). Suppose that $D \in H^1(\Omega)$, for arbitrary constants $n_{il}, K_i > 0$, there exist constants $\delta_0, C > 0$ such that if $\delta \le \delta_0$, then for arbitrary $0 \le \tau \le 1$ there exists a unique solution $\big(\tilde{n}_1^\tau, \tilde{j}_1^\tau, \tilde{n}_2^\tau, \tilde{j}_2^\tau, \tilde{\phi}^\tau\big) \in \big[(\mathcal{B}^2)(\overline{\Omega})\big]^5$ to the BVP (1.11)–(1.12), satisfying the condition (1.13) and the estimates*

$$0 < \frac{1}{2} n_{il} \le \tilde{n}_i^\tau(x) \le 2n_{il}, \quad \forall x \in \overline{\Omega}, \quad i = 1,2, \tag{1.15a}$$

$$\sum_{i=1}^{2} \Big(|\tilde{n}_i^\tau - n_{il}|_2 + |\tilde{j}_i^\tau| \Big) + |\tilde{\phi}^\tau|_2 \le C\delta, \tag{1.15b}$$

where $C > 0$ is independent of δ and $\tau \in [0,1]$.

Remark 1.1. In Theorem 1.1, if $\tau = 0$, then $\big(\tilde{n}_1^0, \tilde{j}_1^0, \tilde{n}_2^0, \tilde{j}_2^0, \tilde{\phi}^0\big)$ is the subsonic stationary solution to the DD model. If $0 < \tau \le 1$, then $\big(\tilde{n}_1^\tau, \tilde{j}_1^\tau, \tilde{n}_2^\tau, \tilde{j}_2^\tau, \tilde{\phi}^\tau\big)$ is the subsonic stationary solution to the HD model.

Theorem 1.2 *(Stability of stationary wave to DD model). Suppose that $D \in H^1(\Omega)$, and the initial data $0 < n_{i0} \in H^2(\Omega)$ is compatible with the boundary data (1.9), for arbitrary constants $n_{il}, K_i > 0$, there exist constants $\delta_1, C, \gamma_1 > 0$ such that if $\sum_{i=1}^{2} \|n_{i0} - \tilde{n}_i^0\|_2 + \delta \le \delta_1$, then there exists a unique global solution $(n_1^0, j_1^0, n_2^0, j_2^0, \phi^0) \in \mathfrak{Y}([0,+\infty))$ to the DD-IBVP (1.7)–(1.9), satisfying the additional regularity $\phi^0 - \tilde{\phi}^0 \in C([0,+\infty); H^4(\Omega))$ and the estimates*

$$0 < \frac{1}{4} n_{il} \le n_i^0(t,x) \le 4n_{il}, \quad i = 1,2, \tag{1.16a}$$

$$\sum_{i=1}^{2} \Big(\|(n_i^0 - \tilde{n}_i^0)(t)\|_2 + \|(j_i^0 - \tilde{j}_i^0)(t)\|_1 \Big) + \|(\phi^0 - \tilde{\phi}^0)(t)\|_4 \le C \sum_{i=1}^{2} \|n_{i0} - \tilde{n}_i^0\|_2 \, e^{-\gamma_1 t}, \tag{1.16b}$$

$$\int_0^t s \sum_{i=1}^{2} \|(n_{itt}^0, n_{ixxt}^0)(s)\|^2 ds \le C \sum_{i=1}^{2} \|n_{i0} - \tilde{n}_i^0\|_2^2 (1+t), \quad \forall t \in [0,+\infty). \tag{1.16c}$$

Remark 1.2. In bipolar case since we have to treat a parabolic system for which the maximum principle is failed to establish the positive lower bound of the density $n_i^0(t,x)$, the smallness assumptions in Theorem 1.2 on the difference of the initial data and stationary solution are necessary. This is the essential difference between our bipolar results in Theorem 1.2 and the unipolar results in [27] for the DD-IBVP.

Theorem 1.3 *(Stability of stationary wave to HD model). Suppose that $D \in H^1(\Omega)$, the initial data $n_{i0}, j_{i0} \in H^2(\Omega)$ satisfy the conditions (1.6) and (1.10), for arbitrary constants $n_{il}, K_i > 0$, there exist constants $\delta_2, C, \gamma_2 > 0$ such that for arbitrary $\tau \in (0,1]$ if $\sum_{i=1}^2 \left(\|n_{i0} - \tilde{n}_i^\tau\|_2 + \|j_{i0} - \tilde{j}_i^\tau\|_1 + \|\tau j_{i0xx}\|\right) + \delta \le \delta_2$, then the time-dependent HD-IBVP (1.3)–(1.5) has a unique global solution $(n_1^\tau, j_1^\tau, n_2^\tau, j_2^\tau, \phi^\tau) \in \left[\mathfrak{X}_2([0,+\infty))\right]^5$ satisfying the condition (1.10), the additional regularity $\phi^\tau - \tilde{\phi}^\tau \in \mathfrak{X}_2^2([0,+\infty))$ and the estimates*

$$0 < \frac{1}{4} n_{il} \le n_i^\tau(t,x) \le 4 n_{il}, \quad i = 1,2, \tag{1.17a}$$

$$\begin{aligned}&\sum_{i=1}^2 \Big(\|(n_i^\tau - \tilde{n}_i^\tau)(t)\|_2 + \|(j_i^\tau - \tilde{j}_i^\tau)(t)\|_1 + \|\tau j_{ixx}^\tau(t)\|\Big) + \|(\phi^\tau - \tilde{\phi}^\tau)(t)\|_4 \\ &\quad \le C \sum_{i=1}^2 \Big(\|n_{i0} - \tilde{n}_i^\tau\|_2 + \|j_{i0} - \tilde{j}_i^\tau\|_1 + \|\tau j_{i0xx}\|\Big) e^{-\gamma_2 t}, \quad \forall t \in [0,+\infty).\end{aligned} \tag{1.17b}$$

Theorem 1.4 *(Relaxation limit of stationary waves). Let the conditions in Theorem 1.1 hold and let $(\tilde{n}_i^\tau, \tilde{j}_i^\tau, \tilde{\phi}^\tau)(x)$ be the stationary HD-solution, $(\tilde{n}_i^0, \tilde{j}_i^0, \tilde{\phi}^0)(x)$ be the stationary DD-solution. Then, for arbitrary constants $n_{il}, K_i > 0$ there exist constants $\delta_3, C > 0$ such that if $\delta \le \delta_3$, then the convergence estimate holds:*

$$\sum_{i=1}^2 \Big(\|\tilde{n}_i^\tau - \tilde{n}_i^0\|_2 + |\tilde{j}_i^\tau - \tilde{j}_i^0|\Big) + \|\tilde{\phi}^\tau - \tilde{\phi}^0\|_4 \le C\delta^2\tau^2, \quad \forall \tau \in (0,1], \tag{1.18}$$

where the constant $C > 0$ is independent of δ and τ.

Theorem 1.5 *(Relaxation limit of global solutions). Assume that the conditions in Theorem 1.2 and Theorem 1.3 hold. Then, for arbitrary constants $n_{il}, K_i > 0$, there exist constants $\delta_4, \gamma_3, C > 0$ such that if*

$$\tau + \delta + \sum_{i=1}^2 \Big(\|n_{i0} - \tilde{n}_i^\tau\|_2 + \|j_{i0} - \tilde{j}_i^\tau\|_1 + \|\tau j_{i0xx}\|\Big) \le \delta_4, \tag{1.19}$$

then the global-in-time HD-solution $(n_i^\tau, j_i^\tau, \phi^\tau)(t,x)$ converges to the global-in-time DD-solution $(n_i^0, j_i^0, \phi^0)(t,x)$ as τ tends to zero. Precisely, for $t \in (0,+\infty)$, the following convergence estimates hold:

$$\sum_{i=1}^2 \|(n_i^\tau - n_i^0)(t)\|_1^2 + \|(\phi^\tau - \phi^0)(t)\|_3^2 \le C\tau^{\gamma_3}, \tag{1.20a}$$

$$\|(j_i^\tau - j_i^0)(t)\|^2 \le \|j_{i0} - j_i^0(0,\cdot)\|^2 e^{-t/\tau^2} + C\tau^{\gamma_3}, \quad i = 1,2, \tag{1.20b}$$

$$\sum_{i=1}^2 \|((n_i^\tau - n_i^0)_{xx}, (j_i^\tau - j_i^0)_x)(t)\|^2 + \|\partial_x^4(\phi^\tau - \phi^0)(t)\|^2 \le C(1+t^{-1})\tau^{\gamma_3}. \tag{1.20c}$$

Now, we illustrate the main ideas and the key technical points in the present paper. Comparing with the unipolar models for semiconductor, the bipolar models are much more complex due to the bipolar coupling structure between the two carriers. The first difficulty arising from the bipolar coupling structure is the construction of the subsonic stationary solutions to the models with non-constant doping profile under the general form of Ohmic contact boundary condition. To solve the stationary problem, we will obtain the Dirichlet boundary value problem of an quasilinear seconder order strongly coupled elliptic system for the stationary densities, which comes no maximum principle applied to establish the positive lower bound of the solutions. Thus, Schauder fixed point argument which is often used in unipolar models no longer applies. Based on some observations, we adopt a new operator method [9,8] to overcome this typical difficulty by using the tools like regular perturbation, linearization and Banach fixed point argument. Meanwhile, we can perform a unified argument for both DD and HD models to construct the subsonic stationary solutions (for details, see the proof of Theorem 1.1). In addition, we further prove the global existence of the solution to the bipolar DD model only if the initial data is close to the stationary solution rather than the case of the large initial data for the unipolar DD model. The difficulty is similar to the stationary problem (see Remark 1.2). Next, we can prove the uniformly (in relaxation time) global existence of the solution to the bipolar HD model by the elaborate energy method, in which we must ensure the generic constants in the energy estimates are independent of the relaxation time. This uniform estimate plays a crucial role in establishing the relaxation limit of the global solution. Furthermore, the relaxation limit of the stationary solutions is also obtained by the standard energy method. Finally, we study the relaxation limit of the global solution, in which the initial layer will occur.

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of the stationary solutions for both DD and HD models by the unified argument. In Section 3, we first show the asymptotic stability of the stationary solution to the DD model in Subsection 3.1 and show the uniformly asymptotic stability of the stationary solution to the HD model in Subsection 3.2. In Section 4, we establish the relaxation limits for the stationary solutions and the global solutions, which are carried out in Subsection 4.1 and Subsection 4.2, respectively.

2. Existence and uniqueness of stationary solution

In this section, we consider the existence of the subsonic stationary solutions to both DD and HD models. We observe that these two problems can be solved by a unified argument. To verify this observation, we give the proof of Theorem 1.1 as follows.

Proof of Theorem 1.1.

Step I. Regular perturbation and linearization.

We first denote the stationary solution to the BVP (1.11) and (1.12) by

$$U(x) = \left(\tilde{n}_1^\tau, \tilde{j}_1^\tau, \tilde{n}_2^\tau, \tilde{j}_2^\tau, \tilde{\phi}^\tau\right)^T(x). \tag{2.1}$$

Observing that if the strength parameter $\delta = 0$, where δ is defined in (1.14), then there exists a unique constant solution to the BVP (1.11) and (1.12), denoted by

$$U(x) \equiv \bar{U} = \left(n_{1l}, 0, n_{2l}, 0, 0\right)^T, \quad \forall \tau \in [0,1]. \tag{2.2}$$

In the case of $0 < \delta \ll 1$, considering the BVP (1.11)–(1.12) as a regular perturbation problem of the BVP (1.11)–(1.12) of $\delta = 0$. To this end, let us introduce the stationary perturbation variables

$$U_\delta(x) := U(x) - \bar{U}, \quad \forall \tau \in [0,1], \tag{2.3}$$

where U_δ can be expressed by

$$U_\delta = \left(n_1^\delta, \tilde{j}_1^\tau, n_2^\delta, \tilde{j}_2^\tau, \tilde{\phi}^\tau\right)^T, \quad n_i^\delta := \tilde{n}_i^\tau - n_{il}.$$

From $\int_0^1$ (1.11b)$/\tilde{n}_i^\tau dx$ and the subsonic condition (1.10), if $\delta \ll 1$, then we can obtain the explicit formula of $\tilde{j}_i^\tau$ in terms of $\tilde{n}_i^\tau = n_i^\delta + n_{il}$:

$$\tilde{j}_i^\tau = J_i[\tilde{n}_i^\tau] := 2B_{ib}\left(\int_0^1 (\tilde{n}_i^\tau)^{-1}dx + \sqrt{\left(\int_0^1 (\tilde{n}_i^\tau)^{-1}dx\right)^2 + 2\tau^2 B_{ib}\left(n_{ir}^{-2} - n_{il}^{-2}\right)}\right)^{-1}, \tag{2.4a}$$

where

$$B_{ib} := (-1)^{i-1}\phi_r - K_i(\ln n_{ir} - \ln n_{il}), \quad i = 1, 2. \tag{2.4b}$$

In addition, solving the BVP (1.11c) and (1.12b) directly yields the explicit formula of $\tilde{\phi}^\tau$ in terms of both $\tilde{n}_1^\tau$ and $\tilde{n}_2^\tau$:

$$\tilde{\phi}^\tau(x) = \Phi[\tilde{n}_1^\tau, \tilde{n}_2^\tau](x) := \int_0^x\int_0^y (\tilde{n}_1^\tau - \tilde{n}_2^\tau - D)(z)dzdy + \left(\phi_r - \int_0^1\int_0^y (\tilde{n}_1^\tau - \tilde{n}_2^\tau - D)(z)dzdy\right)x. \tag{2.5}$$

Based on the explicit formulas (2.4) and (2.5), one can see that once we solve the stationary perturbation densities n_1^δ and n_2^δ, then we can construct the original solution $U = \left(\tilde{n}_1^\tau, \tilde{j}_1^\tau, \tilde{n}_2^\tau, \tilde{j}_2^\tau, \tilde{\phi}^\tau\right)$ directly.

Now we are in the position to solve the stationary perturbation densities n_1^δ and n_2^δ. For simplicity, we adopt the notation

$$W_\delta(x) = \left(n_1^\delta, n_2^\delta\right)^T(x).$$

Then we derive the boundary value problem satisfied by W_δ. In fact, by $\partial_x\left((1.11b)/\tilde{n}_i^\tau\right)$ together with (1.11c), we obtain the equations of $(\tilde{n}_1^\tau, \tilde{n}_2^\tau)$. Next, linearizing the resultant equations around the constant state (n_{1l}, n_{2l}) and noting the B.C. (1.12a), we have

$$\begin{cases} AW_{\delta xx} + BW_\delta = F(W_\delta) + G(x), & x \in \Omega, \quad \text{(a)} \\ W_\delta|_{\partial\Omega} = H(x), & \quad\text{(b)} \end{cases} \tag{2.6}$$

where

$$A := \begin{pmatrix} \frac{K_1}{n_{1l}} & 0 \\ 0 & \frac{K_2}{n_{2l}} \end{pmatrix}, \quad B := \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.7}$$

$$F(W_\delta) := \Big(f_1(W_\delta), f_2(W_\delta)\Big)^T, \quad G(x) = \Big(-(D(x) - \bar{d}), D(x) - \bar{d}\Big)^T, \tag{2.8}$$

$$H(x) = \Big(h_1, h_2\Big)^T(x), \quad h_i(x) = (n_{ir} - n_{il})x, \tag{2.9}$$

$$\begin{aligned} f_i(W_\delta) = \tilde{j}_i^\tau n_{ix}^\delta(\tilde{n}_i^\tau)^{-2} &- \Big[K_i\big((\tilde{n}_i^\tau)^{-1} - n_{il}^{-1}\big) - \tau^2(\tilde{j}_i^\tau)^2(\tilde{n}_i^\tau)^{-3}\Big]n_{ixx}^\delta \\ &- \Big[2\tau^2(\tilde{j}_i^\tau)^2(\tilde{n}_i^\tau)^{-4} - \tilde{S}_i^\tau(\tilde{n}_i^\tau)^{-2}\Big](n_{ix}^\delta)^2, \quad i = 1, 2, \end{aligned} \tag{2.10}$$

$$\tilde{j}_i^\tau = J_i[n_i^\delta + n_{il}], \quad i = 1, 2. \tag{2.11}$$

Let $\lambda := \min\{K_1/n_{1l}, K_2/n_{2l}\} > 0$, then for $\forall\xi \in \mathbb{R}$, we have

$$\xi^T A\xi \geq \lambda|\xi|^2. \tag{2.12}$$

This means that the BVP (2.6) is the Dirichlet BVP of a semilinear strongly elliptic system of seconder order. For classical solutions, the BVP (2.6) together with the explicit formulas (2.4) and (2.5) is equivalent to the original BVP (1.11) and (1.12).

Step II. Banach fixed point argument.

In this step, we use the Banach fixed point theorem to uniquely solve the BVP (2.6). To this end, we first consider the corresponding linear problem

$$\begin{cases} AW_{xx} + BW = R(x), \quad x \in \Omega, & \text{(a)} \\ W|_{\partial\Omega} = H(x). & \text{(b)} \end{cases} \tag{2.13}$$

From the standard L^2-theory of strongly elliptic system: Fredholm alternative (uniqueness implies existence), we find that the linear BVP (2.13) is uniquely solvable and the corresponding strong solution $W \in H^3(\Omega)$ satisfies the elliptic estimate

$$\|W\|_3 \leq C(\|R\|_1 + \|H\|_3), \tag{2.14}$$

provided $R \in H^1(\Omega), H \in H^3(\Omega)$. Since the small parameter τ does not appear in the linear principal part $AW_{xx} + BW$, the elliptic estimate constant $C > 0$ in (2.14) is independent of $\tau \in [0, 1]$.

Based on the structure of the nonlinearity (2.10) and the elliptic estimate (2.14), we introduce a metric space

$$\mathbb{W}[N] := \left\{ W \in H^3(\Omega) \mid \|W\|_3 \leq N\delta, \quad W|_{\partial\Omega} = H \right\} \tag{2.15}$$

equipped with the metric associated with the norm $\|\cdot\|_3$, which will be used in the following Banach fixed point argument. Here the positive constant N will be determined later. In fact, it follows from the trace theorem that $\mathbb{W}[N]$ is a closed subspace of $H^3(\Omega)$ for any $N > 0$ and $\delta \geq 0$. Thus, $\mathbb{W}[N]$ is a complete metric space.

Next, for all $V = \left(m_1^\delta, m_2^\delta\right)^T \in \mathbb{W}[N]$, let $\tilde{k}_i^\tau := J_i[m_i^\delta + n_{il}]$, we have $F(V) \in H^1(\Omega)$ by (2.8). Moreover, let $R := F(V) + G$, one can easily see that $R \in H^1(\Omega)$ if $G \in H^1(\Omega)$. Then we can define a fixed point mapping $S : \mathbb{W}[N] \to H^3(\Omega),\ V \mapsto W =: SV$ by solving the linearized BVP

$$\begin{cases} AW_{xx} + BW = F(V) + G, \quad x \in \Omega, & \text{(a)} \\ W|_{\partial\Omega} = H(x), \quad \forall V \in \mathbb{W}[N]. & \text{(b)} \end{cases} \tag{2.16}$$

Now we tend to determine the positive constant N to ensure that the mapping S is a contraction mapping on $\mathbb{W}[N]$ if $\delta \ll 1$. To this end, we separately show that S is onto and contractive below.

S maps $\mathbb{W}[N]$ into itself: From the definition of the mapping S and the elliptic estimate (2.14), we have

$$\begin{aligned} \|SV\|_3 &\leq C(\|F(V) + G\|_1 + \|H\|_3) \\ &\leq (C_1(N)\delta + C_2)\delta, \end{aligned} \tag{2.17}$$

where we have used the a priori assumption $N\delta \ll 1$ and the estimate of the nonlocal factor $\tilde{k}_i^\tau$ in the nonlinear term $F(V)$

$$|\tilde{k}_i^\tau| = |J_i[m_i^\delta + n_{il}]| \leq C\delta, \quad V = (m_1^\delta, m_2^\delta) \in \mathbb{W}[N].$$

Define

$$N := 2C_2 > 0. \tag{2.18}$$

If

$$\delta \leq C_2/(C_1(2C_2)),$$

then

$$\|SV\|_3 \leq 2C_2\delta = N\delta. \tag{2.19}$$

Thus, S maps $\mathbb{W}[2C_2]$ into itself.

S is contractive in $\mathbb{W}[2C_2]$: For arbitrary $V_1, V_2 \in \mathbb{W}[2C_2]$, we need to estimate $W := SV_1 - SV_2$. To this end, let $R := F(V_1) - F(V_2)$, by definition of the mapping S we know that W satisfies the following BVP

$$\begin{cases} AW_{xx} + BW = R, \quad x \in \Omega, & \text{(a)} \\ W|_{\partial\Omega} = 0. & \text{(b)} \end{cases} \tag{2.20}$$

From the elliptic estimate (2.14), we obtain

$$\begin{aligned} \|SV_1 - SV_2\|_3 &\leq C\|F(V_1) - F(V_2)\|_1 \\ &\leq C_3\delta\|V_1 - V_2\|_3, \\ &\leq \frac{1}{2}\|V_1 - V_2\|_3, \quad \forall V_1, V_2 \in \mathbb{W}[2C_2], \end{aligned} \tag{2.21}$$

where we used the estimate

$$|\tilde{k}_{i1}^\tau - \tilde{k}_{i2}^\tau| = |J_i[m_{i1}^\delta + n_{il}] - J_i[m_{i2}^\delta + n_{il}]| \leq C\delta\|m_{i1}^\delta - m_{i2}^\delta\|_1,$$

for any $V_1 = (m_{11}^\delta, m_{21}^\delta)^T, V_2 = (m_{12}^\delta, m_{22}^\delta)^T \in \mathbb{W}[2C_2]$ and the smallness assumption on the strength parameter $\delta \leq 1/(2C_3)$. Thus, S is a contraction mapping in the complete metric space $\mathbb{W}[2C_2]$.

According to the Banach fixed point theorem, we obtain an unique fixed point $W = \left(n_1^\delta, n_2^\delta\right)^T \in \mathbb{W}[2C_2]$ of the mapping S. By the definition of S, the fixed point W just is the unique solution to BVP (2.6) in $\mathbb{W}[2C_2]$. Therefore, it satisfies the estimate

$$\sum_{i=1}^{2} \|n_i^\delta\|_3 \leq C\delta, \quad \forall \tau \in [0,1], \tag{2.22}$$

where the constant $C > 0$ is independent of δ and τ.

Apparently, $\tilde{n}_i^\tau := n_i^\delta + n_{il}$, $\tilde{j}_i^\tau := J_i[\tilde{n}_i^\tau]$ and $\tilde{\phi}^\tau := \Phi[\tilde{n}_1^\tau, \tilde{n}_2^\tau]$ is the desired solution to the original BVP (1.11) and (1.12), satisfying the condition (1.13) and the estimate (1.15). □

3. Asymptotic stability of the stationary solution

In this section, we consider the asymptotic stability of the subsonic stationary solution $(\tilde{n}_1^\tau, \tilde{j}_1^\tau, \tilde{n}_2^\tau, \tilde{j}_2^\tau, \tilde{\phi}^\tau)$ constructed in Theorem 1.1. Note that $\tau = 0$ is corresponding to the DD-IBVP (1.7)–(1.9) which is of the parabolic–elliptic type. However, $0 < \tau \leq 1$ is corresponding to the HD-IBVP (1.3)–(1.5) which is of the hyperbolic–elliptic type. Due to the essential difference between the system types, we have to establish the stability results separately for $\tau = 0$ and $0 < \tau \leq 1$.

3.1. The DD-IBVP ($\tau = 0$)

In this subsection, we prove the Theorem 1.2. It is worth mentioning that we have to treat a parabolic system rather than a parabolic scalar equation like unipolar case. In our case, there is no maximum principle that can be used to establish the positive lower bound for the carrier density $n_i^0(t,x)$. Thus, we can only obtain the global existence around the stationary solution for bipolar DD model with non-constant doping profile.

Proof of Theorem 1.2.

Step I. Local existence and reformulation.

By a standard iteration scheme and energy method, it is shown that there exists a positive constant T_0 such that the DD-IBVP (1.7)–(1.9) has a unique local solution $(n_1^0, j_1^0, n_2^0, j_2^0, \phi^0) \in \mathfrak{Y}([0,T_0])$.

Next, in order to construct the global solution, we introduce the time-dependent perturbation variables

$$\psi_i^0 := n_i^0 - \tilde{n}_i^0, \quad \eta_i^0 := j_i^0 - \tilde{j}_i^0, \quad \sigma^0 := \phi^0 - \tilde{\phi}^0. \tag{3.1}$$

Then the original DD-IBVP (1.7)–(1.9) is equivalently reformulated as

$$\begin{cases} \psi_{it}^0 - K_i \psi_{ixx}^0 + (-1)^{i-1}(n_i^0\phi_x^0 - \tilde{n}_i^0\tilde{\phi}_x^0)_x = 0, & \text{(a)} \\ \eta_i^0 = (-1)^{i-1}(n_i^0\phi_x^0 - \tilde{n}_i^0\tilde{\phi}_x^0) - K_i\psi_{ix}^0, & \text{(b)} \\ \sigma_{xx}^0 = \psi_1^0 - \psi_2^0, & \text{(c)} \end{cases} \tag{3.2}$$

$$\psi_i^0(0,x) = \psi_{i0}^0(x) := n_{i0}(x) - \tilde{n}_i^0(x), \tag{3.3}$$

$$\psi_i^0(t,0) = \psi_i^0(t,1) = 0, \quad \sigma^0(t,0) = \sigma^0(t,1) = 0, \quad i = 1,2. \tag{3.4}$$

Combining the regularity of the stationary solution and the local existence result above, we immediately obtain the unique local solution $(\psi_1^0, \eta_1^0, \psi_2^0, \eta_2^0, \sigma^0)$ to the perturbation IBVP (3.2)–(3.4) in the same function space $\mathfrak{Y}([0,T_0])$.

The global solution can be constructed by the continuation argument based on the above local existence result and the a priori estimate. To establish the a priori estimate is crucial, and this will be our aim in the next step.

Step II. A priori estimate.

We first make an a priori assumption

$$N_0(T) := \sup_{t\in[0,T]} \sum_{i=1}^{2} \|\psi_i^0(t)\|_2 \ll 1. \tag{3.5}$$

Under the assumption (3.5), we can establish the a priori estimate for the local solution $(\psi_1^0, \eta_1^0, \psi_2^0, \eta_2^0, \sigma^0)(t,x)$ on $[0,T]$ as follows:

$$\sum_{i=1}^{2} \left(\|\psi_i^0(t)\|_2 + \|\eta_i^0(t)\|_1 \right) + \|\sigma^0(t)\|_4 \le C \sum_{i=1}^{2} \|\psi_{i0}^0\|_2 \, e^{-\gamma_1 t}, \quad \forall t \in [0,T]. \tag{3.6}$$

In fact, by (3.2b), (3.2c), (3.4), (1.15) and (3.5), we have

$$\sum_{i=1}^{2} \|\eta_i^0(t)\|_1 + \|\sigma^0(t)\|_4 \le C \sum_{i=1}^{2} \|\psi_i^0(t)\|_2. \tag{3.7}$$

From Sobolev embedding theorem and (3.5), we obtain

$$\sum_{i=1}^{2}\left(|\psi_i^0(t)|_1+|\eta_i^0(t)|_0\right)+|\sigma^0(t)|_3\le CN_0(T). \tag{3.8}$$

By (1.15), (3.5) and equation (3.2a), we have

$$\|\psi_{it}^0(t)\|^2\le C\sum_{i=1}^{2}\|\psi_i^0(t)\|_2^2,\quad i=1,2. \tag{3.9}$$

Performing the procedure (3.2a)/$\tilde{n}_i^0$ yields the working equation

$$\frac{1}{\tilde{n}_i^0}\psi_{it}^0-\frac{K_i}{\tilde{n}_i^0}\psi_{ixx}^0+(-1)^{i-1}\sigma_{xx}^0+\frac{(-1)^{i-1}}{\tilde{n}_i^0}(\phi_{xx}^0\psi_i^0+\phi_x^0\psi_{ix}^0+\tilde{n}_{ix}^0\sigma_x^0)=0. \tag{3.10}$$

Actually, by the following procedures

$$\int_0^1\sum_{i=1}^{2}(3.10)\times(\psi_i^0-\psi_{ixx}^0-\psi_{ixxt}^0)dx, \tag{3.11}$$

together with the smallness condition $N_0(T)+\delta\ll 1$, we can obtain the desired estimate (3.6). Due to the complexity of the calculation, we will check (3.11) step by step in the sequel.

Firstly, by

$$\int_0^1\sum_{i=1}^{2}(3.10)\times\psi_i^0dx,$$

we obtain

$$\int_0^1\sum_{i=1}^{2}(\tilde{n}_i^0)^{-1}\psi_{it}^0\psi_i^0dx-\int_0^1\sum_{i=1}^{2}K_i(\tilde{n}_i^0)^{-1}\psi_{ixx}^0\psi_i^0dx+\int_0^1\sum_{i=1}^{2}(-1)^{i-1}\sigma_{xx}^0\psi_i^0dx$$
$$+\int_0^1\sum_{i=1}^{2}(-1)^{i-1}(\tilde{n}_i^0)^{-1}(\phi_{xx}^0\psi_i^0+\phi_x^0\psi_{ix}^0+\tilde{n}_{ix}^0\sigma_x^0)\psi_i^0dx=0, \tag{3.12}$$

after integration by parts together with (1.15), (3.8) and Poincaré inequality, we have the following estimate if $N_0(T)+\delta\ll 1$,

$$\frac{d}{dt}\int_0^1\sum_{i=1}^{2}\frac{(\psi_i^0)^2}{2\tilde{n}_i^0}dx+C_1\sum_{i=1}^{2}\|\psi_i^0\|_1^2\le 0, \tag{3.13}$$

where we have used the following estimate for the third term (bipolar effect) in the left side of the equation (3.12),

$$\int_0^1\sum_{i=1}^{2}(-1)^{i-1}\sigma_{xx}^0\psi_i^0dx=\int_0^1(\psi_1^0-\psi_2^0)^2dx\ge 0. \tag{3.14}$$

Secondly, from

$$\int_0^1 \sum_{i=1}^2 (3.10) \times (-\psi_{ixx}^0) dx,$$

we have

$$-\int_0^1 \sum_{i=1}^2 (\tilde{n}_i^0)^{-1} \psi_{it}^0 \psi_{ixx}^0 dx + \int_0^1 \sum_{i=1}^2 K_i (\tilde{n}_i^0)^{-1} (\psi_{ixx}^0)^2 dx + \int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xx}^0 \psi_{ixx}^0 dx$$
$$+ \int_0^1 \sum_{i=1}^2 (-1)^i (\tilde{n}_i^0)^{-1} (\phi_{xx}^0 \psi_i^0 + \phi_x^0 \psi_{ix}^0 + \tilde{n}_{ix}^0 \sigma_x^0) \psi_{ixx}^0 dx = 0. \tag{3.15}$$

Similarly, we get

$$\frac{d}{dt} \int_0^1 \sum_{i=1}^2 \frac{(\psi_{ix}^0)^2}{2\tilde{n}_i^0} dx + C_2 \sum_{i=1}^2 \|\psi_{ixx}^0\|^2 \le C(N_0(T) + \delta) \sum_{i=1}^2 \|\psi_i^0\|_2^2, \tag{3.16}$$

where the bipolar effect term in the left side of the equation (3.15) has been treated as follows

$$\int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xx}^0 \psi_{ixx}^0 dx = \int_0^1 (\psi_{1x}^0 - \psi_{2x}^0)^2 dx \ge 0. \tag{3.17}$$

Thirdly, by

$$\int_0^1 \sum_{i=1}^2 (3.10) \times (-\psi_{ixxt}^0) dx,$$

we have

$$-\int_0^1 \sum_{i=1}^2 (\tilde{n}_i^0)^{-1} \psi_{it}^0 \psi_{ixxt}^0 dx + \int_0^1 \sum_{i=1}^2 K_i (\tilde{n}_i^0)^{-1} \psi_{ixx}^0 \psi_{ixxt}^0 dx + \int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xx}^0 \psi_{ixxt}^0 dx$$
$$+ \int_0^1 \sum_{i=1}^2 (-1)^i (\tilde{n}_i^0)^{-1} (\phi_{xx}^0 \psi_i^0 + \phi_x^0 \psi_{ix}^0 + \tilde{n}_{ix}^0 \phi_x^0) \psi_{ixxt}^0 dx = 0, \tag{3.18}$$

and then, by a similar way, we obtain

$$\frac{d}{dt} \int_0^1 \sum_{i=1}^2 \left\{ \left[\frac{K_i}{2\tilde{n}_i^0} (\psi_{ixx}^0)^2 + \frac{(-1)^i}{\tilde{n}_i^0} (\phi_{xx}^0 \psi_i^0 + \phi_x^0 \psi_{ix}^0 + \tilde{n}_{ix}^0 \sigma_x^0) \psi_{ixx}^0 \right] + \frac{1}{2} (\psi_{1x}^0 - \psi_{2x}^0)^2 \right\} dx$$
$$+ C_3 \sum_{i=1}^2 \|\psi_{ixt}^0\|^2 \le C(N_0(T) + \delta) \sum_{i=1}^2 \|\psi_i^0\|_2^2, \tag{3.19}$$

where we have used the following calculation for bipolar effect term,

$$\int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xx}^0 \psi_{ixxt}^0 dx = \frac{d}{dt} \int_0^1 \frac{1}{2} (\psi_{1x}^0 - \psi_{2x}^0)^2 dx. \tag{3.20}$$

Finally, by

$$(3.13) + (3.16) + (3.19)$$

we get

$$\frac{d}{dt} E^0(t) + C_4 \sum_{i=1}^2 \|\psi_i^0(t)\|_2^2 + C_3 \sum_{i=1}^2 \|\psi_{ixt}^0(t)\|^2 \leq 0, \quad \forall t \in [0, T], \tag{3.21}$$

where

$$E^0(t) := \int_0^1 \left\{ \sum_{i=1}^2 \left[\frac{1}{2\tilde{n}_i^0} \Big((\psi_i^0)^2 + (\psi_{ix}^0)^2 + K_i (\psi_{ixx}^0)^2 \Big) + \frac{(-1)^i}{\tilde{n}_i^0} (\phi_{xx}^0 \psi_i^0 + \phi_x^0 \psi_{ix}^0 + \tilde{n}_{ix}^0 \sigma_x^0) \psi_{ixx}^0 \right] + \frac{1}{2} (\psi_{1x}^0 - \psi_{2x}^0)^2 \right\} dx. \tag{3.22}$$

Noting that $C_3 \sum_{i=1}^2 \|\psi_{ixt}^0\|^2 \geq 0$ in (3.21), we have

$$\frac{d}{dt} E^0(t) + C_4 \sum_{i=1}^2 \|\psi_i^0(t)\|_2^2 \leq 0, \quad \forall t \in [0, T]. \tag{3.23}$$

From the definition (3.22) of $E^0(t)$ we know that if $N_0(T) + \delta \ll 1$, then there exist constants $C_{5l}, C_{5r} > 0$ such that

$$C_{5l} \sum_{i=1}^2 \|\psi_i^0(t)\|_2^2 \leq E^0(t) \leq C_{5r} \sum_{i=1}^2 \|\psi_i^0(t)\|_2^2, \quad \forall t \in [0, T]. \tag{3.24}$$

Let

$$\gamma_1 := \frac{C_4}{2C_{5r}} > 0.$$

Then, by (3.23) and (3.24), we have

$$\frac{d}{dt} E^0(t) + 2\gamma_1 E^0(t) \leq 0, \quad \forall t \in [0, T]. \tag{3.25}$$

Applying the Gronwall inequality to (3.25) and using (3.24) again, we obtain

$$\sum_{i=1}^2 \|\psi_i^0(t)\|_2 \leq C_6 \sum_{i=1}^2 \|\psi_{i0}^0\|_2 \, e^{-\gamma_1 t}, \quad \forall t \in [0, T]. \tag{3.26}$$

Combining (3.26) with (3.7), we arrive at the estimate (3.6).

Furthermore, for global solution, by

$$\int_0^t (3.21) ds, \quad \forall t \in [0, +\infty),$$

we have

$$\sum_{i=1}^{2} \|\psi_i^0(t)\|_2^2 + \int_0^t \sum_{i=1}^{2} \|\psi_{ixt}^0(s)\|^2 ds \le C \sum_{i=1}^{2} \|\psi_{i0}^0\|_2^2, \quad \forall t \in [0, +\infty). \tag{3.27}$$

Now, from

$$\int_0^t \int_0^1 \sum_{i=1}^{2} \partial_t (3.2\text{a}) \times s(\psi_{itt}^0 - \psi_{ixxt}^0) dx ds,$$

after a straightforward computation and by using (3.26) and (3.27), we obtain

$$\int_0^t s \sum_{i=1}^{2} \|(\psi_{itt}^0, \psi_{ixxt}^0)(s)\|^2 ds \le C \sum_{i=1}^{2} \|\psi_{i0}^0\|_2^2 (1+t), \quad \forall t \in [0, +\infty). \qquad \square \tag{3.28}$$

3.2. The HD-IBVP ($0 < \tau \le 1$)

In this subsection, we prove the Theorem 1.3. The key ingredient of the proof is to introduce a τ-weighted norm and establish the uniform a priori estimate in both time variable t and relaxation time $\tau \in (0, 1]$. This ensures us to further study the relaxation limit of the global solution.

Proof of Theorem 1.3.

Step I. Reformulation and local existence.

We first introduce the time-dependent perturbation variables

$$\psi_i^\tau := n_i^\tau - \tilde{n}_i^\tau, \quad \eta_i^\tau := j_i^\tau - \tilde{j}_i^\tau, \quad \sigma^\tau := \phi^\tau - \tilde{\phi}^\tau, \quad \forall \tau \in (0, 1]. \tag{3.29}$$

By

$$(1.3\text{a}) - (1.11\text{a}), \quad (1.3\text{b})/n_i^\tau - (1.11\text{b})/\tilde{n}_i^\tau, \quad (1.3\text{c}) - (1.11\text{c}),$$

and initial–boundary conditions (1.4), (1.5) and (1.12), the original HD-IBVP (1.3)–(1.5) can be equivalently reformulated into the following perturbation IBVP:

$$\begin{cases} \psi_{it}^\tau + \eta_{ix}^\tau = 0, & \text{(a)} \\ \tau^2 \left(\frac{\eta_i^\tau + \tilde{j}_i^\tau}{\psi_i^\tau + \tilde{n}_i^\tau} \right)_t + \frac{\tau^2}{2} \left[\frac{(\eta_i^\tau + \tilde{j}_i^\tau)^2}{(\psi_i^\tau + \tilde{n}_i^\tau)^2} - \frac{(\tilde{j}_i^\tau)^2}{(\tilde{n}_i^\tau)^2} \right]_x & \\ \qquad + K_i \Big[\ln \left(\psi_i^\tau + \tilde{n}_i^\tau \right) - \ln \tilde{n}_i^\tau \Big]_x + (-1)^i \sigma_x + \frac{\eta_i^\tau + \tilde{j}_i^\tau}{\psi_i^\tau + \tilde{n}_i^\tau} - \frac{\tilde{j}_i^\tau}{\tilde{n}_i^\tau} = 0, & \text{(b)} \\ \sigma_{xx}^\tau = \psi_1^\tau - \psi_2^\tau, \quad i = 1, 2, & \text{(c)} \end{cases} \tag{3.30}$$

$$(\psi_i^\tau, \eta_i^\tau)(0, x) = (\psi_{i0}^\tau, \eta_{i0}^\tau)(x) := (n_{i0} - \tilde{n}_i^\tau, j_{i0} - \tilde{j}_i^\tau)(x), \tag{3.31}$$

$$\psi_i^\tau(t, 0) = \psi_i^\tau(t, 1) = 0, \quad \sigma^\tau(t, 0) = \sigma^\tau(t, 1) = 0. \tag{3.32}$$

By the standard iteration scheme and energy method, we can establish the local existence result for IBVP (3.30)–(3.32): If the initial data $\psi_{i0}^\tau, \eta_{i0}^\tau \in H^2(\Omega)$ and $\psi_{i0}^\tau + \tilde{n}_i^\tau$, $\eta_{i0}^\tau + \tilde{j}_i^\tau$ satisfy (1.6) and (1.10), then for $\forall \tau \in (0,1]$ there exists a positive constant $T_\tau > 0$ such that the IBVP (3.30)–(3.32) has a unique solution $(\psi_i^\tau, \eta_i^\tau, \sigma^\tau) \in \big(\mathfrak{X}_2 \times \mathfrak{X}_2 \times \mathfrak{X}_2^2\big)([0,T_\tau])$ and $\psi_i^\tau + \tilde{n}_i^\tau$, $\eta_i^\tau + \tilde{j}_i^\tau$ satisfy (1.10).

The uniformly global solution in $\tau \in (0,1]$ can be constructed by the continuation argument based on the above local existence result and the uniform a priori estimate under the appropriate τ-weighted norm (see (1.17b)). To establish the a priori estimate is crucial, and this will be our aim in the next steps.

Before establishing the desired estimate, for arbitrarily fixed $\tau \in (0,1]$, we introduce an a priori assumption

$$N_\tau(T) := \sup_{t\in[0,T]} n_\tau(t) \ll 1, \tag{3.33}$$

where the τ-weighted norm $n_\tau(t)$ is defined as

$$n_\tau(t) := \sum_{i=1}^{2} \Big(\|\psi_i^\tau(t)\|_2 + \|\eta_i^\tau(t)\|_1 + \|\tau\eta_{ixx}^\tau(t)\| \Big). \tag{3.34}$$

From (3.30c) and (3.32), we obtain the elliptic estimate

$$\|\sigma^\tau(t)\|_4 \le C \sum_{i=1}^{2} \|\psi_i^\tau(t)\|_2. \tag{3.35}$$

By (3.30a), we get

$$\|\eta_{ixx}^\tau(t)\| = \|\psi_{ixt}^\tau(t)\|, \quad \|\partial_t^k \eta_{ix}^\tau(t)\| = \|\partial_t^k \psi_{it}^\tau(t)\|, \quad k = 0,1. \tag{3.36}$$

From (3.30b), (3.35), (3.33) and Sobolev embedding theorem, we have

$$\sum_{i=1}^{2} |(\psi_i^\tau, \eta_i^\tau, \tau\psi_{it}^\tau, \tau\eta_{ix}^\tau, \tau^2\eta_{it}^\tau)(t)|_0 + |\sigma^\tau(t)|_3 \le C N_\tau(T), \tag{3.37}$$

$$\|\tau^2\eta_{it}^\tau(t)\| \le C \sum_{i=1}^{2} \Big(\|\psi_i^\tau(t)\|_1 + \|\eta_i^\tau(t)\| \Big) + C(N_\tau(T)+\delta)\|\tau^2\psi_{it}^\tau(t)\|, \quad \forall t \in [0,T], \tag{3.38}$$

where the generic constant $C > 0$ is independent of t and τ. These estimates will be frequently used to establish the basic, higher order and decay estimates in what follows.

Step II. Basic estimate.

Performing the procedure

$$\int_0^1 \sum_{i=1}^{2} (3.30b) \times \eta_i^\tau \, dx,$$

we can obtain the desired basic estimate. Precisely, by $\sum_{i=1}^{2}$ (3.30b) $\times\, \eta_i^\tau$, we have

$$\mathcal{E}_t + \sum_{i=1}^{2} \frac{(\eta_i^\tau)^2}{\tilde{n}_i^\tau} = R_{1x} + R_2, \tag{3.39}$$

where

$$\mathcal{E}(t,x) := \frac{1}{2}(\sigma_x^\tau)^2 + \sum_{i=1}^{2}\left(\frac{\tau^2}{2n_i^\tau}(\eta_i^\tau)^2 + K_i n_i^\tau \Psi\Big(\frac{\tilde{n}_i^\tau}{n_i^\tau}\Big)\right), \quad \Psi(s) := s - 1 - \ln s,$$

$$R_1 := \sigma^\tau \sigma_{xt}^\tau + \sigma^\tau(\eta_1^\tau - \eta_2^\tau) - \sum_{i=1}^{2} K_i(\ln n_i^\tau - \ln \tilde{n}_i^\tau)\eta_i^\tau,$$

$$R_2 := -\sum_{i=1}^{2}\left[\frac{\tau^2(\eta_i^\tau + 2\tilde{j}_i^\tau)}{2(n_i^\tau)^2}\eta_{ix}^\tau\eta_i^\tau + \frac{\tau^2}{2}\left(\frac{(j_i^\tau)^2}{(n_i^\tau)^2} - \frac{(\tilde{j}_i^\tau)^2}{(\tilde{n}_i^\tau)^2}\right)_x \eta_i^\tau + j_i^\tau\left(\frac{1}{n_i^\tau} - \frac{1}{\tilde{n}_i^\tau}\right)\eta_i^\tau\right].$$

Next, by

$$\int_0^1 (3.39)dx$$

and by using the fact that $\int_0^1 R_{1x}dx = 0$, we get

$$\frac{d}{dt}\int_0^1 \mathcal{E}(t,x)dx + \int_0^1 \sum_{i=1}^{2}\frac{(\eta_i^\tau)^2}{\tilde{n}_i^\tau}dx = \int_0^1 R_2 dx. \tag{3.40}$$

Furthermore, if $N_\tau(T) + \delta \ll 1$, then the following estimates hold.

$$\left|\int_0^1 R_2 dx\right| \le C(N_\tau(T) + \delta)\sum_{i=1}^{2}\|(\psi_i^\tau, \eta_i^\tau)(t)\|_1^2, \tag{3.41}$$

$$C_{7l}\sum_{i=1}^{2}\|(\psi_i^\tau, \tau\eta_i^\tau)(t)\|^2 \le \int_0^1 \mathcal{E}(t,x)dx \le C_{7r}\sum_{i=1}^{2}\|(\psi_i^\tau, \tau\eta_i^\tau)(t)\|^2. \tag{3.42}$$

Step III. Higher order estimates.

From

$$-\partial_t^k\Big[\partial_x(1.3\text{b})/n_i^\tau - \partial_x(1.11\text{b})/\tilde{n}_i^\tau\Big], \quad k = 0, 1,$$

we get the working equations used to establish the higher order estimates:

$$(n_i^\tau)^{-1}\tau^2\partial_t^k\psi_{itt}^\tau - \Big[\Big(K_i(n_i^\tau)^{-1} - \tau^2(j_i^\tau)^2(n_i^\tau)^{-3}\Big)\partial_t^k\psi_{ix}^\tau\Big]_x + (-1)^{i+1}\partial_t^k(\psi_1^\tau - \psi_2^\tau) \\ + (n_i^\tau)^{-1}\partial_t^k\psi_{it}^\tau = -\tau^2 2j_i^\tau(n_i^\tau)^{-2}\partial_t^k\psi_{ixt}^\tau + \partial_t^k F_i + L_{ik}, \tag{3.43}$$

where

$$F_i := \tau^2\Big[2(\psi_{it}^\tau)^2(n_i^\tau)^{-2} + 4j_i^\tau(n_i^\tau)^{-3}n_{ix}^\tau\psi_{it}^\tau + 2(j_i^\tau)^2(n_i^\tau)^{-4}(2\tilde{n}_{ix}^\tau + \psi_{ix}^\tau)\psi_{ix}^\tau \\ + 2(2\tilde{j}_i^\tau + \eta_i^\tau)(n_i^\tau)^{-4}\tilde{n}_{ix}^\tau\eta_i^\tau + 2(\tilde{j}_i^\tau)^2(\tilde{n}_{ix}^\tau)^2((n_i^\tau)^{-4} - (\tilde{n}_i^\tau)^{-4}) \\ + ((j_i^\tau)^2(n_i^\tau)^{-3})_x\psi_{ix}^\tau - (2\tilde{j}_i^\tau + \eta_i^\tau)(n_i^\tau)^{-3}\tilde{n}_{ixx}^\tau\eta_i^\tau - (\tilde{j}_i^\tau)^2\tilde{n}_{ixx}^\tau((n_i^\tau)^{-3} - (\tilde{n}_i^\tau)^{-3})\Big]$$

$$+ K_i(n_i^\tau)^{-2} n_{ix}^\tau \psi_{ix}^\tau - K_i \tilde{n}_{ixx}^\tau (\tilde{n}_i^\tau n_i^\tau)^{-1} \psi_i^\tau$$
$$+ (-1)^{i+1} \Big[\tilde{\phi}_x^\tau \tilde{n}_{ix}^\tau (\tilde{n}_i^\tau n_i^\tau)^{-1} \psi_i^\tau - \tilde{n}_{ix}^\tau (n_i^\tau)^{-1} \sigma_x^\tau - \phi_x^\tau (n_i^\tau)^{-1} \psi_{ix}^\tau \Big], \tag{3.44a}$$

$$L_{i0} := 0, \quad L_{i1} := \tau^2 (n_i^\tau)^{-2} \psi_{it}^\tau \psi_{itt}^\tau + \Big[\Big(K_i (n_i^\tau)^{-1} - \tau^2 (j_i^\tau)^2 (n_i^\tau)^{-3} \Big)_t \psi_{ix}^\tau \Big]_x$$
$$+ (n_i^\tau)^{-2} (\psi_{it}^\tau)^2 - \tau^2 \Big(2 j_i^\tau (n_i^\tau)^{-2} \Big)_t \psi_{ixt}^\tau, \qquad i = 1, 2. \tag{3.44b}$$

By the estimates (3.35)–(3.38), we have

$$\|F_i\| \le C(N_\tau(T) + \delta) \sum_{i=1}^{2} \|(\psi_i^\tau, \psi_{ix}^\tau, \tau \psi_{it}^\tau, \tau^2 \eta_i^\tau)\|, \tag{3.45a}$$

$$\|F_{it}\| + \|L_{i1}\| \le C(N_\tau(T) + \delta) \sum_{i=1}^{2} \|(\tau \psi_{itt}^\tau, \psi_{ixt}^\tau, \psi_{it}^\tau, \psi_{ix}^\tau, \psi_i^\tau, \eta_i^\tau)\|, \quad \forall \tau \in (0, 1]. \tag{3.45b}$$

During establishing the higher order estimates, we need to use the homogeneous boundary conditions (3.32) to vanish the boundary terms arising from the integration by parts. To this end, we need to control the spatial derivatives by the time derivatives of the perturbation densities ψ_i^τ. Precisely, if $N_\tau(T) + \delta \ll 1$, we have

$$C_{8l} A(t) \le n_\tau^2(t) \le C_{8r} A(t), \quad \forall t \in [0, T], \tag{3.46}$$

where

$$A(t) := \sum_{i=1}^{2} \|(\tau^2 \psi_{itt}^\tau, \tau \psi_{ixt}^\tau, \psi_{it}^\tau, \psi_{ix}^\tau, \psi_i^\tau, \eta_i^\tau)(t)\|^2. \tag{3.47}$$

From the equation (3.43) with $k = 0$, we obtain the estimates

$$\|\psi_{ixx}^\tau\| \le C \sum_{i=1}^{2} \|(\psi_i^\tau, \psi_{ix}^\tau, \psi_{it}^\tau, \tau^2 \psi_{itt}^\tau, \tau^2 \psi_{ixt}^\tau, \tau^2 \eta_i^\tau)\|, \tag{3.48a}$$

$$\|\tau^2 \psi_{itt}^\tau\| \le C \sum_{i=1}^{2} \|(\psi_i^\tau, \psi_{ix}^\tau, \psi_{it}^\tau, \psi_{ixx}^\tau, \tau^2 \psi_{ixt}^\tau, \tau^2 \eta_i^\tau)\|, \tag{3.48b}$$

which imply the equivalent relation (3.46).

Actually, by the following procedures

$$\int_0^1 \sum_{i=1}^{2} (3.43) \times \big(\partial_t^k \psi_i^\tau + 2\tau^{2k} \partial_t^k \psi_{it}^\tau \big) dx, \quad k = 0, 1, \tag{3.49}$$

together with the smallness condition $N_\tau(T) + \delta \ll 1$, we can obtain the desired higher order estimates. Due to the complexity of the calculation, we will check (3.49) step by step in the sequel.

Firstly, by

$$\int_0^1 \sum_{i=1}^{2} (3.43) \times \partial_t^k \psi_i^\tau dx, \quad k = 0, 1,$$

after integration by parts, we have

$$\frac{d}{dt}I_1^{(k)}(t)+\int_0^1\left\{\sum_{i=1}^2\left[K_i(n_i^\tau)^{-1}(\partial_t^k\psi_{ix}^\tau)^2-\tau^2(n_i^\tau)^{-1}(\partial_t^k\psi_{it}^\tau)^2\right]+(\partial_t^k\psi_1^\tau-\partial_t^k\psi_2^\tau)^2\right\}dx=J_1^{(k)}(t),\tag{3.50}$$

where

$$I_1^{(k)}(t):=\int_0^1\sum_{i=1}^2\left[\tau^2(n_i^\tau)^{-1}\partial_t^k\psi_{it}^\tau\partial_t^k\psi_i^\tau+(2n_i^\tau)^{-1}(\partial_t^k\psi_i^\tau)^2\right]dx,\tag{3.51a}$$

$$\begin{aligned}J_1^{(k)}(t):=&-\int_0^1\sum_{i=1}^2\tau^2(n_i^\tau)^{-1}\psi_{it}^\tau\partial_t^k\psi_{it}^\tau\partial_t^k\psi_i^\tau dx\\&+\int_0^1\sum_{i=1}^2\tau^2\left[2\eta_{ix}^\tau(n_i^\tau)^{-2}\partial_t^k\psi_i^\tau-4j_i^\tau(n_i^\tau)^{-3}n_{ix}^\tau\partial_t^k\psi_i^\tau+2j_i^\tau(n_i^\tau)^{-2}\partial_t^k\psi_{ix}^\tau\right]\partial_t^k\psi_{it}^\tau dx\\&+\int_0^1\sum_{i=1}^2\tau^2(j_i^\tau)^2(n_i^\tau)^{-3}(\partial_t^k\psi_{ix}^\tau)^2dx-\int_0^1\sum_{i=1}^2\frac{1}{2}(n_i^\tau)^{-2}\psi_{it}^\tau(\partial_t^k\psi_i^\tau)^2dx\\&+\int_0^1\sum_{i=1}^2(\partial_t^kF_i+L_{ik})\partial_t^k\psi_i^\tau dx.\end{aligned}\tag{3.51b}$$

Furthermore, by the estimates (1.15), (3.35)–(3.38), (3.45) and Cauchy–Schwarz inequality, we get

$$|J_1^{(0)}|\le\mu\sum_{i=1}^2\|\psi_i^\tau\|^2+C_\mu(N_\tau(T)+\delta)\sum_{i=1}^2\|(\psi_{it}^\tau,\psi_{ix}^\tau,\eta_i^\tau)\|^2,\tag{3.52a}$$

$$|J_1^{(1)}|\le\mu\sum_{i=1}^2\|\psi_{it}^\tau\|^2+C_\mu(N_\tau(T)+\delta)\sum_{i=1}^2\|(\tau\psi_{itt}^\tau,\psi_{ixt}^\tau,\psi_{ix}^\tau,\psi_i^\tau,\eta_i^\tau)\|^2,\tag{3.52b}$$

where $0<\mu\ll1$ will be determined later.

Secondly, by

$$\int_0^1\sum_{i=1}^2(3.43)\times\tau^{2k}\partial_t^k\psi_{it}^\tau dx,\quad k=0,1,$$

after integration by parts, we have

$$\frac{d}{dt}I_2^{(k)}(t)+\int_0^1\sum_{i=1}^2\tau^{2k}(n_i^\tau)^{-1}(\partial_t^k\psi_{it}^\tau)^2dx=J_2^{(k)}(t),\tag{3.53}$$

where

$$\begin{aligned}I_2^{(k)}(t):=\int_0^1\sum_{i=1}^2\Bigg[&\frac{\tau^{2+2k}}{2}(n_i^\tau)^{-1}(\partial_t^k\psi_{it}^\tau)^2+\frac{\tau^{2k}}{2}\Big(K_i(n_i^\tau)^{-1}-\tau^2(j_i^\tau)^2(n_i^\tau)^{-3}\Big)(\partial_t^k\psi_{ix}^\tau)^2\\&+\frac{\tau^{2k}}{2}(\partial_t^k\psi_1^\tau-\partial_t^k\psi_2^\tau)^2\Bigg]dx,\end{aligned}\tag{3.54a}$$

$$
\begin{aligned}
J_2^{(k)}(t) := &-\int_0^1 \sum_{i=1}^2 \frac{\tau^{2+2k}}{2}(n_i^\tau)^{-2}\psi_{it}^\tau(\partial_t^k\psi_{it}^\tau)^2 dx \\
&-\int_0^1 \sum_{i=1}^2 \frac{\tau^{2k}}{2}K_i(n_i^\tau)^{-2}\psi_{it}^\tau(\partial_t^k\psi_{ix}^\tau)^2 dx \\
&-\int_0^1 \sum_{i=1}^2 \frac{\tau^{2+2k}}{2}\Big(2j_i^\tau\eta_{it}^\tau(n_i^\tau)^{-3} - 3(j_i^\tau)^2(n_i^\tau)^{-4}\psi_{it}^\tau\Big)(\partial_t^k\psi_{ix}^\tau)^2 dx \\
&+\int_0^1 \sum_{i=1}^2 \tau^{2+2k}\Big(\eta_{ix}^\tau(n_i^\tau)^{-2} - 2j_i^\tau(n_i^\tau)^{-3}n_{ix}^\tau\Big)(\partial_t^k\psi_{it}^\tau)^2 dx \\
&+\int_0^1 \sum_{i=1}^2 (\partial_t^k F_i + L_{ik})\tau^{2k}\partial_t^k\psi_{it}^\tau dx.
\end{aligned} \tag{3.54b}
$$

Moreover, by the estimates (1.15), (3.35)–(3.38), (3.45) and Cauchy–Schwarz inequality, we obtain

$$
|J_2^{(0)}| \le \mu\sum_{i=1}^2 \|\psi_{it}^\tau\|^2 + C_\mu(N_\tau(T)+\delta)\sum_{i=1}^2 \|(\psi_{ix}^\tau, \psi_i^\tau, \eta_i^\tau)\|^2, \tag{3.55a}
$$

$$
|J_2^{(1)}| \le \mu\sum_{i=1}^2 \|\tau\psi_{itt}^\tau\|^2 + C_\mu(N_\tau(T)+\delta)\sum_{i=1}^2 \|(\sqrt{\tau}\psi_{ixt}^\tau, \psi_{it}^\tau, \psi_{ix}^\tau, \psi_i^\tau, \eta_i^\tau)\|^2, \tag{3.55b}
$$

where $0 < \mu \ll 1$ will be determined later.

Finally, by

$$
(3.50) + 2\times(3.53), \quad k = 0, 1,
$$

we obtain the higher order estimates

$$
\begin{aligned}
\frac{d}{dt}I^{(k)}(t) + \int_0^1 \Bigg\{ &\sum_{i=1}^2 \Big[K_i(n_i^\tau)^{-1}(\partial_t^k\psi_{ix}^\tau)^2 - \tau^2(n_i^\tau)^{-1}(\partial_t^k\psi_{it}^\tau)^2\Big] \\
&+ (\partial_t^k\psi_1^\tau - \partial_t^k\psi_2^\tau)^2 + \sum_{i=1}^2 2\tau^{2k}(n_i^\tau)^{-1}(\partial_t^k\psi_{it}^\tau)^2\Bigg\} dx = J^{(k)}(t),
\end{aligned} \tag{3.56}
$$

where

$$
I^{(k)}(t) := I_1^{(k)}(t) + 2I_2^{(k)}(t), \quad J^{(k)}(t) := J_1^{(k)}(t) + 2J_2^{(k)}(t). \tag{3.57}
$$

Step IV. Decay estimate.

From (3.42), one can see that we only obtain $\|\tau\eta_i^\tau\|^2$ in the energy of basic estimate, it is not enough to close the uniform estimate. Therefore, we must add $\|\eta_i^\tau\|^2$ in the energy of basic estimate. Specifically, by (1.3b) − (1.11b), we have the equation

$$
\eta_i^\tau + \tau^2\eta_{it}^\tau + \Big[\tau^2\Big((j_i^\tau)^2(n_i^\tau)^{-1} - (\tilde{j}_i^\tau)^2(\tilde{n}_i^\tau)^{-1}\Big) + K_i\psi_i^\tau\Big]_x + (-1)^i(n_i^\tau\phi_x^\tau - \tilde{n}_i^\tau\tilde{\phi}_x^\tau) = 0. \tag{3.58}
$$

From

$$\int_0^1 \sum_{i=1}^{2} (3.58) \times \eta_{it}^\tau dx, \tag{3.59}$$

we get

$$\frac{d}{dt} I_3(t) + \int_0^1 \sum_{i=1}^{2} \frac{\tau^2}{2} (\eta_{it}^\tau)^2 dx \le C_9 \sum_{i=1}^{2} \|(\psi_{ixt}^\tau, \psi_{it}^\tau, \psi_{ix}^\tau, \psi_i^\tau, \eta_i^\tau)\|^2, \tag{3.60}$$

where

$$I_3(t) := \int_0^1 \sum_{i=1}^{2} \left[\frac{1}{2} (\eta_i^\tau)^2 + K_i \psi_{ix}^\tau \eta_i^\tau + (-1)^i (n_i^\tau \sigma_x^\tau + \tilde{\phi}_x^\tau \psi_i^\tau) \eta_i^\tau \right] dx. \tag{3.61}$$

Now, we can establish the decay estimate. To this end, by the procedure

$$\left[(3.40) + \sum_{k=0}^{1} (3.56) \right] + \mu_1 (3.60), \quad \text{where } 0 < \mu_1 \ll 1 \text{ will be determined later,}$$

we have

$$\frac{d}{dt} E^\tau(t) + F^\tau(t) \le 0, \quad \forall t \in [0, T], \quad \tau \in (0, 1], \tag{3.62}$$

where

$$E^\tau(t) := \int_0^1 \mathcal{E}(t, x) dx + \sum_{k=0}^{1} I^{(k)} + \mu_1 I_3(t), \tag{3.63a}$$

$$\begin{aligned} F^\tau(t) := & \int_0^1 \sum_{i=1}^{2} \frac{(\eta_i^\tau)^2}{\tilde{n}_i^\tau} dx - \int_0^1 R_2 dx \\ & + \sum_{k=0}^{1} \left\{ \int_0^1 \left\{ \sum_{i=1}^{2} \left[K_i (n_i^\tau)^{-1} (\partial_t^k \psi_{ix}^\tau)^2 - \tau^2 (n_i^\tau)^{-1} (\partial_t^k \psi_{it}^\tau)^2 \right] + (\partial_t^k \psi_1^\tau - \partial_t^k \psi_2^\tau)^2 \right. \right. \\ & \left. \left. + \sum_{i=1}^{2} 2\tau^{2k} (n_i^\tau)^{-1} (\partial_t^k \psi_{it}^\tau)^2 \right\} dx - J^{(k)}(t) \right\} \\ & - C_9 \mu_1 \sum_{i=1}^{2} \|(\psi_{ixt}^\tau, \psi_{it}^\tau, \psi_{ix}^\tau, \psi_i^\tau, \eta_i^\tau)\|^2. \end{aligned} \tag{3.63b}$$

By the estimates (3.41), (3.42), (3.52), (3.55) and Poincaré inequality, there exist constants C_{10l}, C_{10r}, $C_{11} > 0$ such that for $\forall t \in [0, T]$ and $\forall \tau \in (0, 1]$ we have the following equivalent relation if $N_\tau(T) + \delta \ll \mu$, $\mu_1 \ll 1$,

$$C_{10l} A(t) \le E^\tau(t) \le C_{10r} A(t), \tag{3.64}$$

$$C_{11} A(t) \le F^\tau(t). \tag{3.65}$$

By using the Gronwall inequality and the equivalent relation (3.46), we obtain the exponentially decay estimate

$$n_\tau(t) \le C n_\tau(0) e^{-\gamma_2 t}, \quad \forall t \in [0,T], \ \tau \in (0,1], \tag{3.66}$$

where the generic constant $C > 0$ is independent of t and τ. □

4. Relaxation limit

In this section, we discuss the relaxation limit from the HD model to the DD model. Firstly, we show the relaxation limit of the stationary solutions in Subsection 4.1. And then, we study the relaxation limit of the global solutions in Subsection 4.2.

4.1. Stationary solution case

In this subsection, we prove the Theorem 1.4. Since both the global DD-solution and the global HD-solution are constructed near the corresponding stationary solutions, in order to investigate the relaxation limit in the global solution case, we must first consider the relaxation limit in the stationary solution case.

Proof of Theorem 1.4. We first introduce the error variables

$$\tilde{\mathcal{N}}_i^\tau := \tilde{n}_i^\tau - \tilde{n}_i^0, \quad \tilde{\mathcal{J}}_i^\tau := \tilde{j}_i^\tau - \tilde{j}_i^0, \quad \tilde{\Phi}^\tau := \tilde{\phi}^\tau - \tilde{\phi}^0.$$

Note that both $\tilde{j}_i^\tau$ and $\tilde{j}_i^0$ are given by the explicit formula (2.4). Thus, by the mean value theorem and the estimates (1.15), we get

$$|\tilde{j}_i^\tau - \tilde{j}_i^0| \le C\delta(\|\tilde{\mathcal{N}}_i^\tau\| + \delta\tau^2). \tag{4.1}$$

From

$$\frac{(1.11b)}{\tilde{n}_i^\tau} - \frac{(1.11b)|_{\tau=0}}{\tilde{n}_i^0}, \quad (1.11c) - (1.11c)|_{\tau=0},$$

we obtain the working equation

$$K_i\Big[(\tilde{n}_i^\tau)^{-1}\tilde{n}_{ix}^\tau - (\tilde{n}_i^0)^{-1}\tilde{n}_{ix}^0\Big] - \tau^2(\tilde{j}_i^\tau)^2(\tilde{n}_i^\tau)^{-3}\tilde{n}_{ix}^\tau + (-1)^i\tilde{\Phi}_x^\tau = -\Big[\tilde{j}_i^\tau(\tilde{n}_i^\tau)^{-1} - \tilde{j}_i^0(\tilde{n}_i^0)^{-1}\Big], \tag{4.2}$$

$$\tilde{\Phi}_{xx}^\tau = \tilde{\mathcal{N}}_1^\tau - \tilde{\mathcal{N}}_2^\tau. \tag{4.3}$$

Furthermore, $\tilde{\mathcal{N}}_i^\tau$ and $\tilde{\Phi}^\tau$ satisfy the homogeneous boundary conditions

$$\tilde{\mathcal{N}}_i^\tau(0) = \tilde{\mathcal{N}}_i^\tau(1) = 0, \quad \tilde{\Phi}^\tau(0) = \tilde{\Phi}^\tau(1) = 0. \tag{4.4}$$

By the procedure

$$\int_0^1 \sum_{i=1}^2 (4.2)\tilde{\mathcal{N}}_{ix}^\tau dx,$$

we have

$$\underbrace{\int_0^1 \sum_{i=1}^2 K_i\Big[(\tilde{n}_i^\tau)^{-1}\tilde{n}_{ix}^\tau - (\tilde{n}_i^0)^{-1}\tilde{n}_{ix}^0\Big]\tilde{\mathcal{N}}_{ix}^\tau dx}_{\Theta_1} + \underbrace{\int_0^1 \sum_{i=1}^2 (-1)^i \tilde{\Phi}_x^\tau \tilde{\mathcal{N}}_{ix}^\tau dx}_{\Theta_2}$$

$$= \underbrace{\int_0^1 \sum_{i=1}^2 \tau^2 (\tilde{j}_i^\tau)^2 (\tilde{n}_i^\tau)^{-3}\tilde{n}_{ix}^\tau \tilde{\mathcal{N}}_{ix}^\tau dx}_{\Theta_3} - \underbrace{\int_0^1 \sum_{i=1}^2 \Big[\tilde{j}_i^\tau (\tilde{n}_i^\tau)^{-1} - \tilde{j}_i^0 (\tilde{n}_i^0)^{-1}\Big]\tilde{\mathcal{N}}_{ix}^\tau dx}_{\Theta_4}. \tag{4.5}$$

By using the estimate (1.15), (4.1), Poincaré inequality and the smallness condition $\delta \ll 1$, after integration by parts, we can estimate $\Theta_l, l = 1, 2, 3, 4$ as follows.

$$\begin{aligned}
\Theta_1 &= \int_0^1 \sum_{i=1}^2 K_i \Big\{\Big[(\tilde{n}_i^\tau)^{-1} - (\tilde{n}_i^0)^{-1}\Big]\tilde{n}_{ix}^\tau + (\tilde{n}_i^0)^{-1}\tilde{\mathcal{N}}_{ix}^\tau\Big\}\tilde{\mathcal{N}}_{ix}^\tau dx \\
&\geq C_{11}\sum_{i=1}^2 \|\tilde{\mathcal{N}}_{ix}^\tau\|^2 - C\delta \sum_{i=1}^2 \|\tilde{\mathcal{N}}_i^\tau\|_1^2 \\
&\geq C_{11}\sum_{i=1}^2 \|\tilde{\mathcal{N}}_i^\tau\|_1^2,
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\Theta_2 &= \int_0^1 \sum_{i=1}^2 (-1)^{i-1}\tilde{\Phi}_{xx}^\tau \tilde{\mathcal{N}}_i^\tau dx \\
&= \int_0^1 \tilde{\Phi}_{xx}^\tau (\tilde{\mathcal{N}}_1^\tau - \tilde{\mathcal{N}}_2^\tau) dx \\
&= \int_0^1 (\tilde{\mathcal{N}}_1^\tau - \tilde{\mathcal{N}}_2^\tau)^2 dx \geq 0,
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
\Theta_3 &= \int_0^1 \sum_{i=1}^2 \tau^2 (\tilde{j}_i^\tau)^2 (\tilde{n}_i^\tau)^{-3}\tilde{n}_{ix}^\tau \tilde{\mathcal{N}}_{ix}^\tau dx \\
&\leq \int_0^1 \sum_{i=1}^2 \tau^2 \delta^3 \tilde{\mathcal{N}}_{ix}^\tau dx \\
&\leq \mu \sum_{i=1}^2 \|\tilde{\mathcal{N}}_{ix}^\tau\|^2 + C_\mu \delta^6 \tau^4,
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\Theta_4 &= -\int_0^1 \sum_{i=1}^2 \Big[\tilde{\mathcal{J}}_i^\tau (\tilde{n}_i^\tau)^{-1} + \tilde{j}_i^0 (\tilde{n}_i^\tau \tilde{n}_i^0)^{-1}\tilde{\mathcal{N}}_i^\tau\Big]\tilde{\mathcal{N}}_{ix}^\tau dx \\
&\leq C\int_0^1 \sum_{i=1}^2 \Big(|\tilde{\mathcal{J}}_i^\tau \tilde{\mathcal{N}}_{ix}^\tau| + \delta |\tilde{\mathcal{N}}_i^\tau \tilde{\mathcal{N}}_{ix}^\tau|\Big) dx
\end{aligned}$$

$$\begin{aligned}&\le \mu\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_{ix}^{\tau}\|^2 + C_\mu\sum_{i=1}^{2}|\tilde{\mathcal{J}}_i^{\tau}|^2 + C\delta\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_i^{\tau}\|_1^2\\ &\le (\mu + C\delta)\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_i^{\tau}\|_1^2 + C_\mu\delta^2\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_i^{\tau}\|^2 + C_\mu\delta^4\tau^4\\ &\le (\mu + C\delta + C_\mu\delta^2)\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_i^{\tau}\|_1^2 + C_\mu\delta^4\tau^4. \end{aligned} \tag{4.9}$$

Substituting (4.6)–(4.9) into (4.5), we obtain

$$\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_i^{\tau}\|_1 \le C\delta^2\tau^2. \tag{4.10}$$

Substituting (4.10) into (4.1), we get

$$\sum_{i=1}^{2}|\tilde{\mathcal{J}}_i^{\tau}| \le C\delta^2\tau^2. \tag{4.11}$$

Next, solving $\tilde{\mathcal{N}}_{ixx}^{\tau}$ from the equation ∂_x(4.2), and taking the L^2-norm of $\tilde{\mathcal{N}}_{ixx}^{\tau}$, and combining the estimate (4.10) with (4.11), we have

$$\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_{ixx}^{\tau}\| \le C\delta^2\tau^2. \tag{4.12}$$

From (4.10) and (4.12), we obtain

$$\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_i^{\tau}\|_2 \le C\delta^2\tau^2. \tag{4.13}$$

Finally, from (4.3) and (4.4), we get the elliptic estimate

$$\|\tilde{\Phi}^{\tau}\|_4 \le C\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_i^{\tau}\|_2. \tag{4.14}$$

Then, by using the estimates (4.13), (4.11) and (4.14), we get the estimate (1.18). □

4.2. Global solution case

This subsection is devoted to the justification of the relaxation limit in the global solution case. This discussion completes the proof of Theorem 1.5.

Proof of Theorem 1.5. Firstly, from the convergence estimate (1.18) in the stationary solution case, one can see that the following condition

$$\tau + \delta + \sum_{i=1}^{2}\Big(\|n_{i0} - \tilde{n}_i^{\tau}\|_2 + \|j_{i0} - \tilde{j}_i^{\tau}\|_1 + \|\tau j_{i0xx}\|\Big) \ll 1 \tag{4.15}$$

implies the condition

$$\sum_{i=1}^{2} \|n_{i0} - \tilde{n}_i^0\|_2 + \delta \ll 1.$$

Thus, the conditions in Theorem 1.2 and the conditions in Theorem 1.3 hold true at the same time. This ensures that the global HD-solution $(n_i^\tau, j_i^\tau, \phi^\tau)(t,x)$ and the global DD-solution $(n_i^0, j_i^0, \phi^0)(t,x)$ exist at the same time under the condition (4.15).

In addition, in order to establish the convergence estimate in the global solution case, we introduce the error variables

$$\mathcal{N}_i^\tau := n_i^\tau - n_i^0, \quad \mathcal{J}_i^\tau := j_i^\tau - j_i^0, \quad \Phi^\tau := \phi^\tau - \phi^0.$$

By (1.3) − (1.7), we get

$$\mathcal{N}_{it}^\tau + \mathcal{J}_{ix}^\tau = 0, \tag{4.16a}$$

$$\tau^2 j_{it}^\tau + \tau^2 \Big[(j_i^\tau)^2 (n_i^\tau)^{-1} \Big]_x + K_i \mathcal{N}_{ix}^\tau + (-1)^i \Big(\mathcal{N}_i^\tau \phi_x^\tau + n_i^0 \Phi_x^\tau \Big) + \mathcal{J}_i^\tau = 0, \tag{4.16b}$$

$$\Phi_{xx}^\tau = \mathcal{N}_1^\tau - \mathcal{N}_2^\tau. \tag{4.16c}$$

From the initial–boundary conditions (1.8), (1.9), (1.4) and (1.5), we have the homogeneous initial–boundary conditions

$$\mathcal{N}_i^\tau(0,x) = 0, \tag{4.17}$$

$$\mathcal{N}_i^\tau(t,0) = \mathcal{N}_i^\tau(t,1) = 0, \quad \Phi^\tau(t,0) = \Phi^\tau(t,1) = 0. \tag{4.18}$$

By the procedure $(-1) \times \partial_x$(4.16b), we obtain the working equation

$$\tau^2 n_{itt}^\tau - \tau^2 \Big[(j_i^\tau)^2 (n_i^\tau)^{-1} \Big]_{xx} - K_i \mathcal{N}_{ixx}^\tau + (-1)^{i+1} \Big(\mathcal{N}_i^\tau \phi_x^\tau + n_i^0 \Phi_x^\tau \Big)_x + \mathcal{N}_{it}^\tau = 0. \tag{4.19}$$

Finally, from (1.16) and (1.17), there exist constants $C_{12}, C > 0$ which are independent of t, δ and τ such that the following estimates hold.

$$\inf_{x\in\Omega} n_i^0 \geq C_{12}, \tag{4.20a}$$

$$\sum_{i=1}^{2} \Big(\|n_i^0(t)\|_2 + \|j_i^0(t)\|_1 \Big) + |\phi^0(t)|_2 \leq C, \tag{4.20b}$$

$$\int_0^t s \sum_{i=1}^{2} \|(n_{itt}^0, n_{ixxt}^0)(s)\|^2 ds \leq C(1+t), \quad \forall t \in [0, +\infty), \tag{4.20c}$$

and

$$\inf_{x\in\Omega} n_i^\tau, \quad \inf_{x\in\Omega} \left[K_i - \frac{\tau^2 (j_i^\tau)^2}{(n_i^\tau)^2} \right] \geq C_{12}, \tag{4.21a}$$

$$\sum_{i=1}^{2} \Big(\|n_i^\tau(t)\|_2 + \|j_i^\tau(t)\|_1 + \|\tau j_{ixx}^\tau(t)\| + \|\tau^2 n_{itt}^\tau(t)\| \Big) + |\phi^\tau(t)|_2 + |\tau^2 j_{it}^\tau(t)|_0 \leq C, \tag{4.21b}$$

$$\int_0^t \sum_{i=1}^{2} \|(\tau j_{it}^\tau, j_{ixx}^\tau, \tau n_{itt}^\tau)(s)\|^2 ds \leq C, \quad \forall t \in [0, +\infty). \tag{4.21c}$$

Based on the above estimates, we can estimate the error variables. Until now, we have overcome all the difficulties arising from the bipolar effect. In the rest of the proof, there is no essential difference between the bipolar case and the unipolar case [27]. However, the calculations remain complicated, we complete the proof in the next steps:

Step I. By taking

$$\int_0^t \int_0^1 \sum_{i=1}^2 (4.19) \times \mathcal{N}_{it}^\tau dxds$$

we have

$$\begin{aligned} -\int_0^t \int_0^1 \sum_{i=1}^2 K_i \mathcal{N}_{ixx}^\tau \mathcal{N}_{it}^\tau dxds + \int_0^t \int_0^1 \sum_{i=1}^2 (\mathcal{N}_{it}^\tau)^2 dxds \\ = -\int_0^t \int_0^1 \sum_{i=1}^2 \left\{ \tau^2 n_{itt}^\tau - \tau^2 \left[(j_i^\tau)^2 (n_i^\tau)^{-1} \right]_{xx} + (-1)^{i+1} \left(\mathcal{N}_i^\tau \phi_x^\tau + n_i^0 \Phi_x^\tau \right)_x \right\} \mathcal{N}_{it}^\tau dxds. \end{aligned}$$

After integration by parts, and by using (4.17), (4.20) and (4.21), we get

$$\sum_{i=1}^2 \|\mathcal{N}_{ix}^\tau(t)\|^2 + \int_0^t \sum_{i=1}^2 \|\mathcal{N}_{it}^\tau(s)\|^2 ds \le C\left(\tau^2(1+t) + \int_0^t \sum_{i=1}^2 \|\mathcal{N}_i^\tau(s)\|_1^2 ds \right). \tag{4.22}$$

From the homogeneous boundary condition (4.18) and Poincaré inequality, we have

$$\sum_{i=1}^2 \|\mathcal{N}_i^\tau(t)\|_1^2 + \int_0^t \sum_{i=1}^2 \|\mathcal{N}_{it}^\tau(s)\|^2 ds \le C\left(\tau^2(1+t) + \int_0^t \sum_{i=1}^2 \|\mathcal{N}_i^\tau(s)\|_1^2 ds \right). \tag{4.23}$$

By Gronwall inequality, there exist constants $C, \alpha > 0$ which are independent of t, δ and τ such that

$$\int_0^t \sum_{i=1}^2 \|\mathcal{N}_i^\tau(s)\|_1^2 ds \le C\tau^2 e^{\alpha t}. \tag{4.24}$$

Thus,

$$\sum_{i=1}^2 \|\mathcal{N}_i^\tau(t)\|_1^2 + \int_0^t \sum_{i=1}^2 \|\mathcal{N}_{it}^\tau(s)\|^2 ds \le C\tau^2 e^{\alpha t}, \quad \forall t \in [0, +\infty). \tag{4.25}$$

Step II. Since the initial data j_{i0} is not in momentum equilibrium, namely, $\mathcal{J}_i^\tau(0,x) \neq 0$, an initial layer will appear. In order to handle the initial layer, we adopt the time weighted energy method and prove that the layer decays exponentially fast as the relaxation time τ tends to zero and/or time t tends to infinity. Precisely, by

$$\int_0^t \int_0^1 (4.16\text{b}) \times (e^{s/\tau^2} \mathcal{J}_i^\tau) dxds, \quad i = 1, 2,$$

we have

$$\tau^2 e^{t/\tau^2}\int_0^1 \frac{1}{2}(\mathcal{J}_i^\tau)^2 dx + \frac{1}{2}\int_0^t\int_0^1 e^{s/\tau^2}(\mathcal{J}_i^\tau)^2 dxds = \tau^2\int_0^1 \frac{1}{2}(\mathcal{J}_i^\tau)^2(0)dx$$
$$\underbrace{-\int_0^t\int_0^1 e^{s/\tau^2}\left\{\tau^2 j_{it}^0 + \tau^2\Big[(j_i^\tau)^2(n_i^\tau)^{-1}\Big]_x + K_i\mathcal{N}_{ix}^\tau + (-1)^i\Big(\mathcal{N}_i^\tau\phi_x^\tau + n_i^0\Phi_x^\tau\Big)\right\}\mathcal{J}_i^\tau dxds}_{II(t)}. \tag{4.26}$$

From

$$\int_0^t e^{s/\tau^2}(3.21)ds$$

we obtain

$$\int_0^t e^{s/\tau^2}\|n_{ixt}^0(s)\|^2 ds \le Ce^{t/\tau^2}, \quad \forall t\in[0,+\infty), \quad i=1,2. \tag{4.27}$$

By ∂_t(1.7b), (4.20), (4.25) and Cauchy–Schwarz inequality, together with (4.27), we can estimate $II(t)$ in the right side of (4.26) as follows

$$II(t) \le \frac{1}{4}\int_0^t\int_0^1 e^{s/\tau^2}(\mathcal{J}_i^\tau)^2 dxds + C\tau^4 e^{t/\tau^2}e^{\alpha t}. \tag{4.28}$$

Substituting (4.28) into (4.26), and multiplying the resultant inequality by $(\frac{1}{2}\tau^2 e^{t/\tau^2})^{-1}$, we have

$$\|\mathcal{J}_i^\tau(t)\|^2 \le \|\mathcal{J}_i^\tau(0)\|^2 e^{-t/\tau^2} + C\tau^2 e^{\alpha t}, \quad \forall t\in[0,+\infty), \quad i=1,2. \tag{4.29}$$

Step III. By

$$\int_0^t\int_0^1\sum_{i=1}^2 (4.19)\times(-s\mathcal{N}_{ixxt}^\tau)dxds,$$

after integration by parts, together with (4.20c), (4.24), (4.19) and (4.21), the direct calculations lead to

$$\sum_{i=1}^2 \|(\mathcal{N}_{ixx}^\tau, \tau j_{ixx}^\tau)(t)\|^2 \le C\tau e^{\alpha t}t^{-1}, \quad \forall t\in(0,+\infty). \tag{4.30}$$

From

$$\int_0^t\int_0^1\sum_{i=1}^2 (4.19)\times(se^{s/\tau^2}\mathcal{J}_{ix}^\tau)dxds,$$

by a similar way to establish the estimate (4.29), together with (4.20c), (4.24) and (4.30), we obtain

$$\sum_{i=1}^{2} \|\mathcal{J}^\tau_{ix}(t)\|^2 \le C\tau e^{\alpha t} t^{-1}, \quad \forall t \in (0, +\infty). \tag{4.31}$$

Step IV. For arbitrarily fixed τ in condition (1.19), we know that $0 < \tau < 1$. Let

$$T^\tau := -\frac{\ln \tau}{2\alpha} > 0. \tag{4.32}$$

If $0 < t \le T^\tau$, by (4.25), (4.29), (4.30) and (4.31), we get

$$\sum_{i=1}^{2} \|\mathcal{N}^\tau_i(t)\|_1^2 \le C\tau^{3/2}, \tag{4.33a}$$

$$\|\mathcal{J}^\tau_i(t)\|^2 \le \|\mathcal{J}^\tau_i(0)\|^2 e^{-t/\tau^2} + C\tau^{3/2}, \quad i = 1, 2, \tag{4.33b}$$

$$\sum_{i=1}^{2} \|(\mathcal{N}^\tau_{ixx}, \mathcal{J}^\tau_{ix})(t)\|^2 \le C\tau^{1/2} t^{-1}. \tag{4.33c}$$

If $t \ge T^\tau$, by (1.17b), (1.18) and (1.16b), we obtain

$$\begin{aligned}
&\sum_{i=1}^{2} \Big(\|\mathcal{N}^\tau_i(t)\|_2^2 + \|\mathcal{J}^\tau_i(t)\|_1^2\Big) && (4.34)\\
&\le C\sum_{i=1}^{2} \Big(\|(n^\tau_i - \tilde{n}^\tau_i)(t)\|_2^2 + \|(j^\tau_i - \tilde{j}^\tau_i)(t)\|_1^2 + \|\tilde{n}^\tau_i - \tilde{n}^0_i\|_2^2 + \|\tilde{j}^\tau_i - \tilde{j}^0_i\|_1^2 \\
&\qquad\qquad + \|(n^0_i - \tilde{n}^0_i)(t)\|_2^2 + \|(j^0_i - \tilde{j}^0_i)(t)\|_1^2\Big) \\
&\le C\Big(e^{-2\gamma_2 t} + \tau^4 + e^{-2\gamma_1 t}\Big) \\
&\le C\Big(\tau^{\gamma_2/\alpha} + \tau^4 + \tau^{\gamma_1/\alpha}\Big) \\
&\le C\tau^{\gamma_3}, && (4.35)
\end{aligned}$$

where the positive constant γ_3 is given by

$$\gamma_3 := \min\left\{\frac{\gamma_2}{\alpha}, \frac{\gamma_1}{\alpha}, \frac{1}{2}\right\} > 0. \tag{4.36}$$

Finally, combining (4.33), (4.35) and the elliptic estimate $\|\Phi^\tau(t)\|_4^2 \le C\|(\mathcal{N}^\tau_1, \mathcal{N}^\tau_2)(t)\|_2^2$, we arrive at the convergence estimate (1.20) for global solution. □

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