ERRATUM: GLOBAL STABILITY OF MONOSTABLE TRAVELING WAVES FOR NONLOCAL TIME-DELAYED REACTION-DIFFUSION EQUATIONS

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Abstract. This short note is to fix the gap for the proof of Lemma 3.8 in our previous paper [M. Mei, C. Ou and X.-Q. Zhao, *SIAM J. Math. Anal.*, 42 (2010) 2762-2790].

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In our previous paper [2], in order to get the algebraic stability for the critical traveling wavefronts, one of key steps is to build up the decay estimate (see Lemma 3.8 in [2])

$$\|\bar{v}(t)\|_{L^\infty_{w_1}(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}}, \tag{0.1}$$

where $w_1(\xi) = e^{-\lambda_*(\xi - x_0)}$ is the weight function for the critical wave case with $c = c_*$, and $\bar{v}(t,\xi)$ is the solution of

$$\begin{cases} \dfrac{\partial \bar{v}}{\partial t} + c_* \dfrac{\partial \bar{v}}{\partial \xi} - D\dfrac{\partial^2 \bar{v}}{\partial \xi^2} + d'(0)\bar{v} - \varepsilon b'(0)\displaystyle\int_{\mathbb{R}} f_\alpha(y)\bar{v}(t-\tau, \xi - y - c_*\tau)dy = 0, \\ \bar{v}(s,\xi) = \bar{v}_0(s,\xi), \quad s \in [-\tau, 0]. \end{cases} \tag{0.2}$$

This was then proved based on Lemma 3.7 in [2]:

$$\|\hat{v}(t)\|_{L^\infty(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}} e^{k_2 t}, \tag{0.3}$$

where $\hat{v}(t,\xi) := e^{k_2 t} w_1(\xi)\bar{v}(t,\xi)$ satisfies (see (3.47)-(3.48) in [2])

$$\begin{cases} \dfrac{\partial \hat{v}}{\partial t} + k_1 \dfrac{\partial \hat{v}}{\partial \xi} - D\dfrac{\partial^2 \hat{v}}{\partial \xi^2} = \varepsilon b'(0) e^{k_2\tau}\displaystyle\int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_*(y + c_*\tau)}\hat{v}(t-\tau, \xi - y - c_*\tau)dy, \\ \hat{v}(s,\xi) = e^{k_2 s} w_1(\xi)\bar{v}_0(s,\xi) := \hat{v}_0(s,\xi), \quad s \in [-r, 0], \end{cases} \tag{0.4}$$

Here, $k_1 := c_* - 2D\lambda_*$ and $k_2 := c_*\lambda_* - D\lambda_*^2 + d'(0) > 0$. However, the proof of Lemma 3.7 is incorrect. In fact, we converted the working equation (0.4) to the integral form with the regular Green function (the heat kernel without time-delay) $G(t, \xi - \zeta) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(\xi - \zeta + k_1 t)^2}{4Dt}}$, then used the iteration procedure to derive the algebraic convergence rate in the case of critical waves: $C^k(1+t)^{-1/2}$ at the kth iteration. So, the constant coefficient C^k is increasing and unbounded as $k \to \infty$. In order to fix

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such a gap, here, inspired by Mei-Wang [3], we similarly derive the equivalent integral equation with the time-delayed Green function, and show the optimal decay rates of the solutions without iteration. The following is a new proof for Lemma 3.8 based on [3].

Let $\tilde{v}(t,\xi) := w_1(\xi)\bar{v}(t,\xi)$, from (0.2), then $\tilde{v}(t,\xi)$ satisfies

$$\begin{cases} \dfrac{\partial \tilde{v}}{\partial t} + k_1 \dfrac{\partial \tilde{v}}{\partial \xi} - D\dfrac{\partial^2 \tilde{v}}{\partial \xi^2} + k_2 \tilde{v} = \varepsilon b'(0) \displaystyle\int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_*(y+c_*\tau)} \tilde{v}(t-\tau, \xi - y - c_*\tau) dy, \\ \tilde{v}(s,\xi) = w_1(\xi)\bar{v}_0(s,\xi) := \tilde{v}_0(s,\xi), \quad s \in [-\tau, 0], \end{cases} \tag{0.5}$$

Taking Fourier transform to (0.5), and denoting its Fourier transform by $\mathcal{F}[\tilde{v}] = \check{v}(t,\eta)$, as showed in [3], we have

$$\frac{d\check{v}}{dt} + A(\eta)\check{v} = B(\eta)\check{v}(t-\tau,\eta), \quad \text{with} \quad \check{v}(s,\eta) = \check{v}_0(s,\eta), \; s \in [-\tau, 0], \tag{0.6}$$

where

$$A(\eta) = D\eta^2 + k_2 + \mathrm{i}c_*\eta \;\text{ and }\; B(\eta) = \varepsilon b'(0) \int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_*(y+c_*\tau)} e^{-\mathrm{i}(y+c_*\tau)\eta} dy.$$

Thanks to [1], we can solve the above time-delayed ODE (0.6) as

$$\begin{aligned} \check{v}(t,\eta) &= e^{-A(\eta)(t+\tau)} e_\tau^{\mathcal{B}(\eta)t} \check{v}_0(-\tau,\eta) \\ &\quad + \int_{-\tau}^0 e^{-A(\eta)(t-s)} e_\tau^{\mathcal{B}(\eta)(t-\tau-s)} \Big[\frac{d}{ds}\check{v}_0(s,\eta) + A(\eta)\check{v}_0(s,\eta)\Big] ds, \end{aligned} \tag{0.7}$$

where

$$\mathcal{B}(\eta) := B(\eta) e^{A(\eta)\tau}, \tag{0.8}$$

and $e_\tau^{\mathcal{B}(\eta)t}$ is the delayed exponential function in the form

$$e_\tau^{\mathcal{B}(\eta)t} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \le t < 0, \\ 1 + \frac{\mathcal{B}(\eta)t}{1!}, & 0 \le t < \tau, \\ 1 + \frac{\mathcal{B}(\eta)t}{1!} + \frac{\mathcal{B}(\eta)^2(t-\tau)^2}{2!}, & \tau \le t < 2\tau, \\ \vdots & \vdots \\ 1 + \frac{\mathcal{B}(\eta)t}{1!} + \frac{\mathcal{B}(\eta)^2(t-\tau)^2}{2!} + \cdots + \frac{\mathcal{B}(\eta)^m[t-(m-1)\tau]^m}{m!}, & (m-1)\tau \le t < m\tau, \\ \vdots & \vdots \end{cases}$$

Then, by taking the inverse Fourier transform to (0.7), we get

$$\begin{aligned} \tilde{v}(t,\xi) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}x\cdot\eta} e^{-A(\eta)(t+\tau)} e_\tau^{\mathcal{B}(\eta)t} \check{v}_0(-\tau,\eta) d\eta \\ &\quad + \int_{-\tau}^0 \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mathrm{i}x\cdot\eta} e^{-A(\eta)(t-s)} e_\tau^{\mathcal{B}(\eta)(t-\tau-s)} \\ &\qquad\qquad \times \Big[\frac{d}{ds}\check{v}_0(s,\eta) + A(\eta)\check{v}_0(s,\eta)\Big] d\eta ds. \end{aligned} \tag{0.9}$$

Applying Theorem 2.3 in [3] to (0.9), we then obtain the following estimates

$$\|\tilde{v}(t)\|_{L^\infty(\mathbb{R})} \le Ct^{-\frac{1}{2}} e^{-\varepsilon_1(c_1-c_3)t}, \tag{0.10}$$

where $0 < \varepsilon_1 < 1$ is a specified constant, c_1 and c_3 are positive constants given by

$$c_1 := k_2 = c_*\lambda_* - D\lambda_*^2 + d'(0) > 0 \tag{0.11}$$

and

$$\begin{aligned} c_3 :&= \varepsilon b'(0) \int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_*(y_1+c_*\tau)} dy \\ &= \varepsilon b'(0) \int_{\mathbb{R}} f_\alpha(y_1) e^{-\lambda_*(y_1+c_*\tau)} dy_1 \\ &= \varepsilon b'(0) e^{\alpha\lambda_*^2 - \lambda_* c_* \tau} > 0. \end{aligned} \tag{0.12}$$

Notice that, when $c = c_*$ (the critical wave case), we have (see Lemma 2.1 in [2])

$$\varepsilon b'(0) e^{\alpha\lambda_*^2 - \lambda_* c_*\tau} = c_*\lambda_* - D\lambda_*^2 + d'(0), \quad i.e., \ c_1 = c_3.$$

So, from (0.10), we obtain the following algebraic decay

$$\|\tilde{v}(t)\|_{L^\infty(\mathbb{R})} \le Ct^{-\frac{1}{2}},$$

which is equivalent to

$$\|\bar{v}(t)\|_{L^\infty_{w_1}(\mathbb{R})} \le Ct^{-\frac{1}{2}}.$$

Thus, the proof for Lemma 3.8 is complete.

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