ERRATUM: GLOBAL STABILITY OF MONOSTABLE TRAVELING WAVES FOR NONLOCAL TIME-DELAYED REACTION-DIFFUSION EQUATIONS

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In our recent paper [2], in order to get the algebraic stability for critical traveling wavefronts, one of the key steps is to establish the following decay estimate (see [2,Lemma 3.8]):

\[(0.1) \| \hat{v}(t) \|_{L^\infty_1}(\mathbb{R}) \leq C(1+t)^{-\frac{1}{4}}, \]

where \(w_1(\xi) = e^{-\lambda_* (\xi - x_0)}\) is the weight function for the critical wave case with \(c = c_*\), and \(\hat{v}(t,\xi)\) is the solution of

\[(0.2) \begin{cases}
  \frac{\partial \hat{v}}{\partial t} + c_* \frac{\partial \hat{v}}{\partial \xi} - D \frac{\partial^2 \hat{v}}{\partial \xi^2} + d'(0)\hat{v} - \varepsilon b'(0) \int_{\mathbb{R}} f_\alpha(y) \hat{v}(t-\tau,\xi-y-c_*\tau) dy = 0, \\
  \hat{v}(s,\xi) = \hat{v}_0(s,\xi), \quad s \in [-\tau,0].
\end{cases} \]

This was proved with the aid of [2, Lemma 3.7]

\[(0.3) \| \hat{v}(t) \|_{L^\infty(\mathbb{R})} \leq C(1+t)^{-\frac{1}{4}} e^{k_2t}, \]

where \(\hat{v}(t,\xi) := e^{k_2t}w_1(\xi)\hat{v}(t,\xi)\) satisfies (see (3.47)–(3.48) in [2])

\[(0.4) \begin{cases}
  \frac{\partial \hat{v}}{\partial t} + k_1 \frac{\partial \hat{v}}{\partial \xi} - D \frac{\partial^2 \hat{v}}{\partial \xi^2} = \varepsilon b'(0) e^{k_2t} \int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_* (\xi+y-c_*\tau)} \hat{v}(t-\tau,\xi-y-c_*\tau) dy, \\
  \hat{v}(s,\xi) = e^{k_1s}w_1(\xi)\hat{v}_0(s,\xi) := \hat{v}_0(s,\xi), \quad s \in [-\tau,0].
\end{cases} \]

Note that \(k_1 := 2D\lambda_*\) and \(k_2 := c_*\lambda_* - D\lambda_*^2 + d'(0) > 0\). However, the proof of [2, Lemma 3.7] is incorrect. Indeed, we converted the standing equation (0.4) into an integral form with the regular Green function (the heat kernel without time-delay) \(G(t,\xi - \zeta) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(\xi - \zeta + k_1t)^2}{4Dt}}\), then used the iteration procedure to derive

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the algebraic convergence rate in the case of the critical wave: \( C^k(1 + t)^{-1/2} \) at the \( k \)th iteration for \( t \in [(k - 1)\tau, k\tau] \). Thus, the constant coefficient \( C^k \) is increasing and unbounded as \( k \to \infty \). In order to fix such a gap, we derive an equivalent integral equation with the time-delayed Green function and then obtain the decay rates of solutions without using iterations. Inspired by the work of [3], below we provide a new proof for [2, Lemma 3.8].

**Proof of Lemma 3.8.** Here we need to assume that the initial perturbation around the wavefront \( \phi(x + ct) \) satisfies

\[
\phi_0(s, x) - \phi(x + cs) \in C^1([-\tau, 0]; L^1_w(\mathbb{R}) \cap H^1(\mathbb{R})),
\]

where \( w(x) \) is the weight function given by (2.2) in [2].

Let \( \tilde{v}(t, \xi) := w_1(\xi)\tilde{v}(t, \xi) \). It then follows from (0.2) that

\[
\begin{aligned}
\frac{\partial \tilde{v}}{\partial t} + k_1 \frac{\partial^2 \tilde{v}}{\partial \xi^2} - \frac{D}{\partial \xi} \frac{\partial^2 \tilde{v}}{\partial \xi^2} + \epsilon b'(0) \int_\mathbb{R} f_0(y) e^{-\lambda_1(y+c, \tau)} \tilde{v}(t-\tau, \xi - y - c, \eta) dy, \\
\tilde{v}(s, \xi) = w_1(\xi)\tilde{v}_0(s, \xi) = \tilde{v}_0(s, \xi) = \tilde{v}(s, \xi), \quad s \in [-\tau, 0].
\end{aligned}
\]

Taking the Fourier transform to (0.5), we have

\[
\begin{aligned}
\frac{d\tilde{e}}{dt} + A(\eta)\tilde{e} = B(\eta)\tilde{v} - \tau, \eta \quad \text{and} \quad \tilde{v}(s, \eta) = \tilde{v}_0(s, \eta), \quad s \in [-\tau, 0],
\end{aligned}
\]

where \( \tilde{v}(t, \eta) = \mathcal{F}[\tilde{v}] \) is the Fourier transform of \( \tilde{v} \), and

\[
A(\eta) = D\eta^2 + k_2 + ik_1, \quad B(\eta) = \epsilon b'(0) \int_\mathbb{R} f_0(y) e^{-\lambda_1(y+c, \tau)} e^{-\hat{\lambda}(y+c, \tau)} y dy.
\]

It is easy to see that [1, Theorem 1] is still valid provided that the initial function \( \phi(t) \) is continuous on \([-\tau, 0]\), and \( \phi'(t) \) is continuous for all \( t \in [-\tau, 0] \) except for finite many points where both left and right limits of \( \phi'(t) \) exist. Accordingly, we solve the above time-delayed equation (0.6) as

\[
\begin{aligned}
\tilde{v}(t, \eta) &= e^{-(t)(\eta + \tau)} e^{A(\eta)t} \tilde{v}_0(-\tau, \eta) \\
&\quad + \int_{-\tau}^0 e^{-A(\eta)(t-s)} e^{B(\eta)(t-\tau-s)} \left[ \frac{d}{ds} \tilde{v}_0(s, \eta) + A(\eta)\tilde{v}_0(s, \eta) \right] ds,
\end{aligned}
\]

where

\[
B(\eta) := B(\eta)e^{A(\eta)t},
\]

and \( e^{B(\eta)t} \) is the delayed exponential function defined by

\[
\begin{aligned}
e^{B(\eta)t} &= 0, \quad -\infty < t < -\tau, \\
&= 1, \quad -\tau \leq t < 0, \\
&= 1 + \frac{B(\eta)}{1!} t + \frac{B(\eta)^2(t-\tau)^2}{2!}, \quad 0 \leq t < \tau, \\
&= \cdots, \\
&= 1 + \frac{B(\eta)}{1!} t + \frac{B(\eta)^2(t-\tau)^2}{2!} + \cdots + \frac{B(\eta)^m[(t-(m-1)\tau)^m]}{m!}, \quad (m-1)\tau \leq t < m\tau, \\
&= \cdots.
\end{aligned}
\]
Taking the inverse Fourier transform to (0.7), we then get
\[
\tilde{v}(t, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\cdot\eta} e^{-A(\eta)(t+\tau)} e^{B(\eta)t} \tilde{v}_0(-\tau, \eta) d\eta \\
+ \int_{-\tau}^{0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\cdot\eta} e^{-A(\eta)(t-s)} e^{B(\eta)(t-\tau-s)}
\times \left[ \frac{d}{ds} \tilde{v}_0(s, \eta) + A(\eta) \tilde{v}_0(s, \eta) \right] d\eta ds.
\]
(0.9)

By [3, Theorem 2.3], as applied to (0.9), it follows that
\[
\|\tilde{v}(t)\|_{L^\infty(\mathbb{R})} \leq Ct^{-\frac{1}{2}} e^{-\varepsilon_1(c_1-c_3)t},
\]
(0.10)
where \(0 < \varepsilon_1 < 1\) is a specified constant, \(c_1\) and \(c_3\) are positive constants given by
\[
c_1 := k_2 = c_\ast \lambda_\ast - D\lambda^2_\ast + d'(0) > 0,
\]
and
\[
c_3 := \varepsilon b'(0) \int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_\ast (y_1 + c_\ast \tau)} dy \\
= \varepsilon b'(0) \int_{\mathbb{R}} f_{\alpha_1}(y_1) e^{-\lambda_\ast (y_1 + c_\ast \tau)} dy_1 \\
= \varepsilon b'(0) e^{\alpha \lambda^2_\ast - \lambda_\ast c_\ast \tau} > 0.
\]
(0.12)

In the case where \(c = c_\ast\), we see from [2, Lemma 2.1] that
\[
\varepsilon b'(0) e^{\alpha \lambda^2_\ast - \lambda_\ast c_\ast \tau} = c_\ast \lambda_\ast - D\lambda^2_\ast + d'(0),
\]
and hence \(c_1 = c_3\). It then follows from (0.10) that
\[
\|\tilde{v}(t)\|_{L^\infty(\mathbb{R})} \leq Ct^{-\frac{1}{2}},
\]
which is equivalent to
\[
\|\tilde{v}(t)\|_{L^\infty_{-1}(\mathbb{R})} \leq Ct^{-\frac{1}{2}}.
\]
This completes the proof of Lemma 3.8. \(\square\)

REFERENCES