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STATIONARY SOLUTIONS TO HYBRID QUANTUM HYDRODYNAMICAL MODEL OF SEMICONDUCTORS IN BOUNDED DOMAIN

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Abstract. In this paper we study the behaviour of a micro-sized semiconductor device by means of a hybrid model of hydrodynamic equations. First of all, taking into account the quantum effects in the semiconductor device, we derive a new model called the hybrid quantum hydrodynamic model (H-QHD) coupled with the Poisson equation for electric potential. In particular, we write the Bohm potential in a revised form. This new potential is derived heuristically by assuming that the energy of the electrons depends on the charge density n and on ∇n just in a restricted part of the device domain, whereas the remaining parts are modeled classically. Namely, the device is designed with some parts that feel the quantum effects and some parts do not. The main target is to investigate the existence of the stationary solutions for the hybrid quantum hydrodynamic model. Since the quantum effect is regionally degenerate, this will also makes the working equation regionally degenerate regarding its ellipticity, and the corresponding solutions are weak only. This paper seems the first framework to treat the equation with regionally degenerate ellipticity. In order to prove the existence of such weak solutions, we first construct a sequence of smooth QHD solutions, then prove that such a sequence weakly converges and its limit is just our desired weak solution for the hybrid QHD problem. The Debye length limit is also studied. Indeed, we prove that the weak solutions of the hybrid QHD weakly converge to their targets as the spacial Debye length vanishes. Finally, we carry out some numerical simulations for a simple device, which also confirm our theoretical results.

Key words. Hybrid quantum hydrodynamic model, 4th-order degenerate elliptic equations, stationary solutions, existence, uniqueness, classical limit, hybrid limit.

1. Introduction

In the last decades, the characteristic size of semiconductor devices have gradually reduced up to few hundreds of nanometers. Under these conditions, quantum effects can no longer be neglected, because they play an important role in the functioning of the devices. However, quantum effects are usually localized in a small region of the device, while the rest of the domain can be treated classically, with remarkable reduction of the computational costs. Therefore, the hybrid models are developed in order to provide a strictly quantum description wherever necessary.

Simply speaking, the word *hybrid* emphasizes a mathematical approach for which one models a part of the device by using quantum equations (such as Schrödinger equation, quantum drift-diffusion (QDD) or quantum hydrodynamic (QHD)), and the other parts by using classical models, for example hydrodynamical (HD) or drift diffusion (DD). The main problem is which kind of transmission conditions must be prescribed at the interface between classical and quantum zones of the device. The pioneering study in the hybrid coupling between quantum and classical systems is the paper of N. Ben Abdallah [4], where a suitable set of transmission conditions, linking classical Boltzmann equation and stationary Schrödinger equations, is discussed. Since then, the relevant research has gradually become a hot spot. In [7], Baro *et al* study a one-dimensional stationary Schrödinger drift-diffusion

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including collisions. To link quantum zone and classical region, they prescribe the continuity of current density at the interface. In [5] Ben Abdallah, Méhats and Vauchelet introduce a hybrid drift-diffusion-Schrödinger-Poisson (DDSP) model, and later in [24] the optimal parallelization strategy of numerical solutions of the same model is performed. In [3] and [17], the DDSP model is applied to study the electrons transport in strongly confined structures, such as nanotubes. At the interface between classical and quantum domains, they impose the continuity of the total current. In [11], the hydrodynamic hybrid model is studied by prescribing the continuity of the charge density, where a small jump of the current density is accepted and justified from the physical point of view, by using scaling arguments.

As discussed above, many different strategies can be adopted to establish a physically reasonable set of interface conditions. The concept of hybrid model introduces an error at the interface, due to the arbitrary choice to neglect the quantum effects from a certain point on. Therefore, the choice of suitable transmission conditions allows us to preserve the continuity of certain physical quantities while others have to be sacrificed. Hence the great variety of conditions that can be found in the related literature.

In this paper, we first propose a hybrid model matching classical and quantum hydrodynamical equations, that is derived by introducing a modified form of the Bohm potential. As we know, both the classical and the quantum hydrodynamic models have been extensively studied, see, for example [2,8,10,12–16,18–20,22,23] and references therein. However, just very few results are presently available concerning the hybrid approach to the hydrodynamic model. Therefore, it will be quite interesting to theoretically study this hybrid quantum hydrodynamical model (H-QHD model). We introduce a quantum effect function $Q(x)$ where $Q > 0$ holds in the region of the device with quantum effect and $Q = 0$ for the region without quantum effect. As we will show later on, this makes the governing equations regionally degenerated for its ellipticity and the solutions are necessarily weak. Therefore, when studying the H-QHD case, compared with the regular cases of HD and QHD, some peculiar difficulties will appear. More precisely speaking, the governing steady-state equation of the H-QHD will be a 4th-order elliptic equation with regional degeneracy, and its leading coefficients involve $Q(x) \geq 0$ and $Q'(x)$. That is, in some part of the domain the equation is 4th-order elliptic, but in the other part it degenerates to be 2nd-order elliptic. In particular, when $Q(x)$ is the Heaviside function (the physical case), $Q'(x) = \infty$ will be at the jump discontinuous points of $Q(x)$. This makes the theoretical study on the existence of the H-QHD solutions and their regularity to be totally different from both the 4th-order uniform elliptic equation (the QHD model) and the 2th-order uniform elliptic equation (the HD model), and causes us some essential difficulties. To overcome such obstacles, we first introduce a sequence of smooth approximating functions $Q_q(x) \geq q > 0$ satisfying $Q_q \to Q$ as $q \to 0$, which modifies the governing equation to be uniformly elliptic, then we prove the existence of the solutions to the modified H-Q_qHD equation, where, when $q = 0$, we denote the H-Q_0HD as the H-QHD. Then, by rigorously proving the uniform boundedness of the solutions for the H-Q_qHD model with respect to q and by carrying out compactness analysis, we may expect that the approximating (smooth) solutions of the H-Q_qHD model will weakly converge to their target functions, which are just the weak solutions of the original H-QHD model. To the best of our knowledge, this paper is the first framework to treat the equation with regional degeneracy of ellipticity.

The present work is divided into 7 sections. In Section 2 we recall the model introduced in [9] for quantum drift-diffusion model, and derive the new model of hybrid quantum hydrodynamic equation. In Section 3 we state our main theorems: the existence of the smooth solutions to the H-Q_qHD model, the hybrid limit of the H-Q_qHD solutions, the existence of the weak solutions for the H-QHD model; and the zero-space-limit to the weak solutions of the H-QHD model. In Section 4 we prove the existence and uniqueness of the smooth solutions to the H-Q_qHD problem. Furthermore, in Section 5 we show the convergence of the H-Q_qHD model to the fully hybrid model as $Q_q \to Q$. In Section 6 we consider the zero-space-limit as $\lambda \to 0$. Finally, in Section 7 we present some numerical tests in order to validate our model.

2. Derivation of hybrid quantum hydrodynamic (H-QHD) model

H-QHD model. Here we briefly summarize the result described in [9], in order to justify the model and the results presented in this paper from the physical point of view. For more details, see the original paper and the references therein. The main assumption in [9] was that the energy of the electrons depends on ∇n (n is the density of electrons) not in whole domain, as prescribed in [1], but just in a well defined sub-domain. The equation of linear momentum balance for the electrons (neglecting the inertia) reads

$$-\nabla (V+F) + E^e = 0. \tag{1}$$

Here V is the electric potential, F is a generalized chemical potential, E^e is the lowest order for the drag force, which is a function of the electrons velocity v, and the mobility μ, namely

$$E^e = -\frac{v}{\mu}. \tag{2}$$

To include the quantum lowest-order effect, we need to modify the expression of the density energy e. In this general introduction of the model equations, we assume $\Omega \subset \mathbb{R}^3$. We introduce the smooth function $Q:\Omega \to [0,1]$, which indicates where the internal energy depends on the gradient of the charge density. We call $Q(x,y,z)$ the quantum effect function. Therefore, e depends on the charge density n (as in the classical model) and on $\xi := \nabla n$, as follows

$$e_Q := e_Q(n,\xi) = \ln n - Q\frac{\varepsilon^2}{2}\frac{\xi\cdot\xi}{n^2}, \tag{3}$$

where ε is the scaled Plank's constant.

Simply speaking, we have $Q=0$ in the region without quantum effects (classical) and $Q>0$ in the quantum region. The size of the transition region between classical and quantum domain should be approximately $1/\varepsilon$ in order to guarantee a strong coupling between the two domains. Under this prescription, the transition between classical and quantum regimes should be fast enough to not affect the electrons behaviour in the classical and in the quantum domains. Note that, in the standard mathematical approach to the hybrid models, there is no semiclassical zone, thus Q becomes the Heaviside function.

The generalized chemical potential F can be written in terms of e_Q as follows

$$F = \frac{\partial (ne_Q)}{\partial n} - \nabla\cdot\left[n\frac{\partial e_Q}{\partial \nabla n}\right]. \tag{4}$$

From (3), the first term on the right-hand side of (4) can be simply calculated as

$$\frac{\partial\,(ne_Q)}{\partial n} = (\ln n) + \frac{\varepsilon^2}{2} Q \frac{\nabla n \cdot \nabla n}{n^2} + 1, \tag{5}$$

and the second term is

$$\nabla \cdot \left[n \frac{\partial e_Q}{\partial \nabla n} \right] = \nabla \cdot \left[n \frac{\partial}{\partial \nabla n} \left(\ln n + \frac{\varepsilon^2}{2} Q \frac{\nabla n \cdot \nabla n}{n^2} \right) \right], \tag{6}$$

where

$$\nabla \cdot \left[n \frac{\partial}{\partial \nabla n} (\ln n) \right] = \nabla n \cdot \frac{\partial}{\partial \nabla n} (\ln n) + n \frac{\partial}{\partial \nabla n} (\nabla (\ln n)) = 1, \tag{7}$$

and

$$\nabla \cdot \left[n \frac{\partial}{\partial \nabla n} \left(\frac{\varepsilon^2}{2} Q \frac{\nabla n \cdot \nabla n}{n^2} \right) \right] = \varepsilon^2 \left(Q \left(\frac{\Delta n}{n} - \frac{\nabla n \cdot \nabla n}{n^2} \right) + \frac{\nabla Q \cdot \nabla n}{n} \right). \tag{8}$$

Summing up (5)-(8), we get

$$F = \ln n - \varepsilon^2 \left(Q \left(\frac{\Delta n}{n} - \frac{1}{2} \frac{\nabla n \cdot \nabla n}{n^2} \right) + \frac{\nabla Q \cdot \nabla n}{n} \right),$$

or equivalently

$$F = \ln n + 2\varepsilon^2 \left(Q \left(\frac{\Delta(\sqrt{n})}{\sqrt{n}} \right) + \frac{\nabla Q \cdot \nabla(\sqrt{n})}{\sqrt{n}} \right). \tag{9}$$

The formula above provides us with a new intrinsically hybrid representation of the Bohm potential:

$$B[n](x) = 2\varepsilon^2 \left(Q \left(\frac{\Delta(\sqrt{n})}{\sqrt{n}} \right) + \frac{\nabla Q \cdot \nabla(\sqrt{n})}{\sqrt{n}} \right). \tag{10}$$

Introducing (10) in the stationary hydrodynamical equation, we get

$$2\varepsilon^2 n \nabla \left(Q \frac{\Delta \sqrt{n}}{\sqrt{n}} + \frac{\nabla Q \cdot \nabla(\sqrt{n})}{\sqrt{n}} \right) - T \nabla n + \nabla \cdot \left(\frac{J \otimes J}{n} \right) + n \nabla V = \frac{J}{\tau}, \tag{11}$$

where $\tau > 0$ is the relaxation time and J is the current density.

From now on we restrict ourselves to the steady-state one-dimensional case, with $\Omega = [0,1]$. Therefore the previous equation, already divided by n, reads like

$$\begin{cases} 2\varepsilon^2 \left(Q \dfrac{(\sqrt{n})_{xx}}{\sqrt{n}} + Q' \dfrac{(\sqrt{n})_x}{\sqrt{n}} \right)_x - \left(T \ln n + \dfrac{J^2}{2n^2} \right)_x + V_x = \dfrac{J}{\tau n}, \\ J = \mathit{constant}. \end{cases} \tag{12}$$

The above equation is called the hybrid quantum hydrodynamical equation (H-QHD). We consider (12) coupled with the Poisson equation:

$$\lambda^2 V_{xx} = n - C, \tag{13}$$

which provides a description of the self-consistent electrical potential V. As usual $\tau > 0$, $\lambda > 0$ and $T > 0$ are the parameters for the scaled relaxation time, the scaled Debye length, and the scaled temperature respectively. Finally, $C \in L^2(\Omega)$ models the fixed charge background ions in the semiconductor crystal and it is assumed $C(x) \geq C_0 > 0$ for all x in $[0,1]$.

Boundary conditions. The boundary conditions for the stationary problem (12)-(13) are supplemented as follows

$$\textit{contact boundary}: \; n(0) = n(1) = 1, \tag{14}$$

$$\textit{insulation boundary}: \; n_x(0) = n_x(1) = 0, \tag{15}$$

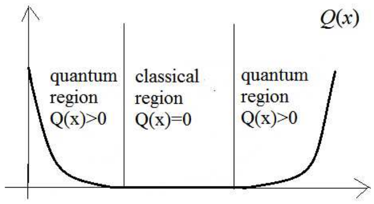

FIGURE 1. Quantum effect function $Q(x)$.

and

$$\textit{electric potential condition}: \; V(0) = V_0, \; V(1) = V_1, \tag{16}$$

where

$$V_0 = -2\varepsilon^2 Q(0)(\sqrt{n})_{xx}(0) + \frac{J^2}{2}. \tag{17}$$

These conditions, as shown in the previous studies for the classical HD model [10, 13, 20] with (14) (or (15)) and (16), and the QHD model [14, 18, 19, 23] with (14)-(16), are necessary to make the steady-state system (12)-(13) well-posed. Let us integrate (12) with respect to x. Then, in view of (16) and (17), we have

$$\begin{aligned} V(x) \;\; &= \;\; -2\varepsilon^2 Q \frac{(\sqrt{n})_{xx}}{\sqrt{n}} - 2\varepsilon^2 Q' \frac{(\sqrt{n})_x}{\sqrt{n}} + \frac{J^2}{2n^2} + T \ln n \\ &\quad\;\; -\frac{J}{\tau} \int_0^x \frac{1}{n} dx, \end{aligned} \tag{18}$$

and we further obtain, by using the boundary conditions (14) and (16), that

$$V_1 = V(1) = -2\varepsilon^2 Q(1)(\sqrt{n})_{xx}(1) + \frac{J^2}{2} - \frac{J}{\tau} \int_0^1 \frac{1}{n} dx. \tag{19}$$

So, for a fixed constant J, the boundary value$V(1) = V_1$ can be explicitly obtained from (19). In this study, we propose the constant J as a parameter and leave $V(1)$ to be a number automatically determined by (19). Namely, throughout the paper, we consider the following boundary problem to the steady-state H-QHD model (12)-(13):

$$\begin{cases} 2\varepsilon^2 \left(Q \frac{(\sqrt{n})_{xx}}{\sqrt{n}} + Q' \frac{(\sqrt{n})_x}{\sqrt{n}} \right)_x - \left(T \ln n + \frac{J^2}{2n^2} \right)_x + V_x = \frac{J}{\tau n}, \\ \lambda^2 V_{xx} = n - C, \\ n(0) = n(1) = 1, \quad n_x(0) = n_x(1) = 0, \quad V(0) = V_0, \quad J = J_0. \end{cases} \tag{20}$$

Here, we treat the really hybrid case: $Q = 0$ in some part of the region and $Q > 0$ in the other part of the region. An example for Q is shown in Figure 1 and in this case the semiconductor device looks like what presented in Figure 2. This is the most physically significant but also mathematically challenging case, and it will be our main target.

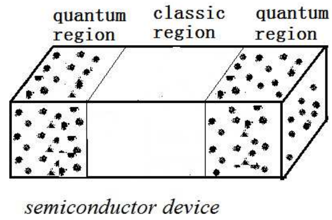

FIGURE 2. Device.

3. Main results

In this section, we are looking for the solution to the hybrid QHD (20) under consideration, with $Q = 0$ in the classical region and $Q > 0$ in the quantum region.

Differentiating $(20)_1$ with respect to x and substituting $V_{xx} = (n - C)/\lambda^2$ from $(20)_2$ to the resultant equation, we first have the following fourth-order differential equation for electronic density n:

$$
(21) \qquad \begin{cases} 2\varepsilon^2 \left(Q \dfrac{(\sqrt{n})_{xx}}{\sqrt{n}} + Q' \dfrac{(\sqrt{n})_x}{\sqrt{n}} \right)_{xx} - \left(\left(\dfrac{T}{n} - \dfrac{J^2}{n^3} \right) n_x \right)_x \\ \qquad + \dfrac{1}{\lambda^2}(n - C) = -\dfrac{J}{\tau n^2} n_x, \\ \lambda^2 V_{xx} = n - C, \\ n(0) = n(1) = 1, \quad n_x(0) = n_x(1) = 0, \quad V(0) = V_0, \quad J = J_0. \end{cases}
$$

As mentioned before, instead of $V(1) = V_1$, we consider $J = J_0$ as an equivalent boundary condition. Throughout the paper, we always assume $\varepsilon < 1$. Here and after, we pay our attention to the case of subsonic flow. To keep the flow subsonic, we need the uniform ellipticity for the above equation, that is, we must restrict it by setting:

$$
(22) \qquad \text{velocity of the flow} := \frac{|J|}{n} < \sqrt{p'(n)} = \sqrt{T} =: \text{sound speed},
$$

where $p(n) = nT$ is the pressure. This is equivalent to have

$$
(23) \qquad \frac{T}{n} - \frac{J^2}{n^3} > 0, \ \textit{i.e.}, \ T > \frac{J^2}{n^2}, \ \text{for } n \text{ under consideration},
$$

which implies the uniform ellipticity of the operator $((\frac{T}{n} - \frac{J^2}{n^3}) n_x)_x$. Both (22) and (23) are equivalent to

$$
(24) \qquad \text{subsonic condition:} \ \ n > \frac{|J_0|}{\sqrt{T}} =: n_\star.
$$

Here $J \equiv J_0$ (a constant).

For the subsonic boundary

$$
n(0) = n(1) = 1 > \frac{|J_0|}{\sqrt{T}},
$$

we need the following compatibility condition:

$$
(25) \qquad |J_0| < \sqrt{T}.
$$

Similarly, we also need a subsonic condition for the doping profile $C(x)$:

$$
(26) \qquad C_0 := \min_{x \in [0,1]} C(x) > n_\star = \frac{|J_0|}{\sqrt{T}}.
$$

Notice that, $Q = 0$ in some part of the domain $[0, 1]$. This makes the 4th order elliptic equation (21) to be degenerate regionally, and causes us a real difficulty to prove the existence of the solution. In order to overcome such an obstacle, for fixed Debye parameter $\lambda > 0$, we first look for the solution $(n_q, V_q)(x)$ to (21) where, instead of Q, we consider a strictly positive function $Q_q \geq q > 0$, such that $Q_q \to Q$ when $q \to 0$ (see Theorem 3.1 for a complete list of the properties that Q_q must satisfy). Then, by taking hybrid limit $q \to 0$, we expect that the solution $(n_q, V_q)(x)$ of the modified QHD converges to the really hybrid solution $(n, V)(x)$ in the weak sense. Furthermore, we will look for the limit solution $(n, V)(x) \to (C, \bar{V})(x)$ as the Debye length $\lambda \to 0$, for a fixed q.

Let $w = \sqrt{n}$, then (21) is reduced to

$$\begin{cases} 2\varepsilon^2 \left(Q\dfrac{w_{xx}}{w} + Q'\dfrac{w_x}{w}\right)_{xx} - 2\left(\left(T - \dfrac{J^2}{w^4}\right)\dfrac{w_x}{w}\right)_x \\ \qquad + \dfrac{1}{\lambda^2}(w^2 - C) = -\dfrac{2J}{\tau w^3} w_x, \\ \lambda^2 V_{xx} = w^2 - C, \\ w(0) = w(1) = 1, \quad w_x(0) = w_x(1) = 0, \quad V(0) = V_0, \quad J = J_0. \end{cases} \tag{27}$$

Here and after, we will mainly focus on the above system (27).

First of all, we consider the modified hybrid QHD equations (H-Q_qHD) where we replace $Q(x)$ by the strictly positive function $Q_q(x)$, and prove the existence of solutions. For fixed $\lambda > 0$, let $(w_q, V_q)(x)$ be the solutions to the so-called H-Q_qHD equations

$$\begin{cases} 2\varepsilon^2 \left(Q_q\frac{(w_q)_{xx}}{w_q} + Q_q'\frac{(w_q)_x}{w_q}\right)_{xx} - 2\left(\left(T - \frac{J^2}{w_q^4}\right)\frac{(w_q)_x}{w_q}\right)_x \\ \qquad + \frac{1}{\lambda^2}(w_q^2 - C) = -\frac{2J}{\tau w_q^3}(w_q)_x, \\ \lambda^2 (V_q)_{xx} = w_q^2 - C, \\ w_q(0) = w_q(1) = 1, \ (w_q)_x(0) = (w_q)_x(1) = 0, \ V_q(0) = V_0, \ J = J_0. \end{cases} \tag{28}$$

We first establish the existence and uniqueness of the solutions for the modified H-Q_qHD equation (28).

Theorem 3.1 (Existence of H-Q_qHD solution)**.** *Under the subsonic conditions* (25) *and* (26)*, assume that $Q_q(x)$ is a non-negative, smooth, and bounded function defined on $\Omega = [0, 1]$ such that*

$$0 < q \leq Q_q \leq 1, \quad \alpha := \max(\|Q_q'\|_\infty, \|Q_q''\|_\infty) < \infty \quad \textit{for all} \quad x \in \Omega, \tag{29}$$

and

$$\varepsilon^2 \max_{x\in\Omega} \frac{|Q_q'|^2}{Q_q} < 4\Big(T - \frac{J_0^2}{\underline{n}^2}\Big), \tag{30}$$

where $\underline{n} := \min\{1, C_0\}$. Then the solution of (28) *exists and $(w_q, V_q) \in H^4(\Omega) \times H^2(\Omega)$.*

Remark 3.2. *To guarantee the condition* (30)*, we may take $\varepsilon \ll 1$. This is usually verified by quantum devices, as $\varepsilon \propto \frac{L\hbar}{\sqrt{mk_BT}}$, where $\hbar$ is the reduced Plank's constant (*$1.055 \cdot 10^{-34}$ *J s), L is the device characteristic length (*125 *nm), m is the effective electron mass (* $\approx 0.63 \cdot 10^{-31}$ *kg) and k_B (* $1.380 \cdot 10^{-23}$ *J/K). Therefore, in many physical situation, $\varepsilon^2 \approx 10^{-3}$.*

Theorem 3.3 (Uniqueness of H-Q_qHD solution)**.** *Assume* (29)*,* (30) *and* (22)*. Let $\varepsilon + |J| \ll 1$, both are independent of q, then the boundary value problem* (28) *admits a unique solution.*

The main purpose in this paper is to investigate the existence of the solution to the really hybrid QHD model (27). Since $Q = 0$ in some part of the domain Ω, this leads the H-QHD system (27) to be regionally degenerate in the 4th order ellipticity as we mentioned before, and it cannot possess smooth solutions. We now define its weak solutions as follows.

Definition 3.4. *A pair of functions* $(w, V)(x)$ *is said to be a weak solution of* (27)*, if it holds*

$$2\varepsilon^2 \int_0^1 \Big(Q\frac{w_{xx}}{w} + Q'\frac{w_x}{w}\Big)\phi_{xx}dx + 2\int_0^1 \Big(\Big(T - \frac{J^2}{w^4}\Big)\frac{w_x}{w}\Big)\phi_x dx + \int_0^1 \frac{1}{\lambda^2}(w^2 - C)\phi dx + \int_0^1 \frac{J}{\tau w^2}\phi_x dx = 0, \tag{31}$$

and

$$\int_0^1 V\phi\, dx = -2\varepsilon^2\int_0^1 Q\frac{w_{xx}}{w}\phi\, dx - 2\varepsilon^2\int_0^1 Q'\frac{w_x}{w}\phi\, dx + \int_0^1 \frac{J^2}{2w^4}\phi\, dx + 2T\int_0^1(\ln w)\phi\, dx - \frac{J}{\tau}\int_0^1\Big(\int_0^x \frac{1}{{w_q}^2(s)}\, ds\Big)\phi\, dx, \tag{32}$$

for any $\phi \in C_0^\infty(\Omega)$.

For the given hybrid quantum effect function $0 \le Q \le 1$, we make the following approximation: let $\{Q_q\}$, $q \in \mathbb{R}_+$ be a q-dependent sequence satisfying:

$$\begin{cases} Q_q \to Q, \quad Q'_q \to Q' \quad \text{uniformly in} \quad \Omega, \quad \text{for} \quad q \to 0, \\ \|Q'_q\|_{L^2} \le \bar{K}, \ \text{uniformly in } q, \\ \varepsilon^2|Q'_q|^2 < Q_q\Big(T - \frac{J^2}{\underline{n}^2}\Big) \ \text{for all } x \in [0,1] \text{ and for all } q \in \mathbb{R}_+ \end{cases} \tag{33}$$

where $\underline{n} > n_\star = \frac{|J|}{\sqrt{T}}$, and $n_\star$ as introduced in (24).

Remark 3.5. *Condition* $(33)_3$ *essentially means that* $|Q'_q|^2/Q_q$ *remains bounded when* $Q_q \to 0$. *We observe that this condition is verified if* Q_q *behaves at least as* $|x - x_0|^m$, *for all* $m \ge 2$, *when* $x \to x_0$. *Finally, we notice that the assumption* $(33)_3$ *is stronger than* (30)*, required in the first part of the paper for* $q > 0$.

Example 3.6. *As an example of a given hybrid quantum effect function, we consider*

$$Q(x) = \begin{cases} 16(x - \frac14)^2, & 0 \le x < \frac14, \\ 0, & \frac14 \le x \le \frac34, \\ 16(x - \frac34)^2, & \frac34 < x \le 1, \end{cases} \tag{34}$$

which satisfies $0 \le Q \le 1$. *Then we may construct a sequence* $\{Q_q\}$ *as*

$$Q_q(x) = \begin{cases} q + 16(1-q)(x - \frac14)^2, & 0 \le x < \frac14, \\ q, & \frac14 \le x \le \frac34, \\ q + 16(1-q)(x - \frac34)^2, & \frac34 < x \le 1, \end{cases} \tag{35}$$

where $0 < q < 1$*. It can be easily verified that* $0 < q \le Q_q \le 1$ *and*

$$\begin{aligned} |Q_q(x) - Q(x)| &= \begin{cases} q[1-16(x-\frac{1}{4})^2], & 0 \le x < \frac{1}{4} \\ q, & \frac{1}{4} \le x \le \frac{3}{4} \\ q[1-16(x-\frac{3}{4})^2], & \frac{3}{4} < x \le 1 \end{cases} \\ &\le q, \quad \textit{for all } x \in [0,1] \end{aligned}$$

and

$$\begin{aligned} |Q_q'(x) - Q'(x)| &= \begin{cases} 32q|x-\frac{1}{4}|, & 0 \le x < \frac{1}{4} \\ 0, & \frac{1}{4} \le x \le \frac{3}{4} \\ 32q|x-\frac{3}{4}|, & \frac{3}{4} < x \le 1 \end{cases} \\ &\le 8q, \quad \textit{for all } x \in [0,1], \end{aligned}$$

which imply the following uniform convergence

$$Q_q \to Q, \quad Q_q' \to Q' \quad \textit{uniformly in } x \in [0,1], \ \textit{as } q \to 0.$$

Moreover, $\|Q_q'\|_{L^2} = \sqrt{\frac{32}{3}}(1-q) \le \sqrt{\frac{32}{3}}$*, and the condition* $(33)_3$ *is equivalent to* $\varepsilon^2(1-q)^2 < \frac{1}{64}(T - \frac{J^2}{\underline{n}^2})$*. This holds automatically by taking* $\varepsilon^2 < \frac{1}{64}(T - \frac{J^2}{\underline{n}^2})$*.*

Now we state the existence of the weak solution to the hybrid quantum hydrodynamic equation (27).

Theorem 3.7 (Hybrid limits and existence of H-QHD solution)**.** *Under the subsonic conditions* (25) *and* (26)*, for given hybrid quantum effect function* $Q \in C^1(0,1)$ *with* $0 \le Q \le 1$*, let us construct a sequence* $\{Q_q\}$ *satisfying* (33)*, and let* $(w_q, V_q)(x)$ *be the solutions to the equation* (28) *corresponding to these selected approximating functions* Q_q*. Then there exists a pair of functions* $(w, V)(x)$ *such that the sequence* $(w_q, V_q)(x)$ *converges to* (w, V) *as follows*

$$\begin{cases} w_q \rightharpoonup w & \textit{in } H^1(\Omega), \\ w_q \to w & \textit{in } C^0(\Omega), \\ V_q \rightharpoonup V & \textit{in } L^2(\Omega), \end{cases} \qquad \textit{as } q \to 0. \tag{36}$$

In particular, such a pair of limits $(w, V)(x)$ *is the weak solution of the H-QHD system* (27)*.*

Finally, we are looking for the zero-space-charge limits by taking the scaled Debye length $\lambda \to 0$. From (20), when we take $\lambda \to 0$, we then formally expect $w_\lambda(x) \to \sqrt{C(x)}$. In fact, we have the following convergence result.

Theorem 3.8 (Zero-space-charge limits for H-Q_qHD)**.** *Let* C*,* $Q \in C^2(\Omega)$ *be given functions such that*

$$\begin{cases} C(0) = C(1) = 1, \ C_x(0) = C_x(1) = 0, \\ 0 < q \le Q(x) \le 1, \\ \varepsilon^2 \max_{x\in\Omega} \frac{|Q'(x)|^2}{Q(x)} < \left(1 - \frac{\varepsilon^2}{8}\right)\left(T - \frac{J^2}{\underline{n}^2}\right) \Big/ 4\left(1 + \frac{\varepsilon^2}{2}\left(1 - \frac{\varepsilon^2}{8}\right)\right), \end{cases} \tag{37}$$

where $\underline{n} > n_\star = |J|/\sqrt{T}$*. Let* (w_λ, V_λ) *be the solution to the problem* (28)*. Then*

$$\begin{aligned} w_\lambda(x) &\rightharpoonup w := \sqrt{C(x)} \quad \textit{in} \quad H^2(\Omega) \\ w_\lambda(x) &\to w := \sqrt{C(x)} \quad \textit{in} \quad C^1(\bar{\Omega}) \\ V_\lambda(x) &\rightharpoonup \tilde{V}(x) \quad \textit{in} \quad L^2(\Omega), \end{aligned}$$

where

$$\tilde{V}(x) = -2\varepsilon^2\left(Q\frac{\sqrt{C}_{xx}}{\sqrt{C}} + Q'\frac{\sqrt{C}_x}{\sqrt{C}}\right) + \frac{J^2}{2C^2} + 2T\ln C + \frac{J}{\tau}\int_0^x \frac{ds}{C(s)}. \tag{38}$$

4. Existence and uniqueness of H-Q_qHD solution

In order to get the existence of the weak solutions (w, V) for the really hybrid quantum hydrodynamic equations (27), as mentioned before, we propose to construct an approximating sequence (w_q, V_q) for the modified hybrid quantum hydrodynamic equations (28). So, in this section, we discuss fixed-point arguments, and prove the existence and uniqueness of the weak solutions to the stationary model (28) with $Q = Q_q \geq q > 0$.

Notice that (28) can be written as

$$2\varepsilon^2\left(Q_q\frac{(w_q)_{xx}}{w_q} + Q_q'\frac{(w_q)_x}{w_q}\right)_{xx} - 2T(\ln w_q)_{xx} - \left(\frac{J^2}{2w_q^4}\right)_{xx} = -\frac{w_q^2 - C}{\lambda^2} + \left(\frac{J}{\tau w_q^2}\right)_x, \tag{39}$$

subjected to the following boundary conditions

$$(w_q)_x(0) = (w_q)_x(1) = 0, \qquad w_q(0) = w_q(1) = 1, \qquad J = J_0. \tag{40}$$

In order to get the existence of the solution, first of all let us prove the following a priori estimates.

Lemma 4.1 (A priori estimates)**.** *Under the subsonic conditions* (25) *and* (26)*, assume that* Q_q *satisfies* (29) *and* (30)*. Let* $w_q \in H^2(\Omega)$ *be the solution of the problem* (39)-(40)*. Then the solution* $w_q(x)$ *is in the subsonic region*

$$w_q(x) \geq \sqrt{\underline{n}} > \sqrt{n_\star} \quad \textit{for } x \in [0,1], \tag{41}$$

and is bounded by

$$\|w_q\|_{L^\infty(\Omega)} \leq w_M. \tag{42}$$

Moreover

$$\varepsilon^2\underline{c_1}\int_0^1 (w_q)_{xx}^2\,dx + \underline{c_2}\int_0^1 (w_q)_x^2\,dx \leq K, \tag{43}$$

where $w_M \geq \sqrt{\underline{n}}$*,* $\underline{c}_1 > 0$*,* $\underline{c}_2 > 0$*, and* $K > 0$ *are constants.*

Proof. Multiplying (39) by $(w_q - 1) \in H_0^1(\Omega)$ and integrating it on the whole domain, we have

$$
\begin{aligned}
(44) \qquad & 2\varepsilon^2 \int_0^1 Q_q \frac{(w_q)_{xx}^2}{w_q}\, dx + 2\int_0^1 \left(T - \frac{J^2}{w_q^4}\right) \frac{(w_q)_x^2}{w_q}\, dx \\
& + 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_q)_x (w_q)_{xx}}{w_q}\, dx \\
& = -\frac{1}{\lambda^2} \int_0^1 (w_q^2 - 1)(w_q - 1)\, dx \\
& + \frac{1}{\lambda^2} \int_0^1 (C-1)(w_q - 1)\, dx - \int_0^1 \frac{J}{\tau w_q^2} (w_q)_x\, dx \\
& =: I_1 + I_2 + I_3.
\end{aligned}
$$

By using the Cauchy inequality, we further have

$$
\begin{aligned}
(45) \qquad I_1 + I_2 \le & -\frac{1}{\lambda^2} \int_0^1 (w_q - 1)^2 (w_q + 1)\, dx + \frac{1}{2\lambda^2} \int_0^1 (C-1)^2\, dx \\
& + \frac{1}{2\lambda^2} \int_0^1 (w_q - 1)^2\, dx \\
\le & -\frac{1}{\lambda^2} \int_0^1 (w_q - 1)^2 \left(w_q + \frac{1}{2}\right) dx + \frac{1}{2\lambda^2} \int_0^1 (C-1)^2\, dx.
\end{aligned}
$$

Clearly,

$$
(46) \qquad I_3 = \int_0^1 \frac{J}{\tau w_q^2} (w_q)_x dx = \frac{J}{\tau w_q} \Big|_{x=0}^{x=1} = 0.
$$

Combining (44)-(46), we obtain

$$
\begin{aligned}
(47) \qquad & 2\varepsilon^2 \int_0^1 Q_q \frac{(w_q)_{xx}^2}{w_q}\, dx + 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_q)_x (w_q)_{xx}}{w_q}\, dx \\
& + 2\int_0^1 \left(T - \frac{J^2}{w_q^4}\right) \frac{(w_q)_x^2}{w_q}\, dx + \frac{1}{\lambda^2} \int_0^1 (w_q - 1)^2 \left(w_q + \frac{1}{2}\right) dx \\
& \le \frac{1}{2\lambda^2} \int_0^1 (C-1)^2\, dx.
\end{aligned}
$$

Obviously, the first three terms of the left hand side in (47) can be read as a quadratic form

$$
\begin{aligned}
(48) \qquad & \int_0^1 \left[2\varepsilon^2 Q_q \frac{(w_q)_{xx}^2}{w_q} + 2\varepsilon^2 Q_q' \frac{(w_q)_x (w_q)_{xx}}{w_q} + 2\Big(T - \frac{J^2}{w_q^4}\Big) \frac{(w_q)_x^2}{w_q}\right] dx \\
& =: \int_0^1 (\mathcal{A}_1 \frac{(w_q)_{xx}^2}{w_q} + \mathcal{B}_1 \frac{(w_q)_x (w_q)_{xx}}{w_q} + \mathcal{C}_1 \frac{(w_q)_x^2}{w_q}) dx \\
& \ge c_1 \int_0^1 \frac{(w_q)_{xx}^2}{w_q} dx + c_2 \int_0^1 \frac{(w_q)_x^2}{w_q} dx
\end{aligned}
$$

for some positive constants c_1 and c_2. It is positively definite because $\mathcal{B}_1^2 - 4\mathcal{A}_1\mathcal{C}_1 < 0$. In fact, from the condition (30), we have

$$\begin{aligned}\mathcal{B}_1^2 - 4\mathcal{A}_1\mathcal{C}_1 &= 4\varepsilon^2\Big[\varepsilon^2|Q_q'|^2 - 4Q_q\left(T - \frac{J^2}{w_q^4}\right)\Big] \\ &< 4\varepsilon^2\Big[\varepsilon^2|Q_q'|^2 - 4Q_q\left(T - \frac{J^2}{\underline{n}^2}\right)\Big] < 0, \quad \text{for } w_q \geq \sqrt{\underline{n}}.\end{aligned}$$

This is always true for a small $\varepsilon > 0$ and should make sense because the condition $\varepsilon \ll 1$ is usually verified by quantum devices.

Substituting (48) into (47), we obtain

$$c_1\int_0^1 \frac{(w_q)_{xx}^2}{w_q}\,dx + c_2\int_0^1 \frac{(w_q)_x^2}{w_q}\,dx \leq \frac{1}{2\lambda^2}\int_0^1 [C-1]^2 dx =: K_0. \tag{49}$$

Moreover, from (47), as $\frac{(w_q)_x^2}{w_q} = 4[(\sqrt{w_q})_x]^2$, we get

$$c_2\int_0^1 [(\sqrt{w_q}-1)_x]^2\,dx \leq K_0. \tag{50}$$

Recalling that for all $x \in (0,1)$ and $f \in H^1(\Omega)$

$$|f(x)| \leq |f(0)| + \int_0^x |f_x(s)|ds \leq |f(0)| + \|f_x\|_{L^2},$$

then (50) easily implies $\|\sqrt{w_q}-1\|_\infty \leq K_1$ where $K_1 = \sqrt{\frac{K_0}{c_2}}$. Thus, (42) is verified by setting $w_M = (1+K_1)^2$, and (43) follows from (49) in view of (42), where $\underline{c_1}$ and $\underline{c_2}$ are positive constants.

Finally, we prove $w_q \geq \sqrt{\underline{n}}$ for all $x \in \Omega$, where $\underline{n} = \min\{1, C_0\}$ is defined in (30). Namely, we prove that the flow under consideration is, indeed, subsonic. Let $(w_q - \sqrt{\underline{n}})^- := \min(0, w_q - \sqrt{\underline{n}})$. Since $w_q|_{\partial\Omega} = 1 > \sqrt{\underline{n}}$, so $(w_q - \sqrt{\underline{n}})^-|_{\partial\Omega} = 0$, and $(w_q - \sqrt{\underline{n}})^- \in H_0^1(\Omega)$. Now, let us consider again the weak formulation of the problem (39) by using the test function $(w_q - \sqrt{\underline{n}})^- = \min(0, w_q - \sqrt{\underline{n}})$, which gives

$$\begin{aligned}&2\varepsilon^2\int_0^1 Q_q\frac{((w_q-\sqrt{\underline{n}})^-)_{xx}^2}{w_q}\,dx + 2\int_0^1\left(T - \frac{J^2}{w_q^4}\right)\frac{((w_q-\sqrt{\underline{n}})^-)_x^2}{w_q}\,dx \\ &+ 2\varepsilon^2\int_0^1 Q_q'\frac{((w_q-\sqrt{\underline{n}})^-)_x((w_q-\sqrt{\underline{n}})^-)_{xx}}{w_q}\,dx \\ &= -\frac{1}{\lambda^2}\int_0^1 (w_q^2 - \sqrt{\underline{n}}^2)(w_q-\sqrt{\underline{n}})^-\,dx \\ &+ \frac{1}{\lambda^2}\int_0^1 (C-\underline{n})(w_q-\sqrt{\underline{n}})^-\,dx - \int_0^1 \frac{J}{\tau w_q^2}((w_q-\sqrt{\underline{n}})^-)_x\,dx \\ &=: L_1 + L_2 + L_3.\end{aligned} \tag{51}$$

One has

$$\begin{aligned}L_1 + L_2 \leq &-\frac{1}{\lambda^2}\int_0^1 ((w_q-\sqrt{\underline{n}})^-)^2(w_q+\sqrt{\underline{n}})\,dx \\ &+\frac{1}{\lambda^2}\int_0^1 (C-\underline{n})(w_q-\sqrt{\underline{n}})^-\,dx.\end{aligned} \tag{52}$$

We observe that, Ω can be written as a disjoint union of $\Omega^\pm$ and isolated points, where $\Omega^+ = \cup_i {\Omega_i}^+$, $\Omega^- = \cup_i {\Omega_i}^-$ and

$${\Omega_i}^+ = \{\forall x \in \Omega \text{ such that } w_q \geq \sqrt{\underline{n}}\}, \quad {\Omega_j}^- = \{\forall x \in \Omega \text{ such that } w_q < \sqrt{\underline{n}}\}.$$

One has

$$
\begin{aligned}
L_3 &= -\int_0^1 \frac{J}{\tau w_q^2}((w_q - \sqrt{\underline{n}})^-)_x \, dx \\
&= -\sum_i \int_{\Omega_i{}^+} \frac{J}{\tau w_q^2}((w_q - \sqrt{\underline{n}})^-)_x \, dx - \sum_j \int_{\Omega_j{}^-} \frac{J}{\tau w_q^2}((w_q - \sqrt{\underline{n}})^-)_x \, dx.
\end{aligned}
$$

Clearly the first sum is zero, therefore

$$
L_3 = -\sum_j \int_{\Omega_j{}^-} \frac{J}{\tau w_q^2}((w-k)^-)_x \, dx.
$$

Consequently, we compute L_3 on each interval $\Omega_j{}^-$. Without loss of generality we just consider a single interval $\Omega_j{}^- = (a_j, b_j)$, which is properly contained in the open interval $(0,1)$. The inequality (43) implies $w_q \in H^2(0,1) \subset C^0(0,1)$, namely w_q is a continuous function in $[a_j, b_j]$. This implies that $w_q(a_j) = w_q(b_j) = \sqrt{\underline{n}}$, and thus

$$
L_3 = -\int_{b_j}^{a_j} \frac{J}{\tau w_q^2}(w_q)_x \, dx = \frac{J}{\tau w_q(b_j)} - \frac{J}{\tau w_q(a_j)} = 0. \tag{53}
$$

Considering (52), and (53) and using (30), there exist non negative constants $\underline{c_1}$, $\underline{c_2}$ and $\underline{c_3}$ such that

$$
\begin{aligned}
&\underline{c_1} \int_0^1 ((w_q - \sqrt{\underline{n}})^-)_{xx}^2 \, dx + \underline{c_2} \int_0^1 ((w_q - \sqrt{\underline{n}})^-)_x^2 \, dx \\
&+ \underline{c_3} \int_0^1 ((w_q - \sqrt{\underline{n}})^-)^2 (w_q + \sqrt{\underline{n}}) \, dx \\
&\le \frac{1}{\lambda^2} \int_0^1 (C - \underline{n})(w_q - \sqrt{\underline{n}})^- \, dx,
\end{aligned} \tag{54}
$$

which implies $(w_q - \sqrt{\underline{n}})^- = 0$ for all $x \in [0,1]$, namely, $w_q \ge \sqrt{\underline{n}} > 0$ for all $x \in [0,1]$. The inequality (41) follows from (54). $\square$

Lemma 4.2. *Under the assumption of Lemma 4.1, the variable u_q, defined as $u_q = \ln n_q$, verifies the following estimate*

$$
\begin{aligned}
&\varepsilon\sqrt{q}\|(u_q)_{xx}\|_{L^2(\Omega)} + \sqrt{T - J^2/\underline{n}}\ \|(u_q)_x\|_{L^2(\Omega)} \\
&\le \varepsilon\|\sqrt{Q_q}(u_q)_{xx}\|_{L^2(\Omega)} + \sqrt{T - J^2/\underline{n}}\ \|(u_q)_x\|_{L^2(\Omega)} \le K_0.
\end{aligned} \tag{55}
$$

Proof. We write (28) in the new variable $u_q = 2\ln w_q$ and we derive it with respect to x

$$
\begin{aligned}
&\varepsilon^2 \left(Q_q \left((u_q)_{xx} + \frac{(u_q)_x^2}{2} \right) + Q_q'(u_q)_x \right)_{xx} + (J^2 e^{-2u_q}(u_q)_x)_x \\
&- T(u_q)_{xx} + \frac{e^{u_q} - C(x)}{\lambda^2} - \left(\frac{J}{\tau} e^{-u_q} \right)_x = 0.
\end{aligned} \tag{56}
$$

The equation (56) is coupled with

$$
u_q(0) = u_q(1) = 0, \qquad (u_q)_x(0) = (u_q)_x(1) = 0. \tag{57}
$$

Following [14], we introduce u_q as a test function in the weak formulation of our problem. Then we have

$$\begin{aligned}
&\varepsilon^2 \int_0^1 Q_q (u_q)_{xx}^2\,dx + \int_0^1 \left(T - \frac{J^2}{e^{2u_q}}\right)(u_q)_x^2\,dx \\
&= -\frac{1}{\lambda^2}\int_0^1 (e^{u_q} - C)u_q\,dx + \frac{J}{\tau}\int_0^1 e^{-u_q}(u_q)_x\,dx \\
&\quad + \varepsilon^2 \int_0^1 Q_q' \frac{(u_q)_x^3}{6}\,dx - \varepsilon^2 \int_0^1 Q_q''(x)\frac{(u_q)_x^2}{2}\,dx \\
&=: N_1 + N_2 + N_3 + N_4.
\end{aligned}$$

Notice that the constant $(e^{-1} + \|C \ln C\|_{L^\infty})$ is an upper bound for the function $u \mapsto -u(e^u - C)$, for all $u \in \mathbb{R}$ and $x \in \Omega$, therefore we obtain

$$N_1 \le \frac{1}{\lambda^2}(e^{-1} + \|C \ln C\|_{L^\infty}).$$

Moreover, it holds

$$N_2 = 0,$$

because of the boundary conditions. Finally, we account for the last two integrals:

$$N_3 + N_4 \le \frac{\varepsilon^2}{6}\|Q_q'\|_\infty \|(u_q)_x\|_\infty^3 + \frac{\varepsilon^2}{2}\|Q_q''\|_\infty \|(u_q)_x\|_\infty^2 \le \frac{\alpha\varepsilon^2}{2}\|(u_q)_x\|_\infty^2 \left(\frac{\|(u_q)_x\|_\infty}{3} + 1\right).$$

In view of the estimate of the terms N_i, we conclude

$$\begin{aligned}
&\varepsilon^2 q \int_0^1 (u_q)_{xx}^2\,dx + \left(T - \frac{J^2}{\underline{n}}\right)\int_0^1 (u_q)_x^2\,dx \\
&\le \varepsilon \|\sqrt{Q_q}(u_q)_{xx}\|_{L^2(\Omega)} + \sqrt{T - J^2/\underline{n}}\;\|(u_q)_x\|_{L^2(\Omega)} \\
&\le K_5,
\end{aligned} \tag{58}$$

and then (55). □

Theorem 4.3 (Existence of H-Q_qHD solutions). *Assuming* (22)*, there exists a weak solution* $u_q \in H^2(\Omega)$ *to the boundary value problem* (56)-(57).

Proof. Since $Q_q \ge q > 0$, equation (56) basically is a QHD model. So the methods used in previous studies [14, 18, 19] are also available for us. Here we adapt the results given by Gyi and Jüngel [14] to our problem. Let us define $\nu \in X = C^{0,1}(\Omega)$. Consider the following linear problem

$$\begin{aligned}
&\varepsilon^2 \left(Q_q\left((u_q)_{xx} + \frac{\sigma}{2}\nu_x^2\right) + Q_q'(u_q)_x\right)_{xx} + \sigma J^2\left(e^{-2\nu}\nu_x\right)_x \\
&\quad - T(u_q)_{xx} + \frac{\sigma}{\lambda^2}\left(\frac{e^\nu - 1}{\nu}u_q + 1 - C\right) - \sigma\frac{J}{\tau}(e^{-\nu})_x = 0
\end{aligned} \tag{59}$$

coupled with the boundary conditions (57), where $\sigma \in [0,1]$. For each $u_q, \phi \in C^{0,1}(\Omega)$, the following bilinear form is continuous and coercive in $C^{0,1}(\Omega)$ for $\phi \in C^{0,1}(\Omega)$:

$$a(u_q, \phi) = \int_0^1 \left(\varepsilon^2\left(Q_q(u_q)_{xx} + Q_q'(u_q)_x\right)\phi_{xx} + T(u_q)_x\phi_x + \frac{\sigma}{\lambda^2}\frac{e^\nu - 1}{\nu}u_q\phi\right)dx$$

and the functional F defined as

$$F(\phi) = \int_0^1 \left(-Q\frac{\varepsilon^2\sigma}{2}\nu_x^2\phi_{xx} + \sigma J^2 e^{-2\nu}\nu_x\phi_x + \frac{\sigma}{\lambda^2}(C-1)\phi \right) dx - \int_0^1 \left(\sigma\frac{J}{\tau}e^{-\nu}\phi_x \right) dx$$

is linear and continuous in $H^2(\Omega)$ for $\phi \in H^2(\Omega)$. By using Lax Milgram Lemma, we get the existence of a unique solution $u \in H^2(\Omega)$ to the boundary value problem (59)-(57). So we have defined a continuous and compact fixed point operator on X

$$S : X \times [0,1] \to X, \ (\nu, \sigma) \to u_q \tag{60}$$

verifying

- $S(\nu, 0) = 0$ for all $\nu \in X$,
- there is a constant $c > 0$ such that

$$\|u\|_X \le c, \tag{61}$$

  for all $(u_q, \sigma) \in X \times [0,1]$ satisfying $S(u_q, \sigma) = u_q$.

For $\sigma = 1$ the inequality (61) follows from the a priori estimates already discussed, whereas for $0 < \sigma < 1$ it can be obtained proceeding in a similar way.

The existence of a fixed point u_q follows applying the Leray-Schauder fixed point theorem. □

Now we prove the uniqueness of subsonic solution to (12)-(14), for sufficiently small values of the current density J.

Theorem 4.4 (Uniqueness of H-Q$_q$HD solutions)**.** *Assume* (29), (30) *and* (22). *Let* $\varepsilon + |J| \ll 1$, *both are independent of q, then the boundary value problem* (56)-(57) *admits unique solution.*

Proof. Let u_q and v_q be two solutions of the boundary problem (56)-(57). Then $u_q - v_q$ satisfies

$$\begin{aligned} &\varepsilon^2 \left(Q_q(u_q - v_q)_{xx}\right)_{xx} + Q_q\varepsilon^2 \left(\frac{(u_q)_x^2}{2} - \frac{(v_q)_x^2}{2} \right)_{xx} + \varepsilon^2(Q_q'(u_q - v_q)_x)_{xx} \\ &\quad - \frac{J^2}{2}(e^{-2u_q} - e^{-2v_q})_{xx} - T(u_q - v_q)_{xx} + \frac{e^{u_q} - e^{v_q}}{\lambda^2} + \frac{J}{\tau}\left(e^{-u_q} - e^{-v_q}\right)_x = 0 \end{aligned} \tag{62}$$

coupled with the following boundary conditions

$$(u_q - v_q)(0) = (u_q - v_q)(1) = 0, \qquad (u_q - v_q)_x(0) = (u_q - v_q)_x(1) = 0. \tag{63}$$

We multiply (62) by $(u_q - v_q) \in H_0^2(\Omega)$ and integrate it by parts on the whole domain

$$\begin{aligned} &\varepsilon^2 \int_0^1 Q_q(u_q - v_q)_{xx}^2\,dx + \frac{\varepsilon^2}{2}\int_0^1 Q_q(u_q + v_q)_x(u_q - v_q)_x(u_q - v_q)_{xx}\,dx \\ &\quad - \frac{\varepsilon^2}{2}\int_0^1 Q_q''(x)(u_q - v_q)_x^2\,dx + T\int_0^1 (u_q - v_q)_x^2\,dx \\ &\quad + \frac{1}{\lambda^2}\int_0^1 (e^{u_q} - e^{v_q})(u_q - v_q)\,dx \\ &= J^2\int_0^1 e^{-2u_q}(u_q - v_q)_x^2 dx + J^2\int_0^1 (e^{-2u_q} - e^{-2v_q})v_{qx}(u_q - v_q)_x dx \\ &\quad + \frac{J}{\tau}\int_0^1 (e^{-u_q} - e^{-v_q})(u_q - v_q)_x\,dx. \end{aligned} \tag{64}$$

We note that

$$\frac{1}{\lambda^2}\int_0^1 (e^{u_q} - e^{v_q})(u_q - v_q)\,dx \geq 0. \tag{65}$$

We recall the estimates (41), (42) for w_q and $u_q = 2\ln w_q$, and the estimate (55) that holds for u_q and v_q, namely,

$$\varepsilon\|\sqrt{Q_q}(u_q)_{xx}\|_{L^2(\Omega)} + \sqrt{T - J^2/\underline{n}}\ \|(u_q)_x\|_{L^2(\Omega)} \leq K_0.$$

From Poincaré inequality:

$$\|(u_q - v_q)\| \leq C\|(u_q - v_q)_x\|,$$

and Sobolev inequality:

$$\|u_q - v_q\|_{L^\infty} \leq \sqrt{2}\|u_q - v_q\|_{L^2}^{1/2}\|(u_q - v_q)_x\|_{L^2}^{1/2},$$

we have

$$J^2\int_0^1 e^{-2u_q}(u_q - v_q)_x^2 dx \leq CJ^2\|(u_q - v_q)_x\|^2, \tag{66}$$

$$\begin{aligned}
&J^2\int_0^1 (e^{-2u_q} - e^{-2v_q})v_{qx}(u_q - v_q)_x dx \\
&\leq J^2\Big(\int_0^1 |(v_q)_x|^2 dx\Big)^{\frac{1}{2}}\Big(\int_0^1 |e^{-2u_q} - e^{-2v_q}|^2|(u_q - v_q)_x|^2 dx\Big)^{\frac{1}{2}} \\
&\leq CJ^2\|(v_q)_x\|\Big(\int_0^1 |u_q - v_q|^2|(u_q - v_q)_x|^2 dx\Big)^{\frac{1}{2}} \\
&\leq CJ^2\|u_q - v_q\|_{L^\infty}\|(u_q - v_q)_x\| \\
&\leq CJ^2\|u_q - v_q\|^{\frac{1}{2}}\|(u_q - v_q)_x\|^{\frac{1}{2}}\|(u_q - v_q)_x\| \\
&\leq CJ^2\|(u_q - v_q)_x\|^{\frac{1}{2}}\|(u_q - v_q)_x\|^{\frac{1}{2}}\|(u_q - v_q)_x\| \\
&= CJ^2\|(u_q - v_q)_x\|^2,
\end{aligned} \tag{67}$$

and

$$\begin{aligned}
&\frac{J}{\tau}\int_0^1 (e^{-u_q} - e^{-v_q})(u_q - v_q)_x\,dx \\
&\leq C|J|\int_0^1 |u_q - v_q||(u_q - v_q)_x| dx \\
&\leq C|J|(\|u_q - v_q\|^2 + \|(u_q - v_q)_x\|^2) \\
&\leq C|J|\|(u_q - v_q)_x\|^2.
\end{aligned} \tag{68}$$

On the other hand, by the properties of the quantum function $Q_q(x)$, and noting the L^2-boundedness for $(u_q)_x$ in (55), we have the following Sobolev inequality

$$\begin{aligned}
\|\sqrt{Q_q}(u_q)_x\|_{L^\infty}^2 &\leq C\|\sqrt{Q_q}(u_q)_x\|\|\sqrt{Q_q}(u_q)_{xx}\| \\
&\leq C\|(u_q)_x\|\|\sqrt{Q_q}(u_q)_{xx}\| \\
&\leq C\|\sqrt{Q_q}(u_q)_{xx}\|.
\end{aligned}$$

This together with the boundeness of $\varepsilon\|\sqrt{Q_q}(u_q)_{xx}\| \le K_0$ and $\varepsilon\|\sqrt{Q_q}(v_q)_{xx}\| \le K_0$ can guarantee

$$
\begin{aligned}
(69)\quad & \frac{\varepsilon^2}{2}\int_0^1 Q_q(x)(u_q+v_q)_x(u_q-v_q)_x(u_q-v_q)_{xx}\,dx \\
&\le \frac{\varepsilon^2}{4}\int_0^1 Q_q(x)|(u_q-v_q)_{xx}|^2dx + \frac{\varepsilon^2}{4}\int_0^1 |\sqrt{Q_q}((u_q+v_q)_x|^2|(u_q-v_q)_x|^2\,dx \\
&\le \frac{\varepsilon^2}{4}\int_0^1 Q_q(x)|(u_q-v_q)_{xx}|^2dx \\
&\quad + C\varepsilon^2\Big(\|\sqrt{Q_q}(u_q)_x\|^2_{L^\infty} + \|\sqrt{Q_q}(v_q)_x\|^2_{L^\infty}\Big)\|(u_q-v_q)_x\|^2 \\
&\le \frac{\varepsilon^2}{4}\int_0^1 Q_q(x)|(u_q-v_q)_{xx}|^2dx \\
&\quad + C\varepsilon^2\Big(\|\sqrt{Q_q}(u_q)_{xx}\| + \|\sqrt{Q_q}(v_q)_x\|\Big)\|(u_q-v_q)_x\|^2 \\
&\le \frac{\varepsilon^2}{4}\int_0^1 Q_q(x)|(u_q-v_q)_{xx}|^2dx + C\varepsilon\|(u_q-v_q)_x\|^2.
\end{aligned}
$$

Substituting (65)-(69) into (64), we obtain

$$
(70)\quad \frac{\varepsilon^2}{2}\int_0^1 Q_q(u_q-v_q)^2_{xx}\,dx + \Big(T - C_1\varepsilon^2 - C_2\varepsilon - C_3J^2 - C_4|J|\Big)\|(u_q-v_q)_x\|^2 \le 0
$$

for some positive constants C_i $(i=1,2,3,4)$. Let $\varepsilon \ll 1$ and $|J| \ll 1$ be independent of q such that

$$
\varepsilon \le \min\Big\{1, \frac{T}{2(C_1+C_2)}\Big\}, \quad |J| \le \min\Big\{1, \frac{T}{2(C_3+C_4)}\Big\},
$$

then (70) with Poincaré inequality implies the uniqueness:

$$
\|u_q-v_q\|^2 + \|(u_q-v_q)_x\|^2 \le 0, \quad \text{namely} \quad u_q - v_q = 0.
$$

The proof is complete. □

Proof of Theorem 3.1. Following [14] and using the regularity of the function Q_q, it is not difficult to show that there exists a solution $u_q \in H^4(\Omega)$ to (56)-(57). Consequently, observing that $w_m^2 \le n_q = e^{u_q} \le w_M^2$, the boundary value problem (12)-(14) admits a unique solution $n_q \in H^4(\Omega)$. Finally, $V_q \in H^2(\Omega)$, thanks to the Poisson equation (13). This concludes the proof. □

Proof of Theorem 3.3. Theorem 4.4 immediately implies Theorem 3.3. □

5. Hybrid limit

In this section, we study the physical case of hybrid quantum hydrodynamic model (27) with $0 \le Q \le 1$, where the quantum effect function is $Q = 0$ for the classical region and $Q > 0$ for the quantum region, just as indicated in Figure 1. Here, we present the main result obtained within this paper: we perform the hybrid limit, namely we study the behaviour of the solution to the problem (28) for $q \to 0$.

Proof of Theorem 3.7. For a given quantum function $Q \in C^1$, first of all, we will construct the approximating functions $\{Q_q\}$ satisfying (33). Let $(w_q, V_q)(x)$ be the solutions to (28) corresponding to Q_q. From now on, all the $q-$independent constants are indicated as $\bar{K}$ or $\bar{c}_i$, and we briefly show that the following $q-$independent *a priori* estimates hold:

$$
(71)\qquad \|w_q\|_{H^1(\Omega)} \le \bar{K}, \quad \|\sqrt{Q_q}w_{q_{xx}}\|_{L^2(\Omega)} \le \bar{K}.
$$

We perform the estimates, as in the proof of Lemma 4.1, in order to obtain the inequality (47). Now we rearrange the first three terms of the left hand side in (47) in a different way, namely:

$$\begin{aligned}
&\int_0^1 \Big[\frac{\varepsilon^2 Q_q}{w_q} {w_q}_{xx}^2 + \frac{2\varepsilon^2 Q_q'}{w_q} {w_q}_x {w_q}_{xx} + \Big(\frac{T}{w_q} - \frac{J^2}{{w_q}^5}\Big){w_q}_x^2\Big]dx \\
&+ \int_0^1 \Big[\frac{\varepsilon^2 Q_q}{w_q} {w_q}_{xx}^2 + \Big(\frac{T}{w_q} - \frac{J^2}{{w_q}^5}\Big){w_q}_x^2\Big]dx \\
&=: \int_0^1 (\mathcal{A}_2 {w_q}_{xx}^2 + \mathcal{B}_2 {w_q}_x {w_q}_{xx} + \mathcal{C}_2 {w_q}_x^2)dx \\
&\quad + \int_0^1 \Big[\frac{\varepsilon^2 Q_q}{w_q} {w_q}_{xx}^2 + \Big(\frac{T}{w_q} - \frac{J^2}{{w_q}^5}\Big){w_q}_x^2\Big]\, dx.
\end{aligned}$$

The first term on the right hand side is positive by (33). In fact, in this case $\mathcal{B}_2^2 - 4\mathcal{A}_2\mathcal{C}_2 < 0$, where

$$\begin{aligned}
\mathcal{B}_2^2 - 4\mathcal{A}_2\mathcal{C}_2 &= \frac{4}{w_q^2}\Big[|\varepsilon^2 Q_q'|^2 - \varepsilon^2 Q_q\left(T - \frac{J^2}{w_q^4}\right)\Big] \\
&< \frac{4\varepsilon^2}{w_q^2}\Big[\varepsilon^2 |Q_q'|^2 - Q_q\left(T - \frac{J^2}{\underline{n}^2}\right)\Big] \\
&< 0, \quad \text{for } n \geq \underline{n}.
\end{aligned}$$

Then, following the proof of Lemma 4.1, in view of (33), we obtain

$$\int_0^1 \frac{\varepsilon^2 Q_q}{w_q} {w_q}_{xx}^2 dx + \int_0^1 \Big(\frac{T}{w_q} - \frac{J^2}{{w_q}^5}\Big){w_q}_x^2\, dx \leq \bar{K}. \tag{72}$$

We observe that $\frac{\varepsilon^2 Q_q}{w_q}{w_q}_{xx}^2 \geq 0$. Moreover, by (22) we can find a positive constant $\bar{c_1}$ $q-$independent, such that $\frac{T}{w_q} - \frac{J^2}{{w_q}^5} \geq \bar{c_1}$. Therefore, following Lemma 4.1, we obtain

$$\bar{c_1}\int_0^1 [(\sqrt{w_q} - 1)_x]^2 \leq \bar{K}, \tag{73}$$

and thus

$$\|w_q\|_{L^\infty(\Omega)} \leq \bar{K}. \tag{74}$$

Using the uniform upper bound for w and the assumption $0 < Q_q \leq 1$, we can rewrite (72) as

$$\bar{c_2}\varepsilon^2 \int_0^1 Q_q {w_q}_{xx}^2 dx + \bar{c_3}\int_0^1 {w_q}_x^2\, dx \leq \bar{K} \tag{75}$$

which obviously implies (71), namely, w_q is uniformly bounded in $H^1(\Omega)$ and $\sqrt{Q_q} w_{qxx}$ is uniformly bounded in $L^2(\Omega)$. Therefore, there exists a $w(x)$ as the hybrid limit of the sequence w_q:

$$w_q \rightharpoonup w \quad \text{in} \quad H^1(\Omega), \tag{76}$$

for $q \to 0$. Since $H^1(\Omega) \hookrightarrow C^0(\Omega)$, we further have

$$w_q \to w \quad \text{in} \quad C^0(\Omega), \tag{77}$$

for $q \to 0$. Now we prove that w is the weak solution of (27), namely, w satisfies (31). Let us consider equation (39). Multiplying (39) by ϕ, where $\phi \in C_0^\infty(\Omega)$ is any given test function, and integrating by parts we have

$$\begin{aligned} &2\varepsilon^2 \int_0^1 \left(Q_q \frac{w_{q_{xx}}}{w_q} + Q_q' \frac{w_{q_x}}{w_q} \right) \phi_{xx} dx + 2T \int_0^1 \frac{w_{q_x}}{w_q} \phi_x dx \\ &- 4 \int_0^1 \left(\frac{J^2}{2{w_q}^4} \right) \frac{w_{q_x}}{w_q} \phi_x dx + \int_0^1 \frac{{w_q}^2 - C}{\lambda^2} \phi dx + \int_0^1 \left(\frac{J}{\tau {w_q}^2} \right) \phi_x dx = 0. \end{aligned} \tag{78}$$

Recalling (71) and that $w_q > \sqrt{\underline{n}} > 0$ (the subsonic condition), in view of (33), the weak form (78) converges in L^2 to the weak form of the limit problem, namely

$$\begin{aligned} &2\varepsilon^2 \int_0^1 \left(Q \frac{w_{xx}}{w} + Q' \frac{w_x}{w} \right) \phi_{xx} dx + 2T \int_0^1 \frac{w_x}{w} \phi_x dx \\ &- 4 \int_0^1 \left(\frac{J^2}{2w^4} \right) \frac{w_x}{w} \phi_x dx + \int_0^1 \frac{w^2 - C}{\lambda^2} \phi dx + \int_0^1 \left(\frac{J}{\tau w^2} \right) \phi_x dx = 0. \end{aligned} \tag{79}$$

Thus, we have proved that w is the weak solution of (27).

Now we consider the expression for the electric potential V_q, obtained by integrating (12) with respect to x and using (14):

$$\begin{aligned} V_q = &- 2\varepsilon^2 Q_q \frac{w_{q_{xx}}}{w_q} - 2\varepsilon^2 Q_q' \frac{w_{q_x}}{w_q} + \frac{J^2}{2{w_q}^4} + 2T \ln w_q \\ &- \frac{J}{\tau} \int_0^x \frac{1}{{w_q}^2} \, dx. \end{aligned} \tag{80}$$

By using assumption (33) and the uniform estimates (71), one has that $\|V_q\|_{L^2} \le \bar{K}$. Therefore, there exists V such that

$$V_q \rightharpoonup V \quad \text{in} \quad L^2(\Omega). \tag{81}$$

Now, we have to prove that the limit V is the weak solution of the hybrid problem. To this end, we multiply (80) by $\phi \in C_0^\infty(\Omega)$ and integrate it in Ω:

$$\begin{aligned} \int_0^1 V_q \phi \, dx = &- 2\varepsilon^2 \int_0^1 Q_q \frac{w_{q_{xx}}}{w_q} \phi \, dx - 2\varepsilon^2 \int_0^1 Q_q' \frac{w_{q_x}}{w_q} \phi \, dx \\ &+ \int_0^1 \frac{J^2}{2{w_q}^4} \phi \, dx + 2T \int_0^1 (\ln w_q) \phi \, dx \\ &- \frac{J}{\tau} \int_0^1 \left(\int_0^x \frac{1}{{w_q}^2} \, ds \right) \phi \, dx. \end{aligned} \tag{82}$$

Due to the uniform estimate in (71) and to the properties of $\{Q_q\}$, it is not difficult to see that, for $q \to 0$, we have

$$\begin{aligned} \int_0^1 V \phi \, dx = &- 2\varepsilon^2 \int_0^1 Q \frac{w_{xx}}{w} \phi \, dx - 2\varepsilon^2 \int_0^1 Q' \frac{w_x}{w} \phi \, dx \\ &+ \int_0^1 \frac{J^2}{2w^4} \phi \, dx + 2T \int_0^1 (\ln w) \phi \, dx \\ &- \frac{J}{\tau} \int_0^1 \left(\int_0^x \frac{1}{{w_q}^2} \, ds \right) \phi \, dx. \end{aligned} \tag{83}$$

Thus, we prove $V_q \rightharpoonup V$ in L^2 and the limit potential V verifies the Poisson equation in the weak sense. From (81) and $n_q = w_q^2$, we prove (36). The proof of Theorem 3.7 is complete. □

6. A zero-space-charge limit for the hybrid model

In this section we discuss the limit $\lambda \to 0$, firstly for the H-Q_qHD model, then for H-QHD equation. In both cases, following the approach proposed by [14], we choose a particular function $C(x)$, which allows to obtain a suitable set of $\lambda-$independent estimates, as in the following theorem.

Theorem 6.1 (Zero-space-charge limits for the modified H-Q_qHD equation with $q > 0$)**.** *Let C, $Q_q \in C^2(\Omega)$ be given functions such that*

$$\begin{cases} C(0) = C(1) = 1, \ C_x(0) = C_x(1) = 0, \\ 0 < q \le Q_q \le 1, \\ \varepsilon^2 \max_{x\in\Omega} \frac{|Q_q'|^2}{Q_q} < \frac{\left(1-\frac{\varepsilon^4}{8}\right)\left(T-\frac{J^2}{\underline{n}^2}\right)}{4\left(1+\frac{\varepsilon^2}{2}\left(1-\frac{\varepsilon^2}{8}\right)\right)}, \end{cases} \tag{84}$$

where $\underline{n} > n_\star = |J|/\sqrt{T}$. Let $(w_{q,\lambda}, V_{q,\lambda})$ be the solution to the problem (28)*. Then*

$$\begin{aligned} w_{q,\lambda}(x) &\rightharpoonup w_q := \sqrt{C(x)} \quad in \quad H^2(\Omega) \\ w_{q,\lambda}(x) &\to w_q := \sqrt{C(x)} \quad in \quad C^1(\bar\Omega) \\ V_{q,\lambda}(x) &\rightharpoonup V_q(x) \quad in \quad L^2(\Omega), \end{aligned} \tag{85}$$

where

$$\begin{aligned} V_q(x) = &-2\varepsilon^2\left(Q_q\frac{\sqrt{C}_{xx}}{\sqrt{C}} + Q_q'\frac{\sqrt{C}_x}{\sqrt{C}}\right) \\ &+ \frac{J^2}{2C^2} + 2T\ln C + \frac{J}{\tau}\int_0^x \frac{ds}{C(s)}. \end{aligned} \tag{86}$$

Proof. As in Section 5, we prove a suitable set of $\lambda-$independent estimates, which allow us to perform the limit $\lambda \to 0$.

Consider (39), multiply it by $(w_{q,\lambda} - \sqrt{C})$ and integrate on Ω. After some calculations we get:

$$\begin{aligned} &2\varepsilon^2\int_0^1 Q_q\frac{(w_{q,\lambda})_{xx}^2}{w_{q,\lambda}}\,dx + 2\int_0^1\left(T-\frac{J^2}{w_{q,\lambda}^4}\right)\frac{(w_{q,\lambda})_x^2}{w_{q,\lambda}}\,dx \\ &+ 2\varepsilon^2\int_0^1 Q_q'\frac{(w_{q,\lambda})_x(w_{q,\lambda})_{xx}}{w_{q,\lambda}}\,dx + \frac{1}{\lambda^2}\int_0^1 (w_{q,\lambda}-\sqrt{C})^2(w_{q,\lambda}+\sqrt{C})\,dx \\ &= 2\varepsilon^2\int_0^1 Q_q\frac{(w_{q,\lambda})_{xx}}{w_{q,\lambda}}\sqrt{C}_{xx}\,dx + 2\int_0^1\left(T-\frac{J^2}{(w_{q,\lambda})^4}\right)\frac{(w_{q,\lambda})_x}{w_{q,\lambda}}\sqrt{C}_x\,dx \\ &+ 2\varepsilon^2\int_0^1 Q'\frac{w_{q,\lambda_x}}{w_{q,\lambda}}\sqrt{C}_{xx}\,dx + \int_0^1\frac{J}{\tau(w_{q,\lambda})^2}(w_{q,\lambda})_x\,dx - \int_0^1\frac{J}{\tau(w_{q,\lambda})^2}\sqrt{C}_x\,dx \\ &=: Y_1+Y_2+Y_3+Y_4+Y_5. \end{aligned} \tag{87}$$

Using the Young's inequality with parameter, the lower $\lambda-$independent bound for $w_{q,\lambda}$ and the properties on the function Q_q, we can estimate the integrals Y_i:

$$
\begin{aligned}
Y_1 &\leq \frac{\varepsilon^4}{4}\int_0^1 Q_q \frac{(w_{q,\lambda})^2_{xx}}{w_{q,\lambda}}\,dx + 4\int_0^1 Q_q \frac{\sqrt{C}^2_{xx}}{w_{q,\lambda}}\,dx \\
&\leq \frac{\varepsilon^4}{4}\int_0^1 Q_q \frac{(w_{q,\lambda})^2_{xx}}{w_{q,\lambda}}\,dx + 4\int_0^1 \frac{\sqrt{C}^2_{xx}}{w_m}\,dx, \\
Y_2 &\leq \int_0^1 \left(T - \frac{J^2}{(w_{q,\lambda})^4}\right)\frac{(w_{q,\lambda})^2_x}{w_{q,\lambda}}\,dx + T\int_0^1 \frac{\sqrt{C}^2_x}{w_m}\,dx \\
&\leq \int_0^1 \left(T - \frac{J^2}{(w_{q,\lambda})^4}\right)\frac{(w_{q,\lambda})^2_x}{w_{q,\lambda}}\,dx + \int_0^1 \frac{\sqrt{C}^2_x}{w_m}\,dx \\
Y_3 &\leq \frac{\varepsilon^4}{4}\int_0^1 [Q_q'(x)]^2 \frac{(w_{q,\lambda})^2_x}{w_{q,\lambda}}\,dx + 4\int_0^1 \frac{[\sqrt{C}_{xx}]^2}{w_{q,\lambda}}\,dx \\
&\leq \frac{\varepsilon^4}{4}\int_0^1 [Q_q'(x)]^2 \frac{(w_{q,\lambda})^2_x}{w_{q,\lambda}}\,dx + 4\int_0^1 \frac{[\sqrt{C}_{xx}]^2}{(w_{q,\lambda})}\,dx,
\end{aligned}
$$

where $w_m = \min\{w_{q,\lambda}\}$. Moreover

$$Y_4 = 0,$$

$$Y_5 \leq \int_0^1 \frac{J^2}{2\tau(w_{q,\lambda})^4}\,dx + \int_0^1 \frac{[\sqrt{C}_x]^2}{2}\,dx \leq \frac{J^2}{2\tau w_m^4} + \int_0^1 \frac{[\sqrt{C}_x]^2}{2}\,dx.$$

In view of the previous estimates, (87) becomes

$$
\begin{aligned}
(88)\quad & 2\varepsilon^2\left(1-\frac{\varepsilon^2}{8}\right)\int_0^1 Q_q\frac{(w_{q,\lambda})^2_{xx}}{w_{q,\lambda}}\,dx \\
&+ \int_0^1 \left[\left(T - \frac{J^2}{(w_{q,\lambda})^4}\right) - \frac{\varepsilon^4 Q_q'^2}{4}\right]\frac{(w_{q,\lambda})^2_x}{w_{q,\lambda}}\,dx \\
&+ 2\varepsilon^2\int_0^1 Q_q'\frac{(w_{q,\lambda})_x(w_{q,\lambda})_{xx}}{w_{q,\lambda}}\,dx + \frac{1}{\lambda^2}\int_0^1((w_{q,\lambda}) - \sqrt{C})^2((w_{q,\lambda}) + \sqrt{C})\,dx \\
&\leq \tilde{K},
\end{aligned}
$$

that implies

$$(89)\qquad \int_0^1 \mathcal{A}_3\frac{(w_{q,\lambda})^2_{xx}}{w_{q,\lambda}}\,dx + \int_0^1 \mathcal{B}_3\frac{(w_{q,\lambda})_x(w_{q,\lambda})_{xx}}{w_{q,\lambda}}\,dx + \int_0^1 \mathcal{C}_3\frac{(w_{q,\lambda})^2_x}{w_{q,\lambda}}\,dx \leq \tilde{K},$$

where

$$
\begin{aligned}
\mathcal{A}_3 &= 2\varepsilon^2\left(1-\frac{\varepsilon^2}{8}\right)Q_q \\
\mathcal{B}_3 &= 2\varepsilon^2 Q_q' \\
\mathcal{C}_3 &= \left[\left(T - \frac{J^2}{(w_{q,\lambda})^4}\right) - \frac{\varepsilon^4 Q_q'^2}{4}\right].
\end{aligned}
$$

The quadratic form in (89) is strictly positive if $\mathcal{B}_3^2 - 4\mathcal{A}_3\mathcal{C}_3 < 0$:

$$\begin{aligned}&\mathcal{B}_3^2 - 4\mathcal{A}_3\mathcal{C}_3 =\\ &= 4\varepsilon^4 Q_q'(x)^2 - 8\varepsilon^2\left(1-\frac{\varepsilon^2}{8}\right)Q_q\left[\left(T-\frac{J^2}{(w_{q,\lambda})^4}\right)-\frac{\varepsilon^4 Q_q'^2}{4}\right] < 0.\end{aligned}$$

The inequality above is verified by

$$\begin{aligned}&4\varepsilon^4[Q_q']^2 + 8\varepsilon^2\left(1-\frac{\varepsilon^2}{8}\right)Q\left[\frac{\varepsilon^4[Q_q'(x)]^2}{4}\right]\\ &< \varepsilon^2\left(1-\frac{\varepsilon^2}{8}\right)Q_q\left[\left(T-\frac{J^2}{(w_{q,\lambda})^4}\right)\right],\end{aligned}$$

which implies, recalling that $Q_q < 1$:

$$4\varepsilon^4[Q_q']^2\left(1+\frac{\varepsilon^2}{2}\left(1-\frac{\varepsilon^2}{8}\right)\right) < \varepsilon^2\left(1-\frac{\varepsilon^2}{8}\right)Q_q\left[\left(T-\frac{J^2}{(w_{q,\lambda})^4}\right)\right]$$

and then, (84).

Proceeding as in Lemma 4.1, assuming (84), we can see that $\|(w_{q,\lambda})\|_\infty \le \tilde{K}$ and $\|(w_{q,\lambda})\|_{H^1} \le \tilde{K}$. Moreover, in view of (87), it is easy to see that

$$\int_0^1 ((w_{q,\lambda}) - \sqrt{C})^2((w_{q,\lambda}) + \sqrt{C}) \le \lambda^2\tilde{K}.$$

Therefore there exists a subsequence $(w_{q,\lambda})$ (not relabeled) such that the first two relations in (85) hold. Following the same idea used in the discussion of the hybrid limit, we can prove that $\|V_{q\lambda}\|_{L^2} \le \tilde{K}$, and then the last limit in (85).

Finally, we have to find that the limit potential for $\lambda \to 0$ is given by (86). Consider (80) multiplied by $\phi \in C_0^\infty(\Omega)$ and integrated it in Ω:

$$\begin{aligned}\int_0^1 V_{q\lambda}\phi\,dx =& -2\varepsilon^2\int_0^1 Q_q\frac{(w_{q,\lambda})_{xx}}{w_{q,\lambda}}\phi\,dx - 2\varepsilon^2\int_0^1 Q_q'\frac{(w_{q,\lambda})_x}{w_{q,\lambda}}\phi\,dx\\ &+\int_0^1 \frac{J^2}{2(w_{q,\lambda})^4}\phi\,dx + 2T\int_0^1 \ln w_{q,\lambda}\phi\,dx\\ &-\frac{J}{\tau}\int_0^1\left(\int_0^x \frac{1}{(w_{q,\lambda})^2}\,ds\right)\phi\,dx.\end{aligned}\tag{90}$$

As a consequence of the uniform estimate derived above, for $\lambda \to 0$, we have

$$\begin{aligned}\int_0^1 V_q\phi\,dx =& -2\varepsilon^2\int_0^1 Q_q\frac{\sqrt{C}_{xx}}{\sqrt{C}}\phi\,dx - 2\varepsilon^2\int_0^1 Q_q'\frac{\sqrt{C}_x}{\sqrt{C}}\phi\,dx\\ &+\int_0^1\frac{J^2}{2C^2}\phi\,dx + T\int_0^1(\ln C)\phi\,dx - \frac{J}{\tau}\int_0^1\left(\int_0^x\frac{1}{C}\,ds\right)\phi\,dx.\end{aligned}\tag{91}$$

This concludes the proof. □

Proof of Theorem 3.8. Let $(w_{q,\lambda}, V_{q,\lambda})(x)$ a the smooth solution for (28). As showed in Theorem 3.1, we have

$$\|w_{q,\lambda}\|_{H^1} \le \tilde{K},\quad \|\sqrt{Q_q}(w_{q,\lambda})_{xx}\|_{L^2} \le \tilde{K},\quad \|V_{q,\lambda}\|_{L^2} \le \tilde{K},$$

and, as showed in Theorem 6.1, we have

$$\int_0^1 (w_{q,\lambda} - \sqrt{C})^2 (w_{q,\lambda} + \sqrt{C})\, dx \le \lambda^2 \tilde{K}$$

and (see (88))

$$\begin{aligned}
&2\varepsilon^2 \left(1 - \frac{\varepsilon^2}{8}\right) \int_0^1 Q_q \frac{(w_{q,\lambda})_{xx}^2}{w_{q,\lambda}}\, dx \\
&+ \int_0^1 \left[\left(T - \frac{J^2}{(w_{q,\lambda})^4} \right) - \frac{\varepsilon^4 Q_q'^2}{4} \right] \frac{(w_{q,\lambda})_x^2}{w_{q,\lambda}}\, dx \\
&+ 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_{q,\lambda})_x (w_{q,\lambda})_{xx}}{w_{q,\lambda}}\, dx \\
&+ \frac{1}{\lambda^2} \int_0^1 ((w_{q,\lambda}) - \sqrt{C})^2 ((w_{q,\lambda}) + \sqrt{C})\, dx \\
&\le \tilde{K}.
\end{aligned} \tag{92}$$

Moreover, taking $q \to 0$, we have

$$\begin{aligned}
&2\varepsilon^2 \left(1 - \frac{\varepsilon^2}{8}\right) \int_0^1 Q \frac{(w_\lambda)_{xx}^2}{w_\lambda}\, dx \\
&+ \int_0^1 \left[\left(T - \frac{J^2}{(w_\lambda)^4} \right) - \frac{\varepsilon^4 Q'^2}{4} \right] \frac{(w_\lambda)_x^2}{w_\lambda}\, dx \\
&+ 2\varepsilon^2 \int_0^1 Q' \frac{(w_\lambda)_x (w_\lambda)_{xx}}{w_\lambda}\, dx \\
&+ \frac{1}{\lambda^2} \int_0^1 ((w_\lambda) - \sqrt{C})^2 ((w_\lambda) + \sqrt{C})\, dx \le \tilde{K}.
\end{aligned} \tag{93}$$

Let us rewrite (93) as follows

$$\begin{aligned}
&\varepsilon^2 \left(1 - \frac{\varepsilon^2}{8}\right) \int_0^1 Q \frac{(w_\lambda)_{xx}^2}{w_\lambda}\, dx \\
&+ (1 - \varepsilon^2) \int_0^1 \left[\left(T - \frac{J^2}{(w_\lambda)^4} \right) - \frac{\varepsilon^4 Q'^2}{4} \right] \frac{(w_\lambda)_x^2}{w_\lambda}\, dx \\
&+ \int_0^1 \left[\mathcal{A}_4 \frac{[(w_\lambda)_{xx}]^2}{w_\lambda} + \mathcal{B}_4 \frac{(w_\lambda)_x (w_\lambda)_{xx}}{w_\lambda} + \mathcal{C}_4 \frac{[(w_\lambda)_x]^2}{w_\lambda} \right] dx \\
&+ \frac{1}{\lambda^2} \int_0^1 ((w_\lambda) - \sqrt{C})^2 ((w_\lambda) + \sqrt{C})\, dx \le \tilde{K},
\end{aligned} \tag{94}$$

where

$$\begin{aligned}
&\mathcal{A}_4 := \varepsilon^2 (1 - \frac{\varepsilon^2}{8}) Q, \\
&\mathcal{B}_4 := 2\varepsilon^2 Q', \\
&\mathcal{C}_4 := \varepsilon^2 [(T - \frac{J^2}{\underline{n}^2}) - \frac{\varepsilon^4 Q'}{4}],
\end{aligned}$$

which satisfy

$$\mathcal{B}_4^2 - 4\mathcal{A}_4 \mathcal{C}_4 < 0$$

due to (37). So, we have

$$\int_0^1 \left[\mathcal{A}_4 \frac{[(w_\lambda)_{xx}]^2}{w_\lambda} + \mathcal{B}_4 \frac{(w_\lambda)_x (w_\lambda)_{xx}}{w_\lambda} + \mathcal{C}_4 \frac{[(w_\lambda)_x]^2}{w_\lambda}\right] dx \geq 0,$$

and, from (94), we further have

$$\begin{aligned}
&\varepsilon^2 \left(1 - \frac{\varepsilon^2}{8}\right) \int_0^1 Q \frac{(w_\lambda)_{xx}^2}{w_\lambda}\, dx \\
&+ (1-\varepsilon^2) \int_0^1 \left[\left(T - \frac{J^2}{(w_\lambda)^4}\right) - \frac{\varepsilon^4 Q'^2}{4}\right] \frac{(w_\lambda)_x^2}{w_\lambda}\, dx \\
&+ \frac{1}{\lambda^2} \int_0^1 ((w_\lambda) - \sqrt{C})^2 ((w_\lambda) + \sqrt{C})\, dx \leq \tilde{K},
\end{aligned} \tag{95}$$

which implies

$$\|w_\lambda\|_{H^1} \leq \tilde{K} \text{ and } \int_0^1 (w_\lambda - \sqrt{C})^2 (w_\lambda + \sqrt{C})\, dx \leq \lambda^2 \tilde{K}.$$

This gives

$$w_\lambda \rightharpoonup \sqrt{C} \text{ in } H^1 \text{ as } \lambda \to 0,$$

and

$$w_\lambda \to \sqrt{C} \text{ in } C^0 \text{ as } \lambda \to 0.$$

Similarly to (82) and (83), we can prove

$$V_\lambda \rightharpoonup \tilde{V} \text{ in } L^2(\Omega) \text{ as } \lambda \to 0,$$

where $\tilde{V}$ is given in (38). The details are omitted. Thus, the proof of Theorem 3.8 is now complete. □

7. Numerical simulations

In the previous sections we have introduced, from the theoretical point of view, a new hybrid model (H-QHD), obtained localizing the quantum effects in a given subset of the device domain. Compared to quantum hydrodynamic model, the H-QHD model (12) has an additional term, namely $\varepsilon^2 Q'(x) \frac{\sqrt{n}_x}{\sqrt{n}}$. Roughly speaking Q' models a semi-classical region linking quantum and classical domains. Obviously, when $Q' = 0$, we obtain again the QHD model.

In this section we test numerically the H-QHD model on a simple device: a $n^+|n|n^+$ transistor. It is characterized by the following typical doping profile:

$$\bar{C}(x) = \begin{cases} C_m, & \forall\, x \in [x_1, x_2] \\ 1, & \forall\, x \in [0, x_1) \text{ and } x \in (x_2, 1] \end{cases} \tag{96}$$

where $C_m < 1$ is a strictly positive constant and $0 < x_1 < x_2 < 1$.

We approximate the step function $\bar{C}(x)$ by using

$$C(x) = 1 - (0.5 - C_m/2)(\tanh(1000(x - 1/3)) - \tanh(1000(x - 2/3))) \qquad x \in [0, 1],$$

and by taking $C_m = 0.2$.

According to the theoretical part, we model as quantum the middle region of the device and as classical the external parts. Namely

$$\begin{cases} \text{Quantum Region} & \forall\, x \in [y_1, y_2] \\ \text{Classical Region} & \forall\, x \in [0, y_1) \text{ and } x \in (y_2, 1], \end{cases} \tag{97}$$

where $0 < y_1 < y_2 < 1$.

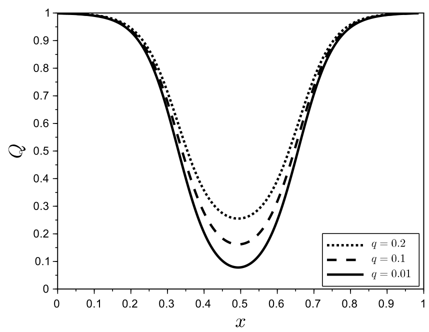

FIGURE 3. Quantum profile function $Q(x)$, for different value of q. In particular we set $q = 0.2$, $q = 0.1$ and $q = 0.01$.

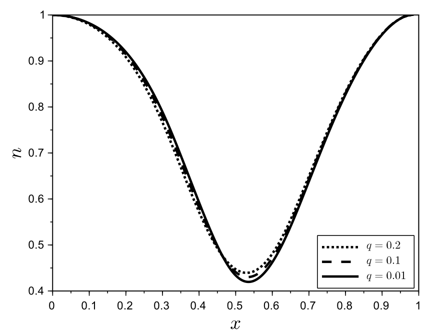

FIGURE 4. Charge density profile $n(x)$, for different value of q. In particular we set $q = 0.2$, $q = 0.1$ and $q = 0.01$.

We consider the ordinary differential equation (56)

$$\begin{aligned}\varepsilon^2 \left(Q \left(u_{xx} + \frac{u_x^2}{2} \right) + Q' u_x \right)_{xx} &+ (J^2 e^{-2u} u_x)_x \\ - T u_{xx} + \frac{e^u - C(x)}{\lambda^2} &- \left(\frac{J}{\tau} e^{-u} \right)_x = 0.\end{aligned}$$

coupled with

$$u(0) = u(1) = 0, \qquad u_x(0) = u_x(1) = 0.$$

The numerical simulation are performed by using COLNEW, a SCILAB function for boundary value problems [6]. For our toy model, it looks reasonable to assume that $Q(x)$ behaves like $C(x)$, therefore we set

$$Q(x) = (0.5 - q/2)((\tanh(10(x - 1/3)) - \tanh(10(x - 2/3))), x \in [0, 1]] \tag{98}$$

where q is the strictly positive minimum of the function $Q(x)$. The behaviour of the function $Q(x)$, for different values of q, is plotted in Figure 3.

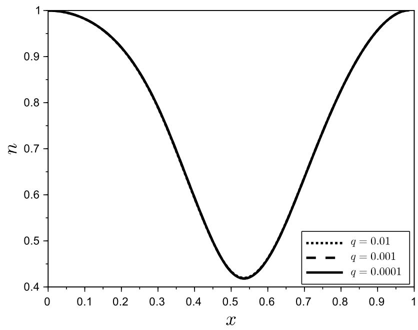

FIGURE 5. Charge density profile $n(x)$, for different value of q. In particular we set $q = 0.01$, $q = 0.001$ and $q = 0.0001$.

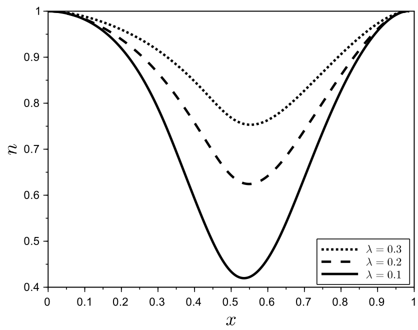

FIGURE 6. Charge density profile $n(x)$, for different value of λ. In particular we set $\lambda = 0.3$, $\lambda = 0.2$ and $\lambda = 0.1$.

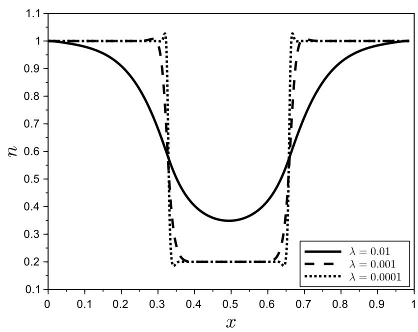

FIGURE 7. Charge density profile $n(x)$, for different value of λ. In particular we set $\lambda = 0.01$, $\lambda = 0.001$ and $\lambda = 0.0001$.

Next we fix the values of the scaled Debye length λ, of the scaled temperature T, of the scaled Plank constant ε, of the current density J, and of the relaxation time τ as follows:

$$\lambda = 0.1, \quad T = 1, \quad \varepsilon = 0.1, \quad J = 0.1, \quad \tau = 0.125.$$

By such a setting, then we can check that the system of the flow is subsonic:

$$C(x) \geq C_m > \frac{J}{\sqrt{T}}, \quad \text{[subsonic doping profile]}$$
$$n(0) = n(1) > \frac{J}{\sqrt{T}}, \quad \text{[subsonic boundary]}$$

and also conditions (33) are verified for all $q \leq 0.5$.

To evaluate the performance of our model, we consider the behaviour of the charge density n for different value of q. In particular, we set $q = 0.2$, $q = 0.1$ and $q = 0.01$ (Figure 4). As expected, reducing the value of q the solution converges to the limit hybrid solution. If we reduce again the value of the parameter q, the solution becomes very close each other and no remarkable difference can be observed numerically, as showed in Figure 5. This verifies numerically the existence of a limit solution discussed in the theoretical part.

In Figure 6 we consider the behaviour of the solution for a fixed value of $q = 0.01$ and different values of the Debye length, namely $\lambda = 0.3$, $\lambda = 0.2$ and $\lambda = 0.1$. As expected, reducing the value of λ the charge density profile becomes more similar to the doping profile $C(x)$.

Finally we investigate numerically the zero-space-charge. In order to verify condition (37), we set the following set of parameters:

$$q = 0.2, \quad T = 50, \quad \varepsilon = 0.1, \quad J = 0.001, \quad \tau = 0.125,$$

and $\lambda = 0.01$, $\lambda = 0.001$ and $\lambda = 0.0001$. The results are summarized in Figure 7. We notice that for $\lambda = 0.001$ and $\lambda = 0.0001$ the charge density profile coincides with the doping profile $C(x)$.

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