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Asymptotic stability of solutions for 1-D compressible Navier–Stokes–Cahn–Hilliard system

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ABSTRACT

This paper is concerned with the evolution of the periodic boundary value problem and the mixed boundary value problem for a compressible mixture of binary fluids modeled by the Navier–Stokes–Cahn–Hilliard system in one dimensional space. The global existence and the large time behavior of the strong solutions for these two systems are studied. The solutions are proved to be asymptotically stable even for the large initial disturbance of the density and the large velocity data. We show that the average concentration difference for the two components of the initial state determines the long time behavior of the diffusive interface for the two-phase flow.

1. Introduction and main result

The evolution for the compressible mixture of binary fluids (e.g. foams, solidification processes, fluid–gas interface etc.) is one of the fundamental problems in hydrodynamical science, and it is attracting more and more attention with the development of modern science and technology. As is well known, the flow of such fluid can be described by the well-known diffusive interface system of equations nonlinearly coupled the Navier–Stokes and Cahn–Hilliard equations, called Navier–Stokes–Cahn–Hilliard system [1,7,22] as follows:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) = \mathrm{div}\mathbb{T}, \\ \partial_t\big(\rho\chi\big) + \mathrm{div}(\rho\chi\mathbf{u}) = \Delta\mu, \\ \rho\mu - \dfrac{1}{\epsilon}\rho\dfrac{\partial f}{\partial\chi} + \epsilon\Delta\chi \in \partial I(\chi), \end{cases} \tag{1.1}$$

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where the unknown functions ρ, $\mathbf{u}$, μ and χ denote the total density, the mean velocity, the chemical potential and the difference of the two components for the fluid mixture respectively. More precisely, the difference of two components for the fluid mixture is $\chi = \chi_1 - \chi_2$, where $\chi_i = \frac{M_i}{M}$, and M_i denotes the mass concentration of the fluid i $(i = 1, 2)$ and mass of the components in the representative material volume V respectively. The total density is given by $\rho = \rho_1 + \rho_2$, where $\rho_i = \frac{M_i}{V}$ denotes the apparent mass density of the fluid i. $\mathbf{u}$ denotes the average velocity given by $\rho\mathbf{u} = \rho_1\mathbf{u}_1 + \rho_2\mathbf{u}_2$, where $\mathbf{u}_i$ is defined as the velocity of the fluid $i = 1, 2$ in the mixed flow. $\partial I(\cdot)$ is the subdifferential of the indicator function $I(\cdot)$ of the set $[-1, 1]$. The Cauchy stress-tensor is represented by

$$\mathbb{T} = \mathbb{S} - \epsilon\big(\nabla\chi \otimes \nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}\big), \tag{1.2}$$

and the conventional Newtonian viscous stress is

$$\mathbb{S} = \nu(\chi)\big((\nabla\mathbf{u} + \nabla^\top\mathbf{u}) - \frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}\big) - p\mathbb{I} + \eta(\chi)\mathrm{div}\mathbf{u}\mathbb{I}, \tag{1.3}$$

where $\mathbb{I}$ is the unit matrix, $\nu(\chi) > 0$, $\eta(\chi) \geq 0$ are defined as viscosity coefficients, $\epsilon > 0$ is the thickness of the diffuse interface of the fluid mixture, and $f = f(\chi)$ is the potential free energy density. We suppose that f satisfies the Ginzburg-Landau double-well potential non-smooth model which is widely used in [3,4,8,9,28,33–35] and the references therein:

$$f(\chi) = \begin{cases} \frac{1}{4}\big(\chi^2 - 1\big)^2, & \text{if } -1 \leq \chi \leq 1, \\ +\infty, & \text{if } |\chi| > 1. \end{cases} \tag{1.4}$$

Physically, the pressure p is given by

$$p = a\rho^\gamma, \quad \gamma \geq 1, \tag{1.5}$$

where a is a positive constant, $\gamma \geq 1$ is the adiabatic constant.

Remark 1.1. Compared to the momentum equation in compressible Navier–Stokes system, an additional force $-\epsilon\big(\nabla\chi \otimes \nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}\big)$ is added in the Cauchy stress tensor, which describes the capillary effect associated with free energy

$$E_{\text{free}}(\rho, \chi) = \int_\Omega \big(\frac{\rho}{\epsilon}f(\chi) + \frac{\epsilon}{2}|\nabla\chi|^2\big)dx. \tag{1.6}$$

To avoid the difficulty of the estimation of density gradient, (1.6) was first presented by Abels-Feireisl [1] which is a simplified model of the following total free energy

$$\tilde{E}_{\text{free}}(\rho, \chi) = \int_\Omega \big(\frac{\rho}{\epsilon}f(\chi) + \frac{\rho\epsilon}{2}|\nabla\chi|^2\big)dx,$$

proposed by Lowengrub and Truskinovsky in [22]. The potential (1.4) is the polynomial non-smooth approximation of the so-called logarithmic potential suggested by Cahn–Hilliard [6]

$$f(\chi) = \frac{1}{2}\theta\Big((1-\chi)\ln(\frac{1-\chi}{2}) + (1+\chi)\ln(\frac{1+\chi}{2})\Big) - \frac{\theta_c}{2}\chi^2,$$

where θ and $\theta_c > 0$ are positive constants. It follows that f is convex on $(-1, 1)$ for $\theta > \theta_c$, and has the bistable form for $\theta < \theta_c$.

Now let us draw the background on the well-posedness of the solutions for the system (1.1). We begin by recalling the compressible Navier–Stokes system. In 1980, Matsumura–Nishida [23] first proved the global existence of smooth solutions for Navier–Stokes equations for initial data close to a non-vacuum equilibrium. Later, Kawashima–Matsumura [16], Matsumura–Nishihara [24,25] obtained the asymptotic stability for the rarefaction waves and shock waves in one-dimensional successively. We also refer to the papers due to Huang–Li–Matsumura [11], Huang–Matsumura [12], Huang–Matsumura–Xin [13], Huang–Wang–Wang–Yang [14], Shi–Yong–Zhang [29,30] and the references therein, in which the asymptotic stability on the superposition of nonlinear waves are studied.

In the case of diffusive interface model for two immiscible fluids, in order to understand the motion of the interfaces between immiscible fluids, phase field models for mixtures of two constituents were studied over the past century which can be traced to van der Waals [32]. Cahn–Hilliard [6] first replaced the sharp interface into the narrow transition layer across which the fluids may mix, and the famous Cahn–Hilliard equations was proposed to describe these diffusive interfaces between two immiscible fluids, Lowengrub–Truskinovsky [22] added the effect of the motion of the particles and the interaction with the diffusion into Cahn–Hilliard equation, and put forward the Navier–Stokes–Cahn–Hilliard equations. The similar result was established by Abels–Feireisl [1], Anderson–Mcfadden–Wheeler [2] and the references therein. Heida–Málek–Rajagopal [10] and Kotschote [17] generalized these models to non-isentropic case. The global existence of weak solutions in three dimensional was obtained by Abels–Feireisl [1], the method they used is the framework which was introduced by Leray [19] and Lions [20]. Kotschote–Zacher [18] established a local existence and uniqueness result for strong solutions. Ding–Li [7] proved the global existence of strong solution in a bounded domain with initial boundary condition (1.9) in one dimension.

In this paper, on the basis of the previous work outlined above, we begin to study the large time behavior for the solutions of the system (1.1) in $\Omega \times (0, +\infty)$

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_x = \nu u_{xx} - \dfrac{\epsilon}{2}\left(\chi_x^2\right)_x, \\ \rho \chi_t + \rho u \chi_x = \mu_{xx}, \\ \rho\mu - \dfrac{\rho}{\epsilon}(\chi^3 - \chi) + \epsilon \chi_{xx} \in \partial I(\chi), \end{cases} \tag{1.7}$$

where $\Omega \subseteq \mathbb{R}, t \in (0, +\infty)$. We study only two model cases for (1.7):

(i) L-periodic boundary case, that is $\Omega = \mathbb{R}$, and

$$\begin{cases} (\rho, u, \chi)(x,t) = (\rho, u, \chi)(x + L, t), & x \in \mathbb{R}, t > 0, \\ (\rho, u, \chi)\big|_{t=0} = (\rho_0, u_0, \chi_0), & x \in \mathbb{R}. \end{cases} \tag{1.8}$$

(ii) Mixed boundary case, that is $\Omega = [0, L]$, and

$$\begin{cases} (u, \chi_x, \mu_x)\big|_{x=0,L} = (0,0,0), & t \geq 0, \\ (\rho, u, \chi)\big|_{t=0} = (\rho_0, u_0, \chi_0), & x \in [0, L]. \end{cases} \tag{1.9}$$

We denote by C and c the positive generic constants without confusion throughout this paper. L^2 denotes the space of Lebesgue measurable functions on $\mathbb{R}$ which are square integrable, with the norm $\|f\| = (\int_0^L |f|^2)^{\frac{1}{2}}$. $H^l(l \geq 0)$ denotes the Sobolev space of L^2-functions f on $\mathbb{R}$ whose derivatives $\partial_x^j f, j = 1, \cdots, l$ are L^2 functions too, with the norm $\|f\|_l = (\sum_{j=0}^{l} \|\partial_x^j f\|^2)^{\frac{1}{2}}$. Let

$$\bar{\rho} = \frac{1}{L}\int_0^L \rho_0 dx, \quad \overline{\rho u} = \frac{1}{L}\int_0^L \rho_0 u_0 dx, \quad \overline{\rho\chi} = \frac{1}{L}\int_0^L \rho_0\chi_0 dx, \quad \bar{u} = \frac{\overline{\rho u}}{\bar{\rho}}, \quad \bar{\chi} = \frac{\overline{\rho\chi}}{\bar{\rho}}, \tag{1.10}$$

and then, for the both cases above, we have

$$\int_0^L (\rho - \bar{\rho})dy = 0, \ \int_0^L (\rho u - \overline{\rho u})dy = 0, \ \int_0^L (\rho\chi - \overline{\rho\chi})dy = 0. \tag{1.11}$$

For the periodic boundary condition (i), because the periodic solutions $(\rho, u, \chi)(x, t)$ of (1.7) with the period L in the whole space $\mathbb{R}$ can be regarded as L-periodic extensions of that on $[0, L]$, we only need to consider the system (1.7) on the bounded interval $[0, L]$. We introduce the Hilbert space $L^2_{\text{per}}(\mathbb{R})$ of locally square integrable functions that are periodic with the period L:

$$L^2_{\text{per}}(\mathbb{R}) = \Big\{g(x)\big|g(x + L) = g(x) \text{ for all } x \in \mathbb{R}, \text{ and } g(x) \in L^2(0, L)\Big\} \tag{1.12}$$

with the norm denoted also by $\|\cdot\|$ (without confusion) which is given by the integral over $[0, L]$, $\|g\| = (\int_0^L |g(x)|^2 dx)^{\frac{1}{2}}$. $H^l_{\text{per}}(\mathbb{R})$ $(l \geq 0)$ denotes the Sobolev space of $L^2_{\text{per}}(\mathbb{R})$-functions g on $\mathbb{R}$ whose derivatives $\partial^j_x g, j = 1, \cdots, l$ are L^2_{per} functions too, with the norm $\|g\|_l = (\sum_{j=0}^l \|\partial^j_x g\|^2)^{\frac{1}{2}}$. Throughout this paper, the initial and boundary data for the density, velocity and concentration difference of two components are assumed to be:

$$\begin{aligned}&(\rho_0, u_0) \in H^2_{\text{per}}, \text{ (periodic boundary case (i))}, \\ &\text{or } \rho_0 \in H^2([0, L]), u_0 \in H^1_0([0, L]) \cap H^2([0, L]), \text{ (mixed boundary case (ii))},\end{aligned} \tag{1.13}$$

$$\begin{aligned}&\chi_0 \in H^4_{\text{per}}, \text{ (periodic case (i))}, \\ &\text{or } \chi_0 \in H^4([0, L]), \text{ (mixed boundary case (i))},\end{aligned} \tag{1.14}$$

$$\inf_{x\in[0,L]} \rho_0 > 0, \ \chi_0 \in [-1, 1], \tag{1.15}$$

$$\rho_t(x, 0) = -\rho_{0x}u_0 - \rho_0 u_{0x}, \tag{1.16}$$

$$u_t(x, 0) = \frac{\nu}{\rho_0} u_{0xx} - u_0 u_{0x} - \frac{p'_\rho(\rho_0)}{\rho_0}\rho_{0x} - \frac{\epsilon}{\rho_{0x}}\chi_{0x}\chi_{0xx}, \tag{1.17}$$

$$\begin{aligned}\chi_t(x, 0) = &-u_0\chi_{0x} - \frac{\epsilon\chi_{0xxxx}}{\rho_0^2} + \frac{2\epsilon\rho_{0x}}{\rho_0^3}\chi_{xxx} + \epsilon\Big(\frac{\rho_{0xx}}{\rho_0^3} - \frac{2\rho_{0x}^2}{\rho_0^4}\Big)\chi_{0xx} \\ &+ \frac{(3\chi_0^2 - 1)\chi_{0xx}}{\epsilon\rho_0} + \frac{6\chi_0\chi_{0x}^2}{\epsilon\rho_0}.\end{aligned} \tag{1.18}$$

Our main result is that we show the average concentration difference for the two components of the initial state determines the long time behavior of the diffusive interface for the two-phase flow. This leads to a separation of two regions for the initial concentration difference: regions A_{stable} and A_{unstable}, which correspond to the stable and unstable regions.

$$A_{\text{stable}} = (-\infty, \frac{\sqrt{3}}{3}) \cup (\frac{\sqrt{3}}{3}, +\infty), \quad A_{\text{unstable}} = (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}). \tag{1.19}$$

Remark 1.2. We rewrite $(1.7)_{3,4}$ as follows

$$\chi_t + u\chi_x + \frac{\epsilon\chi_{xxxx}}{\rho^2} - \frac{2\epsilon\rho_x}{\rho^3}\chi_{xxx} - \epsilon\Big(\frac{\rho_{xx}}{\rho^3} - \frac{2\rho_x^2}{\rho^4}\Big)\chi_{xx} - \frac{3\chi^2 - 1}{\epsilon\rho}\chi_{xx} - \frac{6}{\epsilon\rho}\chi\chi_x^2 \in \partial I(\chi). \tag{1.20}$$

It should be pointed out that the reason for the classification of stable and unstable regions is just that the energy estimates (please refer to (3.5) and (3.6)) ask for the coefficient $\frac{3\chi^2-1}{\epsilon\rho}$ in parabolic equation (1.20) should be positive. If $\chi \in A_{\text{unstable}}$, it is thus physically unstable and mathematically ill-posed for the two-phase flow, and the phase separation will occur in this region.

Now we will give the global existence and asymptotic stability results as follows.

Theorem 1.1 *(Periodic boundary case). Assume that (ρ_0, u_0, χ_0) satisfies the periodic boundary (1.8) with the regularities given in (1.13)–(1.18), then there exists a unique global strong solution (ρ, u, χ) of the system (1.7)–(1.8), satisfying that, for any $T > 0$, there exist positive constants $m, M > 0$ such that*

$$\begin{aligned}
&\rho \in L^\infty(0,T,H^2_{\text{per}}), \rho_t \in L^\infty(0,T;H^1_{\text{per}}),\\
&u \in L^\infty(0,T;H^2_{\text{per}}) \cap L^2(0,T;H^3_{\text{per}}), u_t \in L^\infty(0,T;H^1_{\text{per}}),\\
&\chi \in L^\infty(0,T;H^4_{\text{per}}), \chi_t \in L^\infty(0,T;L^2_{\text{per}}) \cap L^2(0,T;H^2_{\text{per}}),\\
&\mu \in L^\infty(0,T;H^2_{\text{per}}) \cap L^2(0,T;H^4_{\text{per}}),\ \ \mu_t \in L^2(0,T;L^2_{\text{per}}),\\
&\chi \in [-1,1],\ \ m \le \rho \le M,\ \text{ for all } (x,t) \in \mathbb{R} \times [0,T].
\end{aligned} \tag{1.21}$$

Furthermore, if $\bar{\chi} \in A_{stable} \cap [-1,1]$, then there exists a small positive constant ε_1, such that if

$$\|\chi_0 - \bar{\chi}\|_4^2 \le \varepsilon_1, \tag{1.22}$$

then the system (1.7)–(1.8) admits a unique global solution $(\rho, u, \chi)(x,t)$ in $\mathbb{R} \times [0,\infty)$ which satisfies

$$\begin{aligned}
&\rho \in L^\infty(0,+\infty,H^2_{\text{per}}), \rho_t \in L^\infty(0,+\infty;H^1_{\text{per}}),\\
&u \in L^\infty(0,+\infty;H^2_{\text{per}}) \cap L^2(0,+\infty;H^3_{\text{per}}), u_t \in L^\infty(0,+\infty;H^1_{\text{per}}),\\
&\chi \in L^\infty(0,+\infty;H^4_{\text{per}}), \chi_t \in L^\infty(0,+\infty;L^2_{\text{per}}) \cap L^2(0,+\infty);H^2_{\text{per}}),\\
&\mu \in L^\infty(0,+\infty;H^2_{\text{per}}) \cap L^2(0,+\infty;H^4_{\text{per}}),\ \ \mu_t \in L^2(0,+\infty;L^2_{\text{per}}),\\
&\chi \in [-1,1],\quad \text{for all } (x,t) \in \mathbb{R} \times [0,+\infty),
\end{aligned} \tag{1.23}$$

and moreover

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left|(\rho - \bar{\rho}, u - \bar{u}, \chi - \bar{\chi})\right| = 0. \tag{1.24}$$

For the mixed initial boundary value problem, we have the following results concerning the asymptotic behavior of the global classical solutions.

Theorem 1.2 *(Mixed boundary case). Assume that (ρ_0, u_0, χ_0) satisfies the mixed boundary (1.9) with the regularities given in (1.13)–(1.18), then there exists a unique global strong solution (ρ, u, χ) of the system (1.7), (1.9), satisfying that, for any $T > 0$, there exist positive constants $m, M > 0$ such that*

$$\begin{aligned}
&\rho \in L^\infty(0,T,H^2), \rho_t \in L^\infty(0,T;H^1),\\
&u \in L^\infty(0,T;H^1_0 \cap H^2) \cap L^2(0,T;H^3), u_t \in L^\infty(0,T;H^1),\\
&\chi \in L^\infty(0,T;H^4), \chi_t \in L^\infty(0,T;L^2) \cap L^2(0,T;H^2),\\
&\mu \in L^\infty(0,T;H^2) \cap L^2(0,T;H^4),\ \ \mu_t \in L^2(0,T;L^2),\\
&\chi \in [-1,1],\ \ m \le \rho \le M,\ \text{ for all } (x,t) \in [0,L] \times [0,T].
\end{aligned} \tag{1.25}$$

Furthermore, if $\bar{\chi} \in A_{stable} \cap [-1,1]$, then there exists a small positive constant ε_1, such that if

$$\|\chi_0 - \bar{\chi}\|_4^2 \leq \varepsilon_1, \tag{1.26}$$

then the system (1.7)–(1.9) admits a unique global solution $(\rho, u, \chi)(x,t)$ in $[0,L] \times [0,\infty)$ which satisfies

$$\begin{aligned}
&\rho \in L^\infty(0,+\infty, H^2), \rho_t \in L^\infty(0,+\infty; H^1),\\
&u \in L^\infty(0,+\infty; H_0^1 \cap H^2) \cap L^2(0,+\infty; H^3), u_t \in L^\infty(0,+\infty; H^1),\\
&\chi \in L^\infty(0,+\infty; H^4), \chi_t \in L^\infty(0,+\infty; L^2) \cap L^2(0,+\infty); H^2),\\
&\mu \in L^\infty(0,+\infty; H^2) \cap L^2(0,+\infty; H^4),\ \ \mu_t \in L^2(0,+\infty; L^2),\\
&\chi \in [-1,1], \quad \text{for all } (x,t) \in [0,L] \times [0,+\infty),
\end{aligned} \tag{1.27}$$

and moreover

$$\lim_{t\to\infty} \sup_{x\in[0,L]} \left|(\rho - \bar{\rho}, u - \bar{u}, \chi - \bar{\chi})\right| = 0. \tag{1.28}$$

Remark 1.3. In above theorems, the solutions for two different boundary cases are proved to be asymptotically stable even for the large initial disturbance of the density and the large velocity data.

The key point of the proof for the main theorems is to construct some special energy functions and the approximate systems for (1.7) to get the upper and lower bound of the density and the concentration difference. The outline of this paper is organized as follows. In section 2, we construct an approximate system for (1.7), and obtain the local existence of the solutions for the approximate system. In section 3, we obtain the desired a priori estimates on strong solutions of the approximate system and take the limit for the approximate system, then give the proof of the main Theorem 1.1. Moreover, we generalized the Theorem 1.1 to the mixed boundary value problem (1.7) and (1.9), and get the proof of Theorem 1.2.

2. Local existence and uniqueness

In this section, we will give the local existence and uniqueness of the solution for the periodic boundary problem (1.7)–(1.8). For this purpose, we should construct a family of smooth approximate functions for the non-smooth free energy density f (1.4) firstly. This process is strongly motivated by the work of Blowey–Elliott [4]. The polynomial part of the free energy density $\frac{1}{4}(\chi^2-1)^2$ is replaced by the twice continuously differentiable function f_λ $(0 < \lambda < 1)$:

$$f_\lambda(\chi) = \begin{cases}
\frac{1}{2\lambda}(\chi - (1+\frac{\lambda}{2}))^2 + \frac{1}{4}\left(\chi^2-1\right)^2 + \frac{\lambda}{24}, & \chi \geq 1+\lambda,\\
\frac{1}{4}\left(\chi^2-1\right)^2 + \frac{1}{6\lambda^2}(\chi-1)^3, & 1 < \chi \leq 1+\lambda,\\
\frac{1}{4}\left(\chi^2-1\right)^2, & -1 \leq \chi \leq 1,\\
\frac{1}{4}\left(\chi^2-1\right)^2 - \frac{1}{6\lambda^2}(\chi+1)^3, & -1-\lambda < \chi \leq -1,\\
\frac{1}{2\lambda}(\chi + (1+\frac{\lambda}{2}))^2 + \frac{1}{4}\left(\chi^2-1\right)^2 + \frac{\lambda}{24}, & \chi \leq -1-\lambda.
\end{cases} \tag{2.1}$$

Directly by simple calculation, one gets

$$f_\lambda(\chi) \geq \frac{1}{4}\Big(\chi^2 - 1\Big)^2 \geq 0, \ \forall\, 0 < \lambda < 1. \tag{2.2}$$

Moreover, one has

$$\frac{\partial f_\lambda}{\partial \chi}(\chi) = \begin{cases} \frac{1}{\lambda}\big(\chi - (1 + \frac{\lambda}{2})\big) + \chi^3 - \chi, & \chi \geq 1 + \lambda, \\ \frac{1}{2\lambda^2}(\chi - 1)^2 + \chi^3 - \chi, & 1 < \chi < 1 + \lambda, \\ \chi^3 - \chi, & -1 \leq \chi \leq 1, \\ -\frac{1}{2\lambda^2}(\chi + 1)^2 + \chi^3 - \chi, & -1 - \lambda < \chi \leq -1, \\ \frac{1}{\lambda}\big(\chi + (1 + \frac{\lambda}{2})\big) + \chi^3 - \chi, & \chi \leq -1 - \lambda, \end{cases} \tag{2.3}$$

$$\chi \frac{\partial f_\lambda}{\partial \chi}(\chi) \geq \chi^4 - \chi^2, \ \chi^3 \frac{\partial f_\lambda}{\partial \chi}(\chi) \geq \chi^6 - \chi^4, \ \forall\, 0 < \lambda < 1, \tag{2.4}$$

and

$$\frac{\partial^2 f_\lambda}{\partial \chi^2}(\chi) = \begin{cases} \frac{1}{\lambda} + 3\chi^2 - 1, & \chi \geq 1 + \lambda, \\ \frac{1}{\lambda^2}(\chi - 1) + 3\chi^2 - 1, & 1 < \chi < 1 + \lambda, \\ 3\chi^2 - 1, & -1 \leq \chi \leq 1, \\ -\frac{1}{\lambda^2}(\chi + 1) + 3\chi^2 - 1, & -1 - \lambda < \chi \leq -1, \\ \frac{1}{\lambda} + 3\chi^2 - 1, & \chi \leq -1 - \lambda. \end{cases} \tag{2.5}$$

We define the function $\beta_\lambda \in C^1(\mathbb{R})$ as follows

$$\beta_\lambda(\chi) = \lambda\Big(\chi - \chi^3 + \frac{\partial f_\lambda}{\partial \chi}\Big) = \begin{cases} \chi - (1 + \frac{\lambda}{2}), & \chi \geq 1 + \lambda, \\ \frac{1}{2\lambda}(\chi - 1)^2, & 1 < \chi < 1 + \lambda, \\ 0, & -1 \leq \chi \leq 1, \\ -\frac{1}{2\lambda}(\chi + 1)^2, & -1 - \lambda < \chi \leq -1, \\ \chi + (1 + \frac{\lambda}{2}), & \chi \leq -1 - \lambda. \end{cases} \tag{2.6}$$

Lemma 2.1. *Suppose that $0 < \lambda < 1$, then β_λ satisfies the following*

1) β_λ is a Lipschitz continuous function and $0 \leq \beta_\lambda \leq 1$.

2)

$$\beta(\chi) := \lim_{\lambda \to 0} \beta_\lambda(\chi) = \begin{cases} \chi - 1, & \chi > 1, \\ 0, & -1 \leq \chi \leq 1, \\ \chi + 1, & \chi < -1. \end{cases} \tag{2.7}$$

3) $\beta(\chi)$ is a Lipschitz continuous function, and

$$|\beta(\chi) - \beta_\lambda(\chi)| \leq \frac{1}{2}\lambda, \quad |\beta(\chi_1) - \beta(\chi_2)| \leq |\chi_1 - \chi_2|.$$

Making use of the smooth polynomial approximate free energy density f_λ (2.1), we construct an approximate problem for the (1.7)–(1.8) as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_x = \nu u_{xx} - \dfrac{\epsilon}{2}\left(\chi_x^2\right)_x, \\ \rho\chi_t + \rho u \chi_x = \mu_{xx}, \\ \rho\mu = \dfrac{1}{\epsilon}\rho\dfrac{\partial f_\lambda(\rho,\chi)}{\partial\chi} - \epsilon\chi_{xx}, \\ (\rho,u,\chi)(x,t) = (\rho,u,\chi)(x+L,t), \\ (\rho,u,\chi)(x,0) = (\rho_0,u_0,\chi_0). \end{cases} \tag{2.8}$$

For $\forall m>0,\ M>0,\ B>0,\ I\subseteq\mathbb{R}$, we define

$$\begin{aligned} X_{m,M,B}(I) \equiv \Big\{(\rho,u,\chi)\Big|&(\rho,u)\in C^0(I,H^2_{\text{per}}),\chi\in C^0(I;H^4_{\text{per}}), \\ &\rho\in L^2(I;H^2_{\text{per}}), u\in L^2(I;H^3_{\text{per}}),\chi\in L^2(I;H^5_{\text{per}}), \\ &\inf_{t\in I,x\in\mathbb{R}}\rho(x,t)\ge m, \sup_{t\in I}\|(\rho,u)\|_2^2\le M, \sup_{t\in I}\|\chi-\bar{\chi}\|_4^2\le B\Big\}. \end{aligned} \tag{2.9}$$

In particular, for fixed $B>0$, we define

$$X_{0,+\infty,B}(I) \equiv \bigcup_{m>0,M>0} X_{m,M,B}(I). \tag{2.10}$$

Proposition 2.1. *Assume that (ρ_0,u_0,χ_0) satisfies the periodic boundary (1.8) with the regularities given in (1.13)–(1.18). For $\forall m>0,\ M>0$ and $B>0$, if $\inf_{x\in\mathbb{R}}\rho_0(x)\ge m$, $\|(\rho_0,u_0)\|_2^2\le M$, $\|\chi_0\|_4^2\le B$, then there exist a small time $T_*>0$ and a unique strong solution $(\rho_\lambda,u_\lambda,\chi_\lambda)$ to the approximate problem (2.8) with the smooth approximate function f_λ (2.1), such that $(\rho_\lambda,u_\lambda,\chi_\lambda)\in X_{\frac{m}{2},2M,2B}([0,T_*])$.*

Proof. Because of the difficulty with the coefficients of $(2.8)_{3,4}$ when using the result of the linear parabolic equation, we need to approximate the initial conditions. Constructing a approximate function sequence $(\rho_0^\delta,u_0^\delta,\chi_0^\delta)$ for the initial data, $(\rho_0^\delta,u_0^\delta,\chi_0^\delta)\in H^3_{\text{per}}$, $(\rho_0^\delta,u_0^\delta,\chi_0^\delta)$ satisfies the periodic boundary (1.8) with the regularities given in (1.14)–(1.18), and

$$\lim_{\delta\to0}\left(\|\rho_0^\delta-\rho_0\|_2+\|u_0^\delta-u_0\|_2+\|\chi_0^\delta-\chi_0\|_4\right)=0. \tag{2.11}$$

Taking $0<T_0<+\infty$, for $\forall m>0$, $M>0$ and $B>0$, we define

$$\begin{aligned} \tilde{X}_{m,M,B}([0,T_0]) \equiv \Big\{(\rho,u,\chi)\Big|&(\rho,u)\in C^0([0,T_0],H^3_{\text{per}}),\chi\in C^0([0,T_0];H^4_{\text{per}}), \\ &\rho\in L^2([0,T_0];H^3_{\text{per}}), u\in L^2([0,T_0];H^4_{\text{per}}),\chi\in L^2([0,T_0];H^5_{\text{per}}), \\ &\inf_{t\in[0,T_0],x\in\mathbb{R}}\rho(x,t)\ge m,\ \sup_{t\in[0,T_0]}\|(\rho,u)\|_3^2\le M,\ \sup_{t\in[0,T_0]}\|\chi-\bar{\chi}\|_4^2\le B\Big\}. \end{aligned} \tag{2.12}$$

Constructing an iterative sequence $(\rho^{(n)},u^{(n)},\chi^{(n)}), n=0,1,2\cdots$, satisfies $(\rho^{(0)},u^{(0)},\chi^{(0)})=(\rho_0^\delta,u_0^\delta,\chi_0^\delta)$, and the following iterative scheme

$$\begin{cases} \rho_t^{(n)} + (\rho^{(n)} u^{(n-1)})_x = 0, \\ \rho^{(n)} u_t^{(n)} + \rho^{(n)} u^{(n-1)} u_x^{(n)} + p_x^{(n)} = \nu u_{xx}^{(n)} - \frac{\epsilon}{2}\big((\chi_x^{(n)})^2\big)_x, \\ \rho^{(n)} \chi_t^{(n)} + \rho^{(n)} u^{(n-1)} \chi_x^{(n)} = \mu_{xx}^{(n)}, \\ \rho^{(n)} \mu^{(n)} = \frac{1}{\epsilon}\rho^{(n)} \frac{\partial f_\lambda}{\partial \chi}(\chi^{(n-1)}) - \epsilon \chi_{xx}^{(n)}, \\ (\rho^{(n)}, u^{(n)}, \chi^{(n)})(x,t) = (\rho^{(n)}, u^{(n)}, \chi^{(n)})(x+L,t), \\ (\rho^{(n)}, u^{(n)}, \chi^{(n)})\Big|_{t=0} = \big(\rho_0, u_0, \chi_0\big)(x), \end{cases} \tag{2.13}$$

here $p^{(n)} = a\big(\rho^{(n)}\big)^\gamma$. Suppose that $(\rho^{(n-1)}, u^{(n-1)}, \chi^{(n-1)}) \in \tilde{X}_{\frac{m}{2},2M,2B}([0,T_0])$, the existence and uniqueness of the strong solution $\rho^{(n)}$ for the equation $(2.13)_1$ can be obtained by the basic linear hyperbolic theory, we refer to [31] and the references therein. In addition, $\rho^{(n)}$ satisfies

$$\sup_{[0,T_0]} \Big(\|\rho^{(n)}\|_{H^3_{\mathrm{per}}} + \|\rho_t^{(n)}\|_{H^2_{\mathrm{per}}} + \|(\rho^{(n)})^{-1}\|_{L^\infty_{\mathrm{per}}} \Big) \le C(m, M, T_0). \tag{2.14}$$

Further, rewriting $(2.13)_{3,4}$ as

$$\begin{aligned} \chi_t^{(n)} &= -u^{(n-1)} \chi_x^{(n)} - \frac{\epsilon \chi_{xxxx}^{(n)}}{(\rho^{(n)})^2} + \frac{2\epsilon \rho_x^{(n)}}{(\rho^{(n)})^3} \chi_{xxx}^{(n)} + \epsilon\Big(\frac{\rho_{xx}^{(n)}}{(\rho^{(n)})^3} - \frac{2(\rho_x^{(n)})^2}{(\rho^{(n)})^4} \Big) \chi_{xx}^{(n)} \\ &\quad + \frac{\frac{\partial^2 f_\lambda}{\partial \chi^2}(\chi^{(n-1)}) \chi_{xx}^{(n-1)}}{\epsilon \rho^{(n)}} + \frac{\frac{\partial^3 f_\lambda}{\partial \chi^3}(\chi^{(n-1)}) (\chi_x^{(n-1)})^2}{\epsilon \rho^{(n)}}, \end{aligned} \tag{2.15}$$

following from [5], [21] and the references therein, the existence and uniqueness of the solution $\chi^{(n)}$ for (2.15) with the initial boundary value (1.8) can be obtained, and $\chi^{(n)}$ satisfies

$$\sup_{[0,T_0]} \Big(\|\chi^{(n)}\|_{H^4_{\mathrm{per}}} + \|\chi_t^{(n)}\|_{L^2_{\mathrm{per}}} + \|\mu^{(n)}\|_{H^2_{\mathrm{per}}} \Big) \le C(m, M, T_0). \tag{2.16}$$

Moreover, rewriting the $(2.13)_2$ as

$$u_t^{(n)} = \frac{\nu}{\rho^{(n)}} u_{xx}^{(n)} - u^{(n-1)} u_x^{(n)} - \frac{p'_\rho(\rho^{(n)})}{\rho^{(n)}} \rho_x^{(n)} - \frac{\epsilon}{\rho_x^{(n)}} \chi_x^{(n)} \chi_{xx}^{(n)}, \tag{2.17}$$

similar as above, the existence and uniqueness of the solution $u^{(n)}$ for (2.17) with the initial boundary value (1.8) can be obtained, and $u^{(n)}$ satisfies

$$\sup_{[0,T_0]} \Big(\|u^{(n)}\|_{H^3_{\mathrm{per}}} + \|u_t^{(n)}\|_{H^1_{\mathrm{per}}} \Big) \le C(m, M, T_0). \tag{2.18}$$

Now, all we have to do is to show that, there exists a $T_* > 0$ small enough, $(\rho^{(n)}, u^{(n)}, \chi^{(n)}) \in \tilde{X}_{\frac{m}{2},2M,2B}([0,T_*])$. We use mathematical induction and the energy estimate method to prove this assertion. By using $(2.13)_1$, applying the method of characteristics, $\rho^{(n)}$ can be expressed by

$$\rho^{(n)}(x,t) = \rho_0(X(x,t;0)) e^{-\int_0^t u_x^{(n-1)}(X(x,t;s),s)ds}, \tag{2.19}$$

for $(x,t) \in \mathbb{R} \times [0,T_0]$, where $X \in C(\mathbb{R} \times [0,T_0] \times [0,T_0])$ is the solution to the initial value problem

$$\begin{cases} \frac{d}{dt} X(x,t;s) = u^{(n-1)}(X(x,t;s),t), & 0 \le t \le T, \\ X(x,s;s) = x, & 0 \le s \le T, \ x \in \mathbb{R}. \end{cases} \tag{2.20}$$

The regularity of $X(x,t;s)$ gives $\rho^{(n)} \in C^0([0,T_0];H^3)$. Multiplying (2.13)$_1$ by $\rho^{(n)}$ and integrating it by parts, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho^{(n)}\|^2 &\le \frac{1}{2}\int_0^L u_x^{(n-1)}(\rho^{(n)})^2dx + \bar{\rho}\int_0^L |u_x^{(n-1)}||\rho^{(n)}|dx \\ &\le \|u_x^{(n-1)}\|^{\frac{1}{2}}\|u_{xx}^{(n-1)}\|^{\frac{1}{2}}\|\rho^{(n)}\|^2 + \frac{\bar{\rho}}{2}\big(\|u_x^{(n-1)}\|^2 + \|\rho^{(n)}\|^2\big). \end{aligned} \tag{2.21}$$

From Gronwall's inequality, one obtains

$$\|\rho^{(n)}(t)\|^2 \le \big(\|\rho_0\|^2 + \bar{\rho}\int_0^{T_0} \|u_x^{(n-1)}(\tau)\|^2 d\tau\big)e^{\left[\int_0^{T_0}(2\|u_x^{(n-1)}(\tau)^{\frac{1}{2}}\|u_{xx}^{(n-1)}(\tau)\|^{\frac{1}{2}}+1)d\tau\right]}. \tag{2.22}$$

Differentiating both side of (2.13)$_1$ with respect to x, multiplying it by $\rho_x^{(n)}$ and integrating by parts, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho_x^{(n)}\|^2 &\le 2\int_0^L u_x^{(n-1)}(\rho_x^{(n)})^2dx + \int_0^L|\rho^{(n)}-\bar{\rho}||\rho_x^{(n)}||u_{xx}^{(n-1)}|dx + \bar{\rho}\int_0^L|u_{xx}^{(n-1)}||\rho_x^{(n)}|dx \\ &\le 4\|u_x^{(n-1)}\|^{\frac{1}{2}}\|u_{xx}^{(n-1)}\|^{\frac{1}{2}}\|\rho_x^{(n)}\|^2 + 2\|u_{xx}^{(n-1)}\|\|\rho^{(n)}-\bar{\rho}\|^{\frac{1}{2}}\|\rho_x^{(n)}\|^{\frac{3}{2}} \\ &\quad + \bar{\rho}\|u_{xx}^{(n-1)}\|\|\rho_{xx}^{(n)}\|. \end{aligned} \tag{2.23}$$

Similarly, for the second derivatives and third derivatives, one gets

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho_{xx}^{(n)}\|^2 &\le C\Big(\int_0^L|u_x^{(n-1)}|(\rho_{xx}^{(n)})^2dx + \int_0^L|\rho^{(n)}-\bar{\rho}||\rho_{xx}^{(n)}||u_{xxx}^{(n-1)}|dx \\ &\quad + \int_0^L|\rho_x^{(n)}||\rho_{xx}^{(n)}||u_{xx}^{(n-1)}|dx\Big) + \bar{\rho}\int_0^L|u_{xxx}^{(n-1)}||\rho_{xx}^{(n)}|dx \\ &\le C\Big(\|u_x^{(n-1)}\|^{\frac{1}{2}}\|u_{xx}^{(n-1)}\|^{\frac{1}{2}}\|\rho_{xx}^{(n)}\|^2 + \|u_{xxx}^{(n-1)}\|\|\rho^{(n)}-\bar{\rho}\|^{\frac{1}{2}}\|\rho_x^{(n)}\|^{\frac{1}{2}}\|\rho_{xx}^{(n)}\| \\ &\quad + \|u_{xx}^{(n-1)}\|^{\frac{1}{2}}\|u_{xxx}^{(n-1)}\|^{\frac{1}{2}}\|\rho_x^{(n)}\|\|\rho_{xx}^{(n)}\|\Big) + \bar{\rho}\|u_{xxx}^{(n-1)}\|\|\rho_{xx}^{(n)}\|, \end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho_{xxx}^{(n)}\|^2 &\le C\Big(\int_0^L u_x^{(n-1)}(\rho_{xxx}^{(n)})^2dx + \int_0^L|\rho^{(n)}-\bar{\rho}||\rho_{xxx}^{(n)}||u_{xxxx}^{(n-1)}|dx \\ &+ \int_0^L|\rho_x^{(n)}||\rho_{xxx}^{(n)}||u_{xxx}^{(n-1)}|dx + \int_0^L|u_{xx}^{(n-1)}||\rho_{xx}^{(n)}||\rho_{xxx}^{(n)}|dx\Big) + \bar{\rho}\int_0^L|u_{xxxx}^{(n-1)}||\rho_{xxx}^{(n)}|dx \\ &\le C\Big(\|u_x^{(n-1)}\|^{\frac{1}{2}}\|u_{xx}^{(n-1)}\|^{\frac{1}{2}}\|\rho_{xxx}^{(n)}\|^2 + \|u_{xxxx}^{(n-1)}\|\|\rho^{(n)}-\bar{\rho}\|^{\frac{1}{2}}\|\rho_x^{(n)}\|^{\frac{1}{2}}\|\rho_{xxx}^{(n)}\| \\ &+ \|\rho_x^{(n)}\|^{\frac{1}{2}}\|\rho_{xx}^{(n)}\|^{\frac{1}{2}}\|u_{xxx}^{(n-1)}\|\|\rho_{xxx}^{(n)}\| + \|u_{xx}^{(n-1)}\|^{\frac{1}{2}}\|u_{xxx}^{(n-1)}\|^{\frac{1}{2}}\|\rho_{xx}^{(n)}\|\|\rho_{xxx}^{(n)}\|\Big) \\ &+ \bar{\rho}\|u_{xxxx}^{(n-1)}\|\|\rho_{xxx}^{(n)}\|. \end{aligned} \tag{2.25}$$

Hence, adding (2.22)–(2.25), from Gronwall's inequality, one has

$$\|\rho^{(n)}(t)\|_2 \le C\big(M + C(M)T_0\big)e^{C(M)T_0}, \tag{2.26}$$

$$\|\rho^{(n)}_{xxx}(t)\| \le C\big(M + C(M)\sqrt{T_0}\|u^{(n-1)}_{xxxx}\|_{L^2(0,T_0;L^2(\mathbb{R}))}\big)e^{C(M)T_0}, \tag{2.27}$$

and in the same way, one obtains

$$\|\rho^{(n)}_t\|_1 \le C\Big(\big(M + C(M)T_0M\big)e^{C(M)T_0} + 1\Big), \tag{2.28}$$

$$\|\rho^{(n)}_t\|_2 \le C\Big(\big(M + \sqrt{T_0}C(M)\|u^{(n-1)}_{xxxx}\|_{L^2(0,T_0;L^2(\mathbb{R}))}\big)e^{C(M)T_0} + 1\Big). \tag{2.29}$$

From (2.19) and (2.26), for T_0 small enough, which be called as T_1, $\rho^{(n)}$ satisfies

$$\inf_{t\in[0,T_1],x\in\mathbb{R}} \rho^{(n)} \ge \frac{m}{2}, \qquad \sup_{t\in[0,T_1],x\in\mathbb{R}} \|\rho^{(n)} - \bar\rho\|_3^2 \le 2M. \tag{2.30}$$

Multiplying $(2.13)_3$ by $\chi^{(n)}$, integrating over $[0, L]$ by parts, one has

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^L \rho^{(n)}\big(\chi^{(n)}\big)^2 dx &= \int_0^L \mu^{(n)}\chi^{(n)}_{xx} dx \\
&= \frac{1}{\epsilon}\int_0^L \Big(\frac{\partial f_\lambda}{\partial\chi}(\chi^{(n-1)})\Big)\chi^{(n)}_{xx} dx - \epsilon\int_0^L \frac{1}{\rho^{(n)}}(\chi^{(n)}_{xx})^2 dx,
\end{aligned} \tag{2.31}$$

thus

$$\sup_{[0,T_1]} \|(\chi^{(n)} - \bar\chi)(t)\|^2 + \int_0^{T_1} \|\chi^{(n)}_{xx}(\tau)\|^2 d\tau \le B + C(\epsilon, m, M, B)T_1. \tag{2.32}$$

Multiplying $(2.13)_3$ by $\mu^{(n)}$, integrating over $[0, L]$ by parts, one gets

$$\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int_0^L \big|\chi^{(n)}_x\big|^2 dx + \int_0^L |\mu^{(n)}_x|^2 dx \\
&= \frac{1}{\epsilon}\int_0^L \Big(\frac{\partial^2 f_\lambda}{\partial\chi^2}(\rho^{(n-1)}, \chi^{(n-1)})\Big)\chi^{(n-1)}_x\mu^{(n)}_x dx - \frac{\epsilon}{2}\int_0^L u^{(n-1)}_x\big(\chi^{(n)}_x\big)^2 dx \\
&\le \frac{1}{2}\int_0^L (\mu^{(n)}_x)^2 dx + \frac{1}{2\epsilon}\int_0^L \Big(\frac{\partial^2 f_\lambda}{\partial\chi^2}(\rho^{(n-1)}, \chi^{(n-1)})\Big)^2(\chi^{(n-1)}_x)^2 dx \\
&\quad + \frac{\epsilon}{2}\|u^{(n-1)}_x\|^{\frac{1}{2}}\|u^{(n-1)}_{xx}\|^{\frac{1}{2}}\int_0^L \big(\chi^{(n)}_x\big)^2 dx,
\end{aligned}$$

therefore

$$\frac{\epsilon}{2}\frac{d}{dt}\int_0^L \big|\chi^{(n)}_x\big|^2 dx + \frac{1}{2}\int_0^L |\mu^{(n)}_x|^2 dx \le \frac{3B^2(B^2-1)^2}{2\epsilon} + \frac{\epsilon B}{2}\int_0^L \big(\chi^{(n)}_x\big)^2 dx, \tag{2.33}$$

following from Gronwall's inequality, one has

$$\sup_{0\le t\le T_1} \epsilon\|\chi_x^{(n)}\|^2 \le \Big(\delta_0 + \frac{3B^2(B^2-1)^2}{\epsilon}T_1\Big)e^{\epsilon BT_1}, \tag{2.34}$$

substituting the inequality (2.34) into (2.33), one obtains

$$\int_0^{T_1} \|\mu_x^{(n)}\|^2 dx \le \Big(B + \frac{3B^2(B^2-1)^2}{\epsilon}T_1\Big) + \frac{B}{2}\Big(B + \frac{3B^2(B^2-1)^2}{\epsilon}T_1\Big)e^{\epsilon BT_1}T_1, \tag{2.35}$$

combining the inequalities (2.34), (2.35) and $(2.13)_4$, one has

$$\int_0^{T_0}\int_0^L |\chi_{xxx}^{(n)}|^2 dx \le C\big(B + C(\epsilon, m, M, B)T_1\big). \tag{2.36}$$

Differentiating $(2.13)_3$ with respect to t, one gets

$$\rho^{(n)}\chi_{tt}^{(n)} + \rho_t^{(n)}\chi_t^{(n)} + \rho_t^{(n)}u^{(n-1)}\chi_x^{(n)} + \rho^{(n)}u_t^{(n-1)}\chi_x^{(n)} + \rho^{(n)}u^{(n-1)}\chi_{xt}^{(n)} = \mu_{xxt}^{(n)}, \tag{2.37}$$

multiplying (2.37) by $\chi_t^{(n)}$, integrating over $[0, L]$, one has

$$\begin{aligned}
&\frac{1}{2}\int_0^L \rho^{(n)}|\chi_t^{(n)}|^2 dx + \int_0^L \frac{\epsilon}{\rho^{(n)}}|\chi_{xxt}^{(n)}|^2 dx \\
&= -\int_0^L \Big(\rho_t^{(n)}|\chi_t^{(n)}|^2 + \rho_t^{(n)}u^{(n-1)}\chi_x^{(n)}\chi_t^{(n)} + \rho^{(n)}u_t^{(n-1)}\chi_x^{(n)}\chi_t^{(n)} + \big(\frac{1}{\rho^{(n)}}\big)_t\chi_{xx}^{(n)}\chi_{xxt}^{(n)}\Big)dx \\
&\quad + \frac{1}{\epsilon}\int_0^L \Big(\frac{\partial^2 f_\lambda}{\partial\chi^2}(\rho^{(n-1)}, \chi^{(n-1)})\Big)\chi_t^{(n-1)}\chi_{xxt}^{(n)} dx \\
&\le \int_0^L \frac{\epsilon}{2\rho^{(n)}}|\chi_{xxt}^{(n)}|^2 dx + C(M)\int_0^L \rho^{(n)}|\chi_t^{(n)}|^2 + C(\epsilon, M, B)(1+T_1).
\end{aligned} \tag{2.38}$$

Noting that from (2.15), we see that

$$\|\sqrt{\rho^{(n)}}\chi_t^{(n)}(0)\| \le C(\|\rho_0\|_2, \|u_0\|, \|\chi_0\|_4), \tag{2.39}$$

combining with (2.38), and from Gronwall's inequality, one obtains

$$\begin{aligned}
&\sup_{[0,T_1]} \|\chi_t^{(n)}(\tau)\|^2 + \int_0^{T_1} \big(\|\chi_{xxt}^{(n)}(\tau)\|^2 + \|\mu_t^{(n)}(\tau)\|^2 + \|\mu_{xxxx}^{(n)}(\tau)\|^2\big)d\tau \\
&\le C\big(B + C(\epsilon, m, M, B)T_1\big).
\end{aligned} \tag{2.40}$$

Moreover, in the same way, one gets

$$\sup_{[0,T_1]} \|\chi^{(n)}_{xx}(\tau)\|^2 + \int_0^{T_1} \|\chi^{(n)}_{xxxx}(\tau)\|^2 d\tau \le C(B + C(\epsilon, m, M, B)T_1), \tag{2.41}$$

and

$$\sup_{[0,T_1]} \left(\|\mu^{(n)}(t)\|^2 + \|\mu^{(n)}_{xx}(t)\|^2 + \|\chi^{(n)}_{xxx}(t)\|^2 + \|\chi^{(n)}_{xxxx}(t)\|^2\right) + \int_0^{T_1} \|\chi^{(n)}_{xxxxxx}(\tau)\|^2 d\tau$$
$$\le C(B + C(\epsilon, m, M, B)T_1). \tag{2.42}$$

Multiplying $(2.13)_2$ by $u^{(n)}$, integrating over $[0, L]$, one obtains

$$\frac{d}{dt}\int_0^L \frac{1}{2}\rho^{(n)}|u^{(n)}|^2 dx + \int_0^L p^{(n)}_x u^{(n)} dx + \nu \int_0^L |u^{(n)}_x|^2 dx = -\frac{1}{2}\int_0^L \left(\chi^{(n)}_x\right)^2_x u^{(n)} dx. \tag{2.43}$$

We define

$$G(\rho) = \rho \int_{\bar\rho}^{\rho} s^{-2}(p(s) - \bar p) ds, \tag{2.44}$$

where $\bar p = p(\bar\rho)$, so that by $(2.13)_1$, one gets

$$G(\rho^{(n)})_t + \left(G(\rho^{(n)})u^{(n-1)}\right)_x + \left(p(\rho^{(n)}) - \bar p\right)u^{(n-1)}_x = 0. \tag{2.45}$$

Integrating the result and adding it to (2.43), then one has

$$\frac{d}{dt}\int_0^L \left(\frac{1}{2}\rho^{(n)}|u^{(n)}|^2 + G(\rho^{(n)})\right) dx + \nu \int_0^L |u^{(n)}_x|^2 dx = -\frac{1}{2}\int_0^L \left(\chi^{(n)}_x\right)^2_x u^{(n)} dx, \tag{2.46}$$

thus, one obtains

$$\sup_{[0,T_1]} \left(\|u^{(n)}(t)\|^2 + \|\rho^{(n)}(t) - \bar\rho\|^2\right) + \int_0^{T_1} \|u^{(n)}_x(\tau)\|^2 d\tau \le C(M + C(\epsilon, m, M, B)T_1). \tag{2.47}$$

Differentiating both side of $(2.13)_2$ with respect to x, multiplying it by $u^{(n)}_t$, integrating over $[0, L]$ by parts, one has

$$\frac{1}{2}\frac{d}{dt}\|u^{(n)}_x\|^2 + \int_0^L \rho^{(n)}|u^{(n)}_t|^2 dx$$
$$= -\int_0^L \rho^{(n)}u^{(n-1)}u^{(n)}_x u^{(n)}_t dx - \int_0^L p'(\rho^{(n)})\rho^{(n)}_x u^{(n)}_t dx - \epsilon \int_0^L \chi^{(n)}_x \chi^{(n)}_{xx} u^{(n)}_t dx$$
$$\le \frac{1}{2}\int_0^L \rho^{(n)}|u^{(n)}_t|^2 dx + c\|\rho^{(n)}\|^{\frac{1}{2}}\|\rho^{(n)}_x\|^{\frac{1}{2}}\|u^{(n-1)}\|\|u^{(n-1)}_x\| \int_0^L |u^{(n)}_x|^2 dx + C(m, M, B),$$

then from Gronwall's inequality, one gets

$$\sup_{[0,T_1]} \|u_x^{(n)}(t)\|^2 + \int_0^{T_1}\int_0^L \rho^{(n)}|u_t^{(n)}|^2 dxd\tau \le C(M + C(\epsilon, m, M, B)T_1). \tag{2.48}$$

In the same way, for the higher derivative of $u^{(n)}$, one obtains

$$\begin{aligned}&\sup_{[0,T_1]} \big(\|u_t^{(n)}(t)\|^2 + \|u_{xx}^{(n)}(t)\|^2 + \|u_{xt}^{(n)}(t)\|^2 + \|u_{xxx}^{(n)}(t)\|^2\big)\\ &+ \int_0^{T_1}\int_0^L \big(|u_{xt}^{(n)}|^2 + |u_{xxt}^{(n)}|^2 + |u_{xxx}^{(n)}|^2\big) dxd\tau \le C(M + C(\epsilon, m, M, B)T_1).\end{aligned} \tag{2.49}$$

From the energy inequalities (2.26)–(2.30), (2.32)–(2.36), (2.40)–(2.42), (2.47)–(2.49), we can choose T small enough, without loss of generality, say $T_* > 0$, and all the conditions stated in the definition of $\tilde{X}_{\frac{m}{2},2M,2B}([0,T_*])$ are guaranteed. Thereby the iterative sequence $(\rho^{(n)}, u^{(n)}, \chi^{(n)}) \in \tilde{X}_{\frac{m}{2},2M,2B}([0,T_*])$. Moreover, $(\rho^{(n)}, u^{(n)}, \chi^{(n)})$ is the Cauchy sequence in $C^0([0,T_*]; H^1_{\text{per}})\times C^0([0,T_*]; H^1_{\text{per}})\times C^0([0,T_*]; H^1_{\text{per}})$. We define $(\rho_\lambda, u_\lambda, \chi_\lambda)$ the limit of the sequence $(\rho^{(n)}, u^{(n)}, \chi^{(n)})$. It is easy to know that $(\rho_\lambda, u_\lambda, \chi_\lambda)$ is the uniqueness solution of the systems (2.8) in the space $\tilde{X}_{\frac{m}{2},2M,2B}([0,T_*])$. Finally, letting $\delta \to 0$, we can easily derive the limit (v, u, χ) of sequence $(\rho^\delta, u^\delta, \chi^\delta)$ satisfies, $(v, u, \chi) \in X_{m,M,B}([0,T_0])$, and it is a solution of system (2.8) with the initial data (ρ_0, u_0, χ_0). The proof of Proposition 2.1 is completed. □

3. A priori estimates

In this section, we will present the desired estimates and global existence of the solution for the approximate periodic boundary problem (2.8), then we will take the limit of the approximate equation (2.8), and give the proof of the main Theorem 1.1. Furthermore, we will extend these results to mixed boundary problem (1.7), (1.9) and obtain the Theorem 1.2. Based on the local existence and the a priori estimates, we may obtain the global solution by the continuity extension argument developed in the previous papers [12,13], [24–30] and the references therein.

Proposition 3.1. *Let $\bar{\chi} \in A_{stable}\cap[-1,1]$ be fixed. Assume that (ρ_0, u_0, χ_0) satisfies (1.13)–(1.18). Then there exist positive constants C and ε_1 depending only on ν, ϵ, such that if*

$$\|(\chi_0 - \bar{\chi})\|_4^2 \le \varepsilon_1, \tag{3.1}$$

then the periodic boundary problem (1.7), (1.8) has a solution $(\rho, u, \chi) \in X_{0,+\infty,\varepsilon_1}([0,T])$, and

$$\begin{aligned}&\sup_{t\in[0,T]} \big(\|(\rho, u)\|_2^2 + \|\chi(t)\|_4^2 + \|\rho_t\|_2^2 + \|u_t\|_1^2 + \|\chi_t\|^2\big)\\ &\qquad + \int_0^T \big(\|\rho_x(t)\|_1^2 + \|u_x(t)\|_2^2 + \|\chi_x(t)\|_4^2 + \|u_t(t)\|_2^2 + \|\chi_t(t)\|_2^2\big) dt\\ &\le C\Big(\|(\rho_0, u_0)\|_2^2 + \|\chi_0\|_4^2\Big).\end{aligned} \tag{3.2}$$

Proof. For fixed positive constant $\bar{\chi} \in A_{\text{stable}}$, from Sobolev theorem, one can choose B_0 small enough, such that for $0 < B \le B_0$,

$$\chi \in A_{\text{stable}}. \tag{3.3}$$

Multiplying $(2.8)_2$ by u and $(2.8)_3$ by μ, integrating over $[0, L] \times [0, T]$ and adding them up, combining with (2.2), one has

$$\sup_{t\in[0,T]} \int_0^L \Big(\frac{1}{2}\rho u^2 + \frac{\epsilon}{2}|\chi_x|^2 + G(\rho) + \frac{1}{4\epsilon}\rho\big(\chi^2 - 1\big)^2\Big)dx + \int_0^T\int_0^L \big(|\mu_x|^2 + \nu|u_x|^2\big)dx$$
$$\leq \int_0^L \Big(\frac{1}{2}\rho_0 u_0^2 + \frac{\epsilon}{2}|\chi_{0x}|^2 + G(\rho_0) + \frac{1}{4\epsilon}\rho_0\big(\chi_0^2 - 1\big)^2\Big)dx. \tag{3.4}$$

Multiplying $(2.8)_3$ by χ, integrating over $[0, L]$, combining with (2.3) and (2.4), one gets

$$\frac{1}{2}\frac{d}{dt}\int_0^L \rho\chi^2 dx + \frac{1}{\epsilon}\int_0^L (3\chi^2 - 1)\chi_x^2 dx + \epsilon\int_0^L \frac{1}{\rho}\chi_{xx}^2 dx \leq 0, \tag{3.5}$$

by using (3.3), adding (3.4) up, integrating over $[0, T]$, one has

$$\sup_{[0,T]} \|\sqrt{\rho}(\chi(t) - \bar{\chi})\|^2 + \int_0^T \big(\|\chi_x(\tau)\|^2 + \|\frac{1}{\sqrt{\rho}}\chi_{xx}(\tau)\|^2\big)d\tau \leq C\|\sqrt{\rho_0}(\chi_0 - \bar{\chi})\|^2. \tag{3.6}$$

Noting that

$$u_{xx} = -\big[\frac{1}{\rho}(\rho_t + \rho_x u)\big]_x = \big[\rho(\frac{1}{\rho})_t + \rho u(\frac{1}{\rho})_x\big]_x = \rho\frac{d}{dt}(\frac{1}{\rho})_x + \rho u(\frac{1}{\rho})_{xx}, \tag{3.7}$$

then $(2.8)_2$ can be written as

$$(\rho u)_t + (\rho u^2)_x + p'(\rho)\rho_x = \nu\big[\rho\frac{d}{dt}(\frac{1}{\rho})_x + \rho u(\frac{1}{\rho})_{xx}\big] - \frac{\epsilon}{2}\big(\chi_x^2\big)_x. \tag{3.8}$$

Multiplying (3.8) by $(\frac{1}{\rho})_x$, integrating over $[0, L]$, one has

$$\frac{d}{dt}\int_0^L \big(\frac{\nu}{2}\rho\big|(\frac{1}{\rho})_x\big|^2 - \rho u(\frac{1}{\rho})_x\big)dx + \int_0^L \frac{p'(\rho)}{\rho^2}\rho_x^2 dx$$
$$= -\int_0^L \rho u(\frac{1}{\rho})_{xt}dx + \int_0^L (\rho u^2)_x(\frac{1}{\rho})_x dx + \frac{\epsilon}{2}\int_0^L \big(\chi_x^2\big)_x(\frac{1}{\rho})_x dx$$
$$= \int_0^L \Big((\rho u)_x(-\frac{\rho_t}{\rho^2}) + (\rho u^2)_x(-\frac{\rho_x}{\rho^2})\Big)dx + \epsilon\int_0^L \chi_x\chi_{xx}(\frac{1}{\rho})_x dx$$
$$= \int_0^L u_x^2 dx + \epsilon\int_0^L \chi_x\chi_{xx}(\frac{1}{\rho})_x dx. \tag{3.9}$$

Multiplying (3.9) by $\frac{\nu}{2}$, adding up to (3.4) and (3.5), one obtains

$$
\begin{aligned}
&\frac{d}{dt}\int_0^L \Big[\frac{\nu^2}{4}\rho\big|(\frac{1}{\rho})_x\big|^2-\frac{\nu}{2}\rho u(\frac{1}{\rho})_x+\frac{1}{2}\rho u^2+\rho\chi^2+\frac{\epsilon}{2}|\chi_x|^2+G(\rho)+\frac{\rho}{4\epsilon}(\chi^2-1)^2\Big]dx\\
&+\frac{\nu}{2}\int p_\rho'(\rho)\frac{\rho_x^2}{\rho^2}dx+\frac{\nu}{2}\int_0^L|u_x|^2dx+\int_0^L|\mu_x|^2dx+\frac{1}{\epsilon}\int_0^L(3\chi^2-1)\chi_x^2dx+\epsilon\int_0^L\frac{1}{\rho}\chi_{xx}^2dx\\
&=\frac{\nu\epsilon}{2}\int_0^L\chi_x\frac{\chi_{xx}}{\rho^{\frac12}}\rho^{\frac12}\big(\frac{1}{\rho}\big)_xdx\le\frac{\nu\epsilon}{\sqrt{2}}\|\chi_x\|^{\frac12}\|\chi_{xx}\|^{\frac12}\int_0^L\big|\frac{\chi_{xx}}{\rho^{\frac12}}\big|\big|\rho^{\frac12}\big(\frac{1}{\rho}\big)_x\big|dx\\
&\le 2\nu\epsilon\|\chi_x\|\|\chi_{xx}\|+\frac{\nu^2\epsilon}{4}\Big\|\frac{\chi_{xx}}{\rho^{\frac12}}\Big\|^2\Big\|\rho^{\frac12}\big(\frac{1}{\rho}\big)_x\Big\|^2\\
&\le 4\nu^2\epsilon\|\chi_x\|^2+\epsilon\|\rho^{\frac12}\|^2\|\rho^{-\frac12}\chi_{xx}\|^2+\frac{\nu^2\epsilon}{4}\Big\|\frac{\chi_{xx}}{\rho^{\frac12}}\Big\|^2\Big\|\rho^{\frac12}\big(\frac{1}{\rho}\big)_x\Big\|^2,
\end{aligned}\tag{3.10}
$$

combining with (3.3), (3.4), (3.6), (3.10), one gets

$$
\begin{aligned}
&\int_0^L\Big(\rho\big|(\frac{1}{\rho})_x\big|^2+\rho u^2+\rho\chi^2+|\chi_x|^2+G(\rho)+\rho(\chi^2-1)^2\Big)dx\\
&+\int_0^T\int_0^L\Big(p_\rho'(\rho)\frac{\rho_x^2}{\rho^2}+|u_x|^2+|\mu_x|^2+(3\chi^2-1)\chi_x^2+\frac{1}{\rho}\chi_{xx}^2\Big)dxdt\\
&\le C\Big(\|(\rho_0,\chi_0)\|_1^2+\|\sqrt{\rho_0}u_0\|^2\Big).
\end{aligned}\tag{3.11}
$$

Now we're going to give the upper and lower bounds of density. According to the method in Kanel' [15], constructing a function

$$
\Psi(\rho):=\int_{\bar\rho}^{\rho}\frac{\sqrt{G(s)}}{s^{\frac32}}ds.\tag{3.12}
$$

By using the definition of $\bar\rho$ (1.10), we know that for fixed t, there exist a point $x_0\in[0,L]$, $\rho(x_0,t)=\bar\rho$, and one has

$$
\begin{aligned}
|\Psi(\rho(x,t))|&=\Big|\int_{x_0}^{x}\frac{\partial}{\partial s}\Psi(\rho(s,t))ds\Big|\le\int_0^L\Big|\frac{G(\rho)^{\frac12}}{\rho^{\frac32}}\rho_x\Big|dx\\
&\le\Big(\int_0^L G(\rho)dx\Big)^{\frac12}\Big(\int_0^L\rho\big(\frac{1}{\rho}\big)_x^2dx\Big)^{\frac12}\le C(\rho_0,u_0,\chi_0).
\end{aligned}\tag{3.13}
$$

On the other hand, noting that the definition of (2.44), one gets

$$
G(\rho):=\begin{cases}
a\Big(\rho\ln\rho-\rho(\ln\bar\rho+1)+\bar\rho\Big)\begin{cases}\sim\rho\ln\rho,\ \rho\to0^+,\\ \sim\rho\ln\rho,\ \rho\to+\infty,\end{cases} & \gamma=1,\\
a\Big(\frac{\rho^\gamma}{\gamma-1}-\frac{\gamma\bar\rho^{\gamma-1}}{\gamma-1}\rho+\bar\rho^\gamma\Big)\begin{cases}\sim\rho,\quad\rho\to0^+,\\ \sim\rho^\gamma,\ \rho\to+\infty,\end{cases} & \gamma>1,
\end{cases}\tag{3.14}
$$

and then

$$\Psi(\rho) := \int_{\bar{\rho}}^{\rho} \frac{\sqrt{G(s)}}{s^{\frac{3}{2}}} ds \begin{cases} \longrightarrow +\infty, & \rho \to +\infty, \\ \longrightarrow -\infty, & \rho \to 0^+. \end{cases} \tag{3.15}$$

Taking advantage of (3.13), there exist two positive constants $\underline{m} > 0$ and $\overline{M} > 0$, satisfy

$$0 < \underline{m} \leq \rho \leq \bar{M} < +\infty. \tag{3.16}$$

From $(2.8)_4$, one derives that

$$\frac{\epsilon}{\rho}\chi_{xx} = \frac{1}{\epsilon}\frac{\partial f_\lambda}{\partial \chi}(\chi) - \mu, \tag{3.17}$$

differentiating (3.17) with respect to x, one has

$$\epsilon\chi_{xxx} = -\rho\mu_x + \frac{\epsilon\rho_x}{\rho}\chi_{xx} + \frac{\rho}{\epsilon}\frac{\partial^2 f_\lambda}{\partial \chi^2}\chi_x, \tag{3.18}$$

then directly from (3.11) and (3.16), one gets at once

$$\int_0^T \|\chi_{xxx}\|^2 dt \leq C\Big(\|(\rho_0,\chi_0)\|_1^2 + \|u_0\|^2\Big). \tag{3.19}$$

Multiplying $(2.8)_2$ by $-\frac{1}{\rho}u_{xx}$, integrating over $\mathbb{R}$ by parts, one gets

$$\begin{aligned}
&\Big(\int \frac{1}{2}u_x^2 dx\Big)_t + \nu\int_0^L \frac{1}{\rho}u_{xx}^2 dx = \int_0^L u_{xx}\Big(\frac{p'_\rho}{\rho}\rho_x + uu_x + \frac{\epsilon}{\rho}\chi_x\chi_{xx}\Big)dx \\
&\leq \frac{\nu}{4}\int_0^L \frac{1}{\rho}u_{xx}^2 dx + \frac{c_1}{\nu}\Big(\int_0^L (p'_\rho)^2\frac{\rho_x^2}{\rho}dx + \int \rho u^2 u_x^2 dx + \int \frac{1}{\rho}\chi_x^2\chi_{xx}^2 dx\Big) \\
&\leq \frac{\nu}{4}\int_0^L \frac{1}{\rho}u_{xx}^2 dx + \frac{c_1}{\nu}\Big(\int_0^L (p'_\rho)^2\frac{\rho_x^2}{\rho}dx + \|u_x\|\|u_{xx}\|\int_0^L \rho u^2 dx + \|\chi_x\|\|\chi_{xx}\|\int_0^L \frac{1}{\rho}\chi_{xx}^2 dx\Big) \\
&\leq \frac{\nu}{2}\int_0^L \frac{1}{\rho}u_{xx}^2 dx + \frac{c_2}{\nu}\Big(\int_0^L p'_\rho(\rho)\frac{\rho_x^2}{\rho^2}dx + \|u_x\|^2 + \int_0^L \frac{1}{\rho}\chi_{xx}^2 dx\Big),
\end{aligned} \tag{3.20}$$

integrating over $[0,T]$, from (3.11) and (3.19), one has

$$\sup_{[0,T]}\|u_x(t)\|^2 + \int_0^T \|u_{xx}(t)\|^2 dt \leq C\Big(\|(\rho_0,u_0)\|_1^2 + \|\chi_0\|_2^2\Big). \tag{3.21}$$

Multiplying $(2.8)_3$ by χ_t, integrating over $[0,L]$, one gets

$$\int_0^L \rho\chi_t^2 dx + \int_0^L \rho u\chi_x\chi_t dx = \int_0^L \mu\chi_{xxt} dx$$

$$= -\frac{1}{\epsilon}\int_0^L \frac{\partial^2 f_\lambda}{\partial\chi^2}\chi_x\chi_{xt}dx - \frac{\epsilon}{2}\frac{d}{dt}\int_0^L \frac{1}{\rho}\chi_{xx}^2 dx + \epsilon\int_0^L (\frac{1}{\rho})_t\chi_{xx}^2 dx, \tag{3.22}$$

and consequently

$$\begin{aligned}
&\int_0^L \rho\chi_t^2 dx + \frac{1}{2\epsilon}\frac{d}{dt}\int_0^L \frac{\partial^2 f_\lambda}{\partial\chi^2}\chi_x^2 dx + \frac{\epsilon}{2}\frac{d}{dt}\int_0^L \frac{1}{\rho}\chi_{xx}^2 dx \\
&= -\int_0^L \rho u\chi_x\chi_t dx + \frac{\epsilon}{2}\int_0^L (\frac{1}{\rho})_t\chi_{xx}^2 dx \\
&\le \frac{1}{2}\int_0^L \rho\chi_t^2 dx + C\|\chi_x\|\|\chi_{xx}\|\int_0^L \rho u^2 dx + \frac{\epsilon}{2}\Big|\int_0^L (\frac{u_x}{\rho}\chi_{xx}^2 + \frac{\rho_x u}{\rho^2}\chi_{xx}^2)dx\Big| \\
&\le \frac{1}{2}\int_0^L \rho\chi_t^2 dx + C\|\chi_x\|\|\chi_{xx}\| + C\|\chi_{xx}\|^{\frac{1}{2}}\|\chi_{xxx}\|^{\frac{1}{2}}\Big(\int_0^L u_x^2 dx + \int_0^L \chi_{xx}^2 dx\Big) \\
&\quad + C\|\chi_{xx}\|^{\frac{1}{2}}\|\chi_{xxx}\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|u_x\|^{\frac{1}{2}}(\int_0^L \rho_x^2 dx + \int_0^L \chi_{xx}^2 dx) \\
&\le C\Big(\|\chi_x\|^2 + \|\chi_{xx}\|^2 + \|\chi_{xx}\|^{\frac{1}{2}}\|\chi_{xxx}\|^{\frac{1}{2}}(\int_0^L u_x^2 dx + \int_0^L \rho_x^2 dx + \int_0^L \chi_{xx}^2 dx)\Big) \\
&\quad + \frac{1}{2}\int_0^L \rho\chi_t^2 dx,
\end{aligned} \tag{3.23}$$

that is, combining with (3.11), (3.19) and (2.5), one obtains

$$\sup_{[0,T]}\Big(\|\chi_x(t)\|^2 + \|\chi_{xx}(t)\|^2\Big) + \int_0^T \|\chi_t(t)\|^2 dt \le C\|\chi_0\|_2^2. \tag{3.24}$$

Differentiating $(2.8)_3$ with respect of t, one has

$$\rho\chi_{tt} + \rho_t\chi_t + \rho_t u\chi_x + \rho u_t\chi_x + \rho u\chi_{xt} = \mu_{xxt}, \tag{3.25}$$

multiplying (3.25) by χ_t, integrating over $[0, L]$, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L \rho\chi_t^2 dx + \int_0^L \frac{1}{\rho}\chi_{xxt}^2 dx \\
&= -2\int_0^L \rho u\chi_t\chi_{xt}dx + \int_0^L \rho_x u^2\chi_t\chi_x dx + \int_0^L \rho u_x u\chi_x\chi_t dx \\
&\quad - \int_0^L \rho u_t\chi_x\chi_t dx + \frac{1}{\epsilon}\int_0^L \frac{\partial^2 f_\lambda}{\partial\chi^2}\chi_t\chi_{xxt}dx - \int_0^L \frac{u_x}{\rho}\chi_{xx}\chi_{xxt}dx - \int_0^L \frac{\rho_x}{\rho^2}u\chi_{xx}\chi_{xxt}dx
\end{aligned}$$

$$\leq \frac{1}{4}\int_0^L \chi_{xxt}^2 dx + \frac{1}{4}\int_0^L \rho u_t^2 dx + C\int_0^L \left(\chi_t^2 + \chi_{xx}^2\right) dx, \tag{3.26}$$

on the other hand, multiplying $(2.8)_2$ by u_t, integrating over $[0, L]$, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2 dx + \int_0^L \rho u_t^2 dx = -\int_0^L \rho u u_x u_t dx dt - \int_0^L p'_\rho \rho_x u_t dx dt - \epsilon\int_0^L \chi_x \chi_{xx} u_t dx dt\\ &\leq \frac{1}{4}\int_0^L \rho u_t^2 dx + C\int_0^L \left(u_x^2 + \rho_x^2 + \chi_{xx}^2\right) dx,\end{aligned} \tag{3.27}$$

summing (3.22), (3.26) and (3.27) up, making use of (3.11), (3.19), (3.21) and (2.5), one obtains

$$\begin{aligned}&\sup_{[0,T]} \left(\|\chi_x(t)\|_1^2 + \|\chi_t(t)\|^2 + \|u_x(t)\|^2\right) + \int_0^T \left(\|\chi_t(t)\|^2 + \|\chi_{xxt}^2(t)\|^2 + \|u_t(t)\|^2\right) dx\\ &\leq C\Big(\|(\rho_0, u_0)\|_1^2 + \|\chi_0\|_2^2 + \|\chi_{0t}\|^2\Big).\end{aligned} \tag{3.28}$$

Differentiating $(2.8)_3$ with respect of x, multiplying by μ_{xxx}, one gets

$$\sup_{[0,T]} \|\chi_{xxx}(t)\|^2 + \int_0^T \|\mu_{xxx}\|^2 dx \leq C\|\chi_0 - \bar{\chi}\|_3^2. \tag{3.29}$$

Directly from (2.8), by using the estimates (3.11), (3.21), (3.28), one has

$$\begin{aligned}&\sup_{[0,T]} \left(\|\rho_t(t)\|^2 + \|\mu(t)\|^2 + \|\mu_{xx}(t)\|^2\right) + \int_0^T \left(\|\mu_t(t)\|^2 + \|\mu_{xxx}(t)\|^2\right) dt\\ &\leq C\Big(\|(\rho_0, u_0)\|_1^2 + \|\chi_0\|_2^2 + \|\chi_{0t}\|^2\Big).\end{aligned} \tag{3.30}$$

Differentiating $(2.8)_2$ with respect of t, one derives

$$\rho_t u_t + \rho u_{tt} + \rho_t u u_x + \rho u_t u_x + \rho u u_{xt} + p_{xt} = \nu u_{xxt} - \epsilon \chi_{xt}\chi_{xx} - \epsilon \chi_x \chi_{xxt}. \tag{3.31}$$

Multiplying (3.31) by u_t, integrating over $[0, L] \times [0, T]$ by parts, making use of (3.11), (3.19), (3.21) and (3.28), one gets

$$\sup_{[0,T]} \|u_t(t)\|^2 + \int_0^T\int_0^L u_{xt}^2 dx dt \leq C\Big(\|(\rho_0 - \bar{\rho}, u_0)\|_1^2 + \|\chi_0 - \bar{\chi}\|_2^2 + \|\chi_{0t}\|^2 + \|u_{0t}\|^2\Big). \tag{3.32}$$

Differentiating $(2.8)_2$ with respect of x, one derives

$$\rho_x u_t + \rho u_{xt} + \rho_x u u_x + \rho u_x u_x + \rho u u_{xx} + p_{xx} = \nu u_{xxx} - \epsilon \chi_{xx}^2 - \epsilon \chi_x \chi_{xxx}. \tag{3.33}$$

Multiplying (3.33) by u_{xxx}, integrating over $[0, L] \times [0, T]$ by parts, making use of (3.11), (3.19), (3.21) and (3.28), (3.32), one gets

$$\sup_{[0,T]} \|u_{xx}(t)\|^2 + \int_0^T \int_0^L u_{xxx}^2 dxdt \le C\Big(\|\rho_0\|_1^2 + \|(u_0,\chi_0)\|_2^2 + \|\chi_{0t}\|^2 + \|u_{0t}\|^2\Big). \tag{3.34}$$

Differentiating (3.8) with respect of x, one derives

$$(\rho u)_{xt} + (\rho u^2)_{xx} + p_{xx} = \nu\big[\rho \frac{d}{dt}(\frac{1}{\rho})_x + \rho u(\frac{1}{\rho})_x\big]_x - \frac{\epsilon}{2}\big(\chi_x^2\big)_{xx}. \tag{3.35}$$

Multiplying (3.35) and (3.33) by $\big(\frac{1}{\rho}\big)_{xx}$, u_x respectively, integrating over $[0, L]$ and summing, one obtains

$$\begin{aligned}&\sup_{[0,T]} \Big(\|u_x(t)\|^2 + \|\rho_{xx}(t)\|^2\Big) + \int_0^T \big(\|\rho_{xx}(\tau)\|^2 + \|u_{xx}(\tau)\|^2\big)d\tau \\ &\le C\Big(\|\rho_0\|_2^2 + \|u_0\|_1^2 + \|\chi_0\|_3^2\Big).\end{aligned} \tag{3.36}$$

Furthermore, similar to the proof above, one has

$$\sup_{[0,T]} \|\chi_{xxxx}(t)\|^2 + \int_0^T \|\mu_{xxxx}\|^2 dx \le C\|\chi_0 - \bar{\chi}\|_4^2. \tag{3.37}$$

Now we will let $\lambda \to 0$ in approximate problem (2.8). For $0 < \lambda < 1$, from the results above, there exists a solution $(\rho_\lambda, u_\lambda, \chi_\lambda)$ of approximate problem (2.8), satisfies $(\rho_\lambda, u_\lambda, \chi_\lambda) \in X_{0,+\infty,B}([0, +\infty))$. Multiplying $(2.8)_4$ by $\beta_\lambda(\chi_\lambda)$, integrating over $[0, L]$, one has

$$\begin{aligned}&\epsilon \int_0^L \rho_\lambda (\chi_\lambda)_x (\beta_\lambda)_x dx + \frac{1}{\epsilon\lambda} \int_0^L \rho_\lambda \beta_\lambda^2 dx \\ &= \int_0^L \rho_\lambda \mu_\lambda \beta_\lambda dx - \int_0^L \frac{\rho_\lambda}{\epsilon}(\chi_\lambda^3 - \chi_\lambda)\beta_\lambda dx \\ &\le C\epsilon\lambda\Big(\int_0^L \rho_\lambda \mu_\lambda^2 dx + \frac{1}{\epsilon^2}\int_0^L \rho_\lambda\Big(\frac{\partial f_\lambda}{\partial \chi}(\rho_\lambda, \chi_\lambda)\Big)dx\Big) + \frac{1}{2\epsilon\lambda}\int_0^L \rho_\lambda \beta_\lambda^2 dx,\end{aligned} \tag{3.38}$$

combining with

$$\begin{aligned}\epsilon \int_0^L \rho_\lambda (\chi_\lambda)_x (\beta_\lambda)_x dx &= \epsilon \int_0^L \rho_\lambda \beta_\lambda'(\chi_\lambda)_x^2 dx \\ &\ge \epsilon \int_0^L \rho_\lambda (\beta_\lambda')^2 (\chi_\lambda)_x^2 dx \ge \epsilon \int_0^L \rho_\lambda (\beta_\lambda)_x^2 dx,\end{aligned} \tag{3.39}$$

one obtains

$$\|\beta_\lambda\|_{L^2(0,T;L^2)} \le C\lambda, \text{ and } \|(\beta_\lambda)_x\|_{L^2(0,T;L^2)} \le C\lambda^{\frac{1}{2}}, \tag{3.40}$$

thus, letting $\lambda \to 0$, one has $\beta(\chi) = 0$, that is,

$$-1 \le \chi \le 1. \tag{3.41}$$

Moreover, by using the compactness theory and the Lions–Aubin argument, combining with the results above, it is easy to know that the system (1.7)–(1.8) has a strong solution (ρ, u, χ) when $\lambda \to 0$. Therefore, the proof of Proposition 3.1 is completed. □

Now we will give the proof of Theorem 1.1. For the global existence, we only need to get rid of the small condition (3.1) in the Proposition 3.1. Noting that the small condition for the initial value χ only used in (3.5) to get the estimate (3.6), rewriting (3.5) as follows

$$\frac{1}{2}\frac{d}{dt}\int_0^L \rho\chi^2 dx + \frac{1}{\epsilon}\int_0^L 3\chi^2\chi_x^2 dx + \epsilon\int_0^L \frac{1}{\rho}\chi_{xx}^2 dx \le \frac{1}{\epsilon}\int_0^L \chi_x^2 dx, \tag{3.42}$$

integrating over $[0, T]$, one has

$$\frac{1}{2}\int_0^L \rho\chi^2(t) dx + \frac{1}{\epsilon}\int_0^L 3\chi^2\chi_x^2 dx + \epsilon\int_0^T\int_0^L \frac{1}{\rho}\chi_{xx}^2 dx = \frac{1}{\epsilon}\int_0^T\int_0^L \chi_x^2 dx + \frac{1}{2}\int_0^L \rho\chi_0^2 dx, \tag{3.43}$$

by using (3.4) and Gronwall's inequality, one obtains similar energy estimates without the small condition (3.1), but the constant C depending on T, and the global existence for (1.7)–(1.8) is obtained, the uniqueness of this solution can be obtained by the classical method, we omit it. At last, it suffices to present the large time behavior for the solution (ρ, u, χ) of system (1.7)–(1.8). By using the energy inequality (3.2), combining with Sobolev embedding theorem for periodic functions, one has at once

$$\begin{aligned}\sup_{x\in[0,L]}\Big|(\rho-\bar\rho, \rho u-\overline{\rho u}, \rho\chi-\overline{\rho\chi})(x,t)\Big| &\le L\|(\rho_x, (\rho u)_x, (\rho\chi)_x)(t)\| \\ &\longrightarrow 0, \quad \text{as } t \to +\infty,\end{aligned} \tag{3.44}$$

thus, (1.24) is obtained. Moreover, by the similar way above, we have the proof of Theorem 1.2, the details are omitted here.

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