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Convergence to Diffusion Waves of the Solutions for Benjamin-Bona-Mahony-Burgers Equations

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Abstract

This paper is concerned with the large-time behavior of solution of the Cauchy problem for the Benjamin-Bona-Mahony-Burgers equation. We prove that the solution unique globally exists and time-asymptotically tends to its corresponding diffusion wave, when the initial perturbation is small enough. The corresponding diffusion wave is constructed by the heat equation or the Burgers equation. In particular, we obtain the convergence rates in L^q-spaces ($2 \le q \le \infty$). The mathematical proof is based on the Fourier transform method and the energy method. Furthermore, we take the numerical computations on such a problem. The numerical simulations show that the convergence rates obtained theoretically seem to be sharp.

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1 Introduction and Main Result

Considered here is the time-asymptotic behavior of solutions to the Cauchy problem of the Benjamin-Bona-Mahony-Burgers (BBM-B) equations in the form

$$\begin{cases} u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + u^p u_x = 0 \\ u|_{t=0} = u_0(x) \to 0, \quad \text{as } x \to \pm\infty \end{cases} \tag{1.1}$$

where $x \in \boldsymbol{R}$, $t > 0$, $p \geq 1$ is integer, $\alpha > 0$ and β are any given constants. Without loss of generality, we may let $\alpha = 1/2$ here and after here, because we may make a suitable scale to variables $x \to 2\alpha x$ and $t \to 2\alpha t$ such that Eq. (1.1) becomes

$$u_t - \frac{1}{4\alpha^2} u_{xxt} - \frac{1}{2} u_{xx} + \beta u_x + u^p u_x = 0.$$

Since D. H. Pergrine [27], T. B. Benjamin, J. L. Bona and J. J. Mahony [2] proposed the alternative regularized long-wave equations for the physical phenomenon of bore propagation and water waves as follows

$$u_t - u_{xxt} + u_x + u u_x = 0,$$

so-called the Benjamin-Bona-Mahony (BBM) equation, this subject has become a hot spot and attracted many mathematicians and physicists. There are a number of works on the time-asymptotic behavior of solutions, see [1-6,8,19-23,28] and the references therein. The asymptotic state of the BBM-B solution $u(x,t)$ is usually considered as zero in the previous works. However, we find that its corresponding diffusion wave, a solution of corresponding parabolic partial differential equation to (1.1), is a better asymptotic profile than the 0 in such a sense that the convergence of $u(x,t)$ toward the diffusion wave is faster than that of $u(x,t)$ toward the 0. This will be theoretically proved and numerically experimented in the following four sections, which is our main goal in the present paper.

At first, let us recall the so-called diffusion waves. Let p (≥ 1) be integer and let

$$f(u) = \beta u + \frac{u^{p+1}}{p+1}, \tag{1.2}$$

we consider the following parabolic equation

$$\begin{cases} \theta_t - \frac{1}{2}\theta_{xx} + [f(0) + f'(0)\theta + \frac{1}{2}f''(0)\theta^2]_x = 0 \\ \theta|_{t=0} = \theta_0(x) \to 0, \quad \text{as } x \to \pm\infty. \end{cases} \tag{1.3}$$

We call this solution as the diffusion wave to the BBM-B equation (1.1). Note that

$$f'(0) = \beta \ , \quad f''(0) = \begin{cases} 1 \ , & for \ \ p = 1, \\ 0 \ , & for \ \ p \geq 2, \end{cases} \tag{1.4}$$

when $p \geq 2$, Eq. (1.3) is equivalent to

$$\begin{cases} \theta_t - \frac{1}{2}\theta_{xx} + \beta\theta_x = 0, \\ \theta|_{t=0} = \theta_0(x), \end{cases} \tag{1.5}$$

which is a linear heat equation, and has a unique solution in the form

$$\theta(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\beta t-y)^2}{2t}} \theta_0(y) dy. \tag{1.6}$$

This solution is called the linear diffusion wave for Eq. (1.1) with $p \geq 2$.

When $p = 1$, we may reduce (1.3) into

$$\begin{cases} \theta_t - \frac{1}{2}\theta_{xx} + \beta\theta_x + \theta\theta_x = 0, \\ \theta|_{t=0} = \theta_0(x). \end{cases} \tag{1.7}$$

By using the Hopf-Cole transformation

$$\theta(x,t) = -(\ln\varphi)_x, \quad i.e., \quad \varphi(x,t) = e^{-\int_{-\infty}^{x}\theta(\xi,t)d\xi} \tag{1.8}$$

to Eq. (1.7), then it can be reduced to

$$\begin{cases} \varphi_t + \beta\varphi_x - \frac{1}{2}\varphi_{xx} = 0, \\ \varphi|_{t=0} = e^{-\int_{-\infty}^{x}\theta_0(y)dy} =: \varphi_0(x). \end{cases} \tag{1.9}$$

It is well-known that the above linear heat equation has a unique solution

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-\beta t-y)^2}{2t}}\varphi_0(y)dy. \tag{1.10}$$

Thus, from (1.8) and (1.10), we obtain the solution for (1.7)

$$\begin{aligned} \theta(x,t) &= -(\ln\varphi)_x = -\frac{\varphi_x}{\varphi} \\ &= \frac{\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\beta t-y)^2}{2t}\right)\exp\left(-\int_{-\infty}^{y}\theta_0(\eta)d\eta\right)\theta_0(y)dy}{\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\beta t-y)^2}{2t}\right)\exp\left(-\int_{-\infty}^{y}\theta_0(\eta)d\eta\right)dy}. \end{aligned} \tag{1.11}$$

This solution is called the nonlinear diffusion wave for Eq. (1.1) with $p = 1$.

For a parabolic conservation law, Chern and Liu [7], Jeffrey and Zhao [11] studied that, for a given initial data $\theta_0(x) \in L^1 \cap H^2$, the diffusion wave solutions of (1.3) decay in the form

$$\|\theta(t)\|_{L^2} = O(t^{-1/4}),$$

furthermore, if

$$\int_{-\infty}^{\infty}\theta_0(x)\,dx = 0,$$

then

$$\|\theta(t)\|_{L^2} = O(t^{-3/4}).$$

On the other hand, for the BBM-B equation (1.1), corresponding to the restriction on the initial data

$$\int_{-\infty}^{\infty}u_0(x)\,dx \neq 0 \quad \text{or} \quad = 0,$$

the same decay rates

$$\|u(t)\|_{L^2} = O(t^{-1/4}) \quad \text{or} \quad O(t^{-3/4})$$

were also showed by Amick-Bona-Schonbek [1], Bisognin- Menzala [3,4], Bona-Luo [5,6], Dix [8], Mei [20,21], Namkin-Shishmarov [24] and Zhang [29], and so on.

In the present paper, we are interested in the convergence of solution $u(x,t)$ of the BBM-B equation (1.1) to the corresponding diffusion wave solution $\theta(x,t)$ of the parabolic equation (1.3), when the initial perturbation is small. Explicitly speaking, in the case of $\int_{-\infty}^{\infty}\theta_0(x)dx \neq 0$ and $\int_{-\infty}^{\infty}u_0(x)dx \neq 0$, according to the previous results mentioned above, the diffusion waves $\theta(x,t)$ and the unique solution $u(x,t)$ of (1.1) have the same decay rates $\|\theta(t)\|_{L^2} = 0(t^{-1/4})$ and $\|u(t)\|_{L^2} = O(t^{-1/4})$, which implies naturally that $\|(u-\theta)(t)\|_{L^2} = O(t^{-1/4})$, however, this decay rate is not satisfactory if the initial perturbation $u_0(x)-\theta_0(x)$ satisfies

$$\int_{-\infty}^{\infty}[u_0(x) - \theta_0(x)]\, dx = 0. \tag{1.12}$$

In fact, we expect that the solution $u(x,t)$ converges to the corresponding diffusion wave $\theta(x,t)$ faster than $O(t^{-1/4})$ in L^2-sense, namely, we will prove that

$$\|(u-\theta)(t)\|_{L^2} = \begin{cases} O(t^{-(3/4)+\sigma}), & \text{for } p=1 \\ O(t^{-3/4}\log(2+t)), & \text{for } p=2 \\ O(t^{-3/4}), & \text{for } p\geq 3 \end{cases}$$

where σ is any given positive constant, of course, we may let $0<\sigma\ll 1$. This means that the diffusion wave $\theta(x,t)$ is a better asymptotic profile of $u(x,t)$ than the 0.

Now let us compare our results with the previous interesting works in this field. In the framework [7] by I-L. Chern and T.-P. Liu, they investigated that the solution of the Cauchy problem to a $n\times n$ hyperbolic system of viscous conservation laws including the single equation case as follows

$$u_t + f(u)_x - u_{xx} = 0 \quad \text{with} \quad f''(0)\neq 0, \quad u(x,0) = u_0(x),$$

tends to the nonlinear diffusion wave for the Burgers equation

$$\theta_t + f'(0)\theta_x + f''(0)\theta\theta_x - \theta_{xx} = 0, \quad \theta(x,0) = \theta_0(x),$$

under the restriction $\int_{-\infty}^{\infty}[u_0(x)-\theta_0(x)]dx = 0$. The L^2-convergence rates of $u(x,t)$ to $\theta(x,t)$ is $O(t^{-(3/4)+\sigma})$ for any given small constant $\sigma>0$. This is quite same to our result in the case $p=1$. Recently, G. Karch in [12] and [13] studied the convergence to the diffusion wave for the KdV-Burgers equation

$$u_t + uu_x - u_{xx} + u_{xxx} = 0$$

and the BBM-Burgers equation

$$u_t + uu_x - u_{xx} - u_{xxt} = 0$$

respectively, and showed the L^2-convergence rate to be $O(t^{-1/4})$. But it is less than ours for the case $p=1$.

When $p > 1$, it is clear that $|u^p u_x|$ is much smaller than $|uu_x|$ if $|u(x,t)|$ is small, namely, $|u^p u_x|$ decays much faster. On the other hand, u_{xxt} is also a faster decay term. Therefore, to consider the long time asymptotic behavior of the solution to Eq. (1.1), the effects by both of $u^p u_x$ and u_{xxt} in Eq. (1.1) for $p > 1$ may be deleted, and the main controlling part of the BBM-B equation (1.1) is expected to be the linear part

$$u_t + \beta u_x - \frac{1}{2} u_{xx} = 0.$$

This is why we can expect that the solutions $u(x,t)$ of the BBM-B equations for $p \geq 2$ converge to the linear diffusion waves $\theta(x,t)$. We further prove that the decay rates are faster than that for $p = 1$. Indeed, as stated above, we obtain the L^2-decay rates $O(t^{-3/4}\log(2+t))$ and $O(t^{-3/4})$ for $p = 2$ and $p \geq 3$, respectively. Regarding the convergence of solutions to diffusion waves for other types of viscous hyperbolic conservation laws, we refer to those works in [7,10,15-18,25,26].

Now let

$$\varepsilon := \int_{-\infty}^{\infty} (|\theta_0(x)| + |x\theta_0(x)|)\, dx < +\infty,$$

we are going to state the main results as follows.

Theorem 1.1 *Suppose that (1.12) and*

$$v_0(y) := \int_{-\infty}^{x} [u_0(y) - \theta_0(y)]dy \in W^{3,1}(\boldsymbol{R}) \tag{1.14}$$

hold. Then there exists a positive constant δ_0 such that when $\|v_0\|_{W^{3,1}} + \varepsilon \leq \delta_0$, then the Cauchy problem (1.1) has a unique global solution $u(x,t)$ satisfying

$$u(x,t) - \theta(x,t) \in C(0,\infty; H^1(\boldsymbol{R}))$$

and the followings:

(i) If $p = 1$, for any $\sigma > 0$, then the following estimates hold

$$\begin{aligned} &\|(u-\theta)(t)\|_{L^2} = O(1)(1+t)^{-\frac{3}{4}+\sigma}, \\ &\|(u-\theta)_x(t)\|_{L^2} = O(1)(1+t)^{-1+\sigma}, \qquad (1.15)_1 \\ &\|(u-\theta)(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{7}{8}+\sigma}. \end{aligned}$$

(ii) If $p = 2$, the convergence rates are as follows

$$\begin{aligned} &\|(u-\theta)(t)\|_{L^2} = O(1)(1+t)^{-\frac{3}{4}}\log(2+t), \\ &\|(u-\theta)_x(t)\|_{L^2} = O(1)(1+t)^{-1}, \qquad (1.15)_2 \\ &\|(u-\theta)(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{7}{8}}\sqrt{\log(2+t)}. \end{aligned}$$

(iii) If $p \geq 3$, the convergence to the diffusion wave are much faster as follows

$$\begin{aligned} &\|(u-\theta)(t)\|_{L^2} = O(1)(1+t)^{-\frac{3}{4}}, \\ &\|(u-\theta)_x(t)\|_{L^2} = O(1)(1+t)^{-\frac{5}{4}}, \\ &\|(u-\theta)(t)\|_{L^\infty} = O(1)(1+t)^{-1}. \end{aligned} \tag{1.15$_3$}$$

Using L^2, L^∞-results in Theorem 1.1 and the interposing inequality

$$\|f\|_{L^q} \leq \|f\|_{L^\infty}^{(q-2)/q}\|f\|_{L^2}^{2/q}, \quad \text{for} \quad 2 \leq q \leq \infty,$$

we can obtain immediately L^q-decay rates as follows.

Corollary 1.1 *Under the assumptions in Theorem 1.1, it follows*

$$\|(u-\theta)(t)\|_{L^q} = \begin{cases} O(1)(1+t)^{-(7/8)+(1/4q)+\sigma}, & \text{for } p=1 \\ O(1)(1+t)^{-(7/8)+(1/4q)}(\log(2+t))^{(1/2)+(1/q)}, & \text{for } p=2 \\ O(1)(1+t)^{-1+(1/2q)}, & \text{for } p \geq 3 \end{cases} \tag{1.16}$$

for $2 \leq q \leq \infty$.

For the mathematical proof of Theorem 1.1, as showed in [20,21], we are going to adopt the Fourier transform method and the energy method. These will be carried out in Sections 2 and 3. Finally, in Section 4, we take the numerical computations on the two Cauchy problems of BBM-Burgers equation and the corresponding parabolic equation (diffusion wave's equation), respectively. The numerical simulations show that the convergence rates obtained theoretically to the diffusion waves seem to be sharp.

Notations. We now make some notation for simplicity. C always denotes some positive constants without confusion. $\partial_x^k f := \partial^k f/\partial x^k$. L^p presents the Lebesque integral space with the norm $\|\cdot\|_{L^p}$. Especially, L^2 is the square integral space with the norm $\|\cdot\|_{L^2}$, and L^∞ is the essential bounded space with the norm $\|\cdot\|_{L^\infty}$. H^k and $W^{k,p}$ denote the usual Sobolev space with the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W^{k,p}}$, respectively. Suppose that $f(x) \in L^1 \cap L^2(\boldsymbol{R})$, we define the Fourier transforms of $f(x)$ as follows:

$$F[f](\xi) \equiv \hat{f} = \int_{\boldsymbol{R}} f(x)e^{-ix\xi}dx.$$

Let T and B be a positive constant and a Banach space, respectively. $C^k(0,T;B)$ $(k \geq 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$, and $L^2(0,T;B)$ denotes the space of B-valued L^2-functions on $[0,T]$. The corresponding spaces of B-valued function on $[0,\infty)$ are defined similarly.

2 Reformulation to the Original Problem

From Eqs. (1.1) and (1.3), we have

$$(u-\theta)_t - u_{xxt} - \frac{1}{2}(u-\theta)_{xx} + [f(u) - f(0) - f'(0)\theta - \frac{1}{2}f''(0)\theta^2]_x = 0. \tag{2.1}$$

Since $\theta(\pm\infty, t) = 0$, and we expect $u(\pm\infty, t) = 0$, $u_x(\pm\infty, t) = 0$, then after integrating (2.1) over $(-\infty, \infty)$, we have formally

$$\frac{d}{dt}\int_{-\infty}^{\infty} [u(x,t) - \theta(x,t)]\, dx = 0. \tag{2.2}$$

Thanks to the essential assumption (1.12), we integrate (2.2) over $[0, t]$ with respect to t to have

$$\int_{-\infty}^{\infty} [u(x,t) - \theta(x,t)]dx = \int_{-\infty}^{\infty} [u_0(x) - \theta_0(x)]dx = 0. \tag{2.3}$$

Therefore, it is natural to introduce

$$v(x,t) = \int_{-\infty}^{x} [u(y,t) - \theta(y,t)]dy, \quad i.e., \quad v_x(x,t) = u(x,t) - \theta(x,t), \tag{2.4}$$

which satisfies

$$v_{xt} - v_{xxxt} - \theta_{xxt} - \frac{1}{2}v_{xxx} + [f(\theta + v_x) - f(0) - f'(0)\theta - \frac{1}{2}f''(0)\theta^2]_x = 0. \tag{2.5}$$

Integrating it over $(-\infty, x]$ with respect to x, and noting $u(\pm\infty, t) = 0$, $\theta(\pm\infty, t) = 0$, we obtain

$$\begin{cases} v_t - v_{xxt} - \frac{1}{2}v_{xx} - \theta_{xt} + f(\theta + v_x) - f(0) - f'(0)\theta - \frac{1}{2}f''(0)\theta^2 = 0, \\ v|_{t=0} = \int_{-\infty}^{x} [u_0(y) - \theta_0(y)]dy = v_0(x). \end{cases} \tag{2.6}$$

Noting $f(u) = \beta u + \frac{u^{p+1}}{p+1}$, we have from (2.6) that, if $p = 1$

$$\begin{cases} v_t - v_{xxt} - \frac{1}{2}v_{xx} + \beta v_x = F_1, \\ v|_{t=0} = v_0(x), \end{cases} \tag{2.7$_1$}$$

where

$$F_1 = \theta_{xt} - [\frac{1}{2}(\theta + v_x)^2 - \frac{1}{2}\theta^2] = \theta_{xt} - \frac{1}{2}v_x^2 - \theta v_x; \tag{2.8$_1$}$$

and if $p \geq 2$,

$$\begin{cases} v_t - v_{xxt} - \frac{1}{2}v_{xx} + \beta v_x = F_p, \\ v|_{t=0} = v_0(x), \end{cases} \tag{2.7$_p$}$$

where

$$F_p = \theta_{xt} - \frac{1}{p+1}(\theta + v_x)^{p+1} = \theta_{xt} - \frac{1}{p+1}\sum_{i=0}^{p+1} a_i \theta^i v_x^{p+1-i}, \quad p \geq 2 \qquad (2.8)_p$$

for some positive constants $a_i = C_{p+1}^i$.

Now we are going to state the following theorem for the Cauchy problem (2.6) which implies Theorem 1.1 directly.

Theorem 2.1 *Under the same assumptions in Theorem 1.1. Then there exits a positive constant δ_1, such that when $\|v_0\|_{W^{3,1}} + \varepsilon < \delta_1$, then the Cauchy problem $(2.7)_p$ $(p \geq 1)$ has a unique global solution $v(x,t)$ satisfying*

$$v(x,t) \in C(0,\infty; H^2(\boldsymbol{R})).$$

Furthermore, we have the following estimates.

1. *When $p = 1$, for any given $\sigma > 0$, the solution $v(x,t)$ of $(2.7)_1$ satisfies*

$$\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\|\partial_x^j v(t)\|_{L^2} + (1+t)^{1-\sigma}\|v_{xx}(t)\|_{L^2} \leq C(\|v_0\|_{W^{3,1}} + \varepsilon). \qquad (2.10)_1$$

2. *When $p = 2$, the solution $v(x,t)$ of $(2.7)_2$ satisfies*

$$\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}}\log^{-1}(2+t)\|\partial_x^j v(t)\|_{L^2} + (1+t)\|v_{xx}(t)\|_{L^2} \leq C(\|v_0\|_{W^{3,1}} + \varepsilon). \quad (2.10)_2$$

3. *When $p \geq 3$, the solution $v(x,t)$ of $(2.7)_p$ $(p \geq 3)$ satisfies*

$$\sum_{j=0}^{2}(1+t)^{\frac{2j+1}{4}}\|\partial_x^j v(t)\|_{L^2} \leq C(\|v_0\|_{W^{3,1}} + \varepsilon). \qquad (2.10)_3$$

By Theorem 2.1 and the well-known inequalities

$$\|v(t)\|_{L^\infty} \leq \sqrt{2}\|v(t)\|_{L^2}^{1/2}\|v_x(t)\|_{L^2}^{1/2},$$

$$\|v_x(t)\|_{L^\infty} \leq \sqrt{2}\|v_x(t)\|_{L^2}^{1/2}\|v_{xx}(t)\|_{L^2}^{1/2},$$

we can obtain the decay rates for $\|v(t)\|_{L^\infty}$ and $\|v_x(t)\|_{L^\infty}$ as follows.

Corollary 2.1 *Under the assumptions in Theorem 1.1, it follows*

$$\|v(t)\|_{L^\infty} = \begin{cases} O(1)(1+t)^{-(\frac{1}{2}-\sigma)}, & \text{for } p = 1 \\ O(1)(1+t)^{-\frac{1}{2}}\log(2+t), & \text{for } p = 2 \\ O(1)(1+t)^{-\frac{1}{2}}, & \text{for } p \geq 3 \end{cases} \qquad (2.10)_4$$

and

$$\|v_x(t)\|_{L^\infty} = \begin{cases} O(1)(1+t)^{-(\frac{7}{8}-\sigma)}, & for\ \ p=1 \\ O(1)(1+t)^{-\frac{7}{8}}\sqrt{\log(2+t)}, & for\ \ p=2 \\ O(1)(1+t)^{-1}, & for\ \ p\geq 3. \end{cases} \qquad (2.10)_5$$

Once Theorem 2.1 is proved, by Theorem 2.1 and Corollary 2.1, and noting (2.4), i.e., $v_x(x,t) = u(x,t) - \theta(x,t)$, we can prove Theorem 1.1 immediately. Therefore, to prove Theorem 2.1 is our main purpose in the rest of this paper. We are going to prove it based on the following local existence and the *a priori* estimates by the continuation argument. Before stating these two results, we now define the solution spaces as follows, for any $T > 0$ and given $\delta > 0$,

$$X_p(0,T) = \left\{v \in C(0,\infty; H^2)\middle| M_p(T) \leq \delta\right\}, \quad p \geq 1,$$

where

$$M_1(T) = \sup_{0\leq t\leq T} \left\{\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\|\partial_x^j v(t)\|_{L^2} + (1+t)^{1-\sigma}\|v_{xx}(t)\|_{L^2}\right\}$$

$$M_2(T) = \sup_{0\leq t\leq T} \left\{\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}}\log^{-1}(2+t)\|\partial_x^j v(t)\|_{L^2} + (1+t)\|v_{xx}(t)\|_{L^2}\right\}$$

$$M_p(T) = \sup_{0\leq t\leq T} \sum_{j=0}^{2}(1+t)^{\frac{2j+1}{4}}\|\partial_x^j v(t)\|_{L^2}, \quad \text{for } p \geq 3.$$

Now we give the theorems of local existence and *a priori* estimates.

Proposition 2.2 (local existence) *Suppose that* $v_0 \in H^2$ *holds, then there exists a positive constant* T_0 *such that the Cauchy problem* $(2.7)_p$ $(p \geq 1)$ *has a unique solution* $v(x,t) \in X_p(0,T_0)$ *satisfying* $M_p(T_0) \leq 2M_p(0)$ *for all* $p \geq 1$.

Proposition 2.3 (a priori estimate) *Let* T *be a positive constant, and* $v(x,t) \in X_p(0,T)$ $(p \geq 1)$ *be a solution of the Cauchy problem* $(2.7)_p$. *Suppose that the assumptions in Theorem 1.1 hold, then there exist positive constants* δ_2 *and* C *independent of* T *such that if* $M_p(t) \leq \delta_2$, *then for* $t \in [0,T]$ *the following estimates hold:*

1. *When* $p = 1$, *for any* $\sigma > 0$

$$\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}-\sigma}\|\partial_x^j v(t)\|_{L^2} + (1+t)^{1-\sigma}\|v_{xx}(t)\|_{L^2} \leq C(\|v_0\|_{W^{3,1}} + \varepsilon). \qquad (2.11)_1$$

2. When $p = 2$,

$$\sum_{j=0}^{1}(1+t)^{\frac{2j+1}{4}}\log^{-1}(2+t)\|\partial_x^j v(t)\|_{L^2} + (1+t)\|v_{xx}(t)\|_{L^2} \le C(\|v_0\|_{W^{3,1}} + \varepsilon). \quad (2.11)_2$$

3. When $p \ge 3$,

$$\sum_{j=0}^{2}(1+t)^{\frac{2j+1}{4}}\|\partial_x^j v(t)\|_{L^2} \le C(\|v_0\|_{W^{3,1}} + \varepsilon). \quad (2.11)_3$$

As showed in [21], using the continuation argument based on Propositions 2.2 and 2.3, we can prove Theorem 2.1. We omit the details. So, to prove Propositions 2.2 and 2.3 is our goal. Since Proposition 2.1 can be proved in the standard way, our main effort will be made on the proof of Proposition 2.2 in the next section.

3 A Priori Estimates

For the cases $p \ge 1$, by use of the Fourier transform to $(2.7)_p$ $(p \ge 1)$, we obtain

$$\hat{v}_t - (\mathrm{i}\xi)^2\hat{v}_t - \frac{1}{2}(\mathrm{i}\xi)^2\hat{v} + \mathrm{i}\xi\beta\hat{v} = \widehat{F}_p, \quad (3.1)$$

namely,

$$\hat{v}_t + \frac{\frac{1}{2}\xi^2 + \mathrm{i}\beta\xi}{1+\xi^2}\hat{v} = \frac{\widehat{F}_p}{1+\xi^2}.$$

Thus we have

$$\hat{v}(\xi,t) = e^{-A(\xi)t}\hat{v}_0 + \int_0^t e^{-A(\xi)(t-s)}\frac{\widehat{F}_p(\xi,s)}{1+\xi^2}ds \quad (3.2)$$

where

$$A(\xi) = \frac{\frac{1}{2}\xi^2 + \mathrm{i}\beta\xi}{1+\xi^2}, \quad B(\xi) = Re A(\xi) = \frac{\frac{1}{2}\xi^2}{1+\xi^2}. \quad (3.3)$$

Taking the inverse Fourier transform to (3.2), we have

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\xi x}e^{-A(\xi)t}\hat{v}_0(\xi)d\xi + \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} e^{\mathrm{i}\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F}_p(\xi,s)}{1+\xi^2}d\xi ds. \quad (3.4)$$

Differentiating it with respect to x, we have

$$\begin{aligned}\partial_x^j v_x(x,t) = &\ \tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty}(\mathrm{i}\xi)^j e^{\mathrm{i}\xi x}e^{-A(\xi)t}\hat{v}_0(\xi)d\xi \\ &-\tfrac{1}{2\pi}\textstyle\int_0^t\int_{-\infty}^{\infty}(\mathrm{i}\xi)^j e^{\mathrm{i}\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F}_p(\xi,s)}{1+\xi^2}d\xi ds.\end{aligned} \quad (3.5)$$

Now we give several preparation lemmas as follows.

Lemma 3.1 **([7, 11])** *Let $\theta(x,t)$ be the diffusion waves of (1.3). If*

$$\varepsilon=\int_{-\infty}^{\infty}(|\theta_0(x)|+|x\theta_0(x)|)dx<+\infty. \tag{3.6}$$

then

$$\|\partial_x^j\theta(t)\|_{L^2}=O(1)\varepsilon(1+t)^{-\frac{2j+1}{4}}, \tag{3.7}$$

$$\|\theta(t)\|_{L^q}=O(1)\varepsilon(1+t)^{-\frac{q-1}{2q}},\quad 1\le q\le\infty \tag{3.8}$$

$$\|\partial_x^j\theta_t(t)\|_{L^1}=O(1)\varepsilon(1+t)^{-1-\frac{j}{2}} \tag{3.9}$$

hold for all $t\ge 0$.

Lemma 3.2 **([28])** *Let $a>0$ and $b>0$ be constants. If $\max(a,b)>1$, then*

$$\int_0^t(1+t-s)^{-a}(1+s)^{-b}ds\le C(1+t)^{-\min(a,b)}. \tag{3.10}$$

If $\max(a,b)=1$, then

$$\int_0^t(1+t-s)^{-a}(1+s)^{-b}ds\le C(1+t)^{-\min(a,b)}\log(2+t). \tag{3.11}$$

If $\max(a,b)<1$, then

$$\int_0^t(1+t-s)^{-a}(1+s)^{-b}ds\le C(1+t)^{1-a-b}. \tag{3.12}$$

The applications of this lemma can be found in many works, e.g. in [19-22].

Lemma 3.3 **([20,21])** *It holds*

$$\int_{-\infty}^{\infty}\frac{|\xi|^j e^{-CB(\xi)t}}{(1+\xi^2)(1+|\xi|)^j}d\xi\le C(1+t)^{-\frac{j+1}{2}}. \tag{3.13}$$

Lemma 3.4 **([20,21])** *If $v_0\in W^{3,1}(\boldsymbol{R})$, then*

$$\Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^j e^{ix\xi}e^{-A(\xi)t}\hat{v}_0(\xi)d\xi\Big\|_{L^2}\le C\|v_0\|_{W^{j+1,1}}(1+t)^{-\frac{2j+1}{4}} \tag{3.14}$$

for $j=0,1,2$.

Lemma 3.5 *Suppose that $v(x,t)\in X_1(0,T)$, then*

$$\int_0^t\Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^j e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F}_1(\xi,s)}{1+\xi^2}d\xi\Big\|_{L^2}ds$$
$$\le C[\varepsilon+(\varepsilon+\delta M_1(T))^2](1+t)^{-\frac{2j+1}{4}+\sigma} \tag*{(3.16)$_j$}$$

for $j=0,1$, and

$$\int_0^t\Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^2 e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F}_1(\xi,s)}{1+\xi^2}d\xi\Big\|_{L^2}ds$$
$$\le C[\varepsilon+(\varepsilon+\delta M_1(T))^2](1+t)^{-1+\sigma}. \tag*{(3.16)$_2$}$$

Proof. Let $v(x,t) \in X_1(0,T)$. Since

$$|F_1| \le |\theta_{xt}| + \frac{1}{2}|v_x|^2 + |v_x\theta|, \tag{3.17}$$

$$|F_{1x}| \le |\theta_{xxt}| + |v_{xx}||v_x| + |v_{xx}\theta| + |v_x\theta_x|, \tag{3.18}$$

using Lemma 3.1, the properties of Fourier transform and the definition of $M_1(T)$, we obtain

$$\begin{aligned}
\sup_{\xi\in R}|\widehat{F}_1(\xi,s)| &\le \textstyle\int_{-\infty}^{\infty}|F_1(x,s)|dx \\
&\le \textstyle\int_{-\infty}^{\infty}[|\theta_{xt}| + \frac{1}{2}|v_x|^2 + |v_x\theta|]dx \\
&\le C[\|\theta_{xt}(s)\|_{L^1} + \|v_x(s)\|_{L^2}^2 + \|v_x(s)\|_{L^2}\|\theta(s)\|_{L^2}] \\
&\le C[\varepsilon(1+s)^{-\frac{3}{2}} + M_1(T)^2(1+s)^{-(\frac{3}{2}-2\sigma)} \\
&\quad +\varepsilon M_1(T)(1+s)^{-(1-\sigma)}] \\
&\le C[\varepsilon + (\varepsilon + M_1(T))^2](1+s)^{-(1-\sigma)}
\end{aligned} \tag{3.19}$$

provided $0 < \sigma \le 1/2$, and

$$\begin{aligned}
\sup_{\xi\in R}|\xi||\widehat{F}_1(\xi,s)| &\le \textstyle\int_{-\infty}^{\infty}|F'_{1x}(x,s)|dx \\
&\le \textstyle\int_{-\infty}^{\infty}[|\theta_{xxt}| + |v_{xx}||v_x| + |v_{xx}\theta| + |v_x\theta_x|]dx \\
&\le \|\theta_{xxt}(s)\|_{L^1} + \|v_{xx}(s)\|_{L^2}\|v_x\|_{L^2} + \|v_{xx}(s)\|_{L^2}\|\theta(s)\|_{L^2} \\
&\quad +\|v_x(s)\|_{L^2}\|\theta_x(s)\|_{L^2} \\
&\le C\varepsilon(1+s)^{-2} + M_1(T)^2(1+s)^{-2(1-\sigma)} + \varepsilon M_1(T)(1+s)^{-(\frac{3}{2}-\sigma)} \\
&\le C[\varepsilon + (\varepsilon + M_1(T))^2](1+s)^{-(\frac{3}{2}-\sigma)}
\end{aligned} \tag{3.20}$$

for $0 < \sigma \le 1/2$.

Applying the Parseval's equality and using (3.19), (3.20), Lemma 3.3 and Lemma 3.2, we then have

$$
\begin{aligned}
\int_0^t & \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}\xi x} \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F}_1(\xi, s) d\xi \right\|_{L^2} ds \\
&= \int_0^t \left\| \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F}_1(\xi, s) \right\|_{L^2} ds \\
&= \int_0^t \left(\int_{-\infty}^{\infty} \frac{e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} |\widehat{F}_1(\xi, s)|^2 d\xi \right)^{\frac{1}{2}} ds \\
&\leq \int_0^t \sup_{\xi \in R} |\widehat{F}_1(\xi, s)| \left(\int_{-\infty}^{\infty} \frac{e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} d\xi \right)^{\frac{1}{2}} ds \\
&\leq C[\varepsilon + (\varepsilon + M_1(T))^2] \int_0^t (1+s)^{-(1-\sigma)} (1+t-s)^{-\frac{1}{4}} ds \\
&\leq C[\varepsilon + (\varepsilon + M_1(T))^2] (1+t)^{1-(1-\sigma)-\frac{1}{4}} \\
&= C[\varepsilon + (\varepsilon + M_1(T))^2] (1+t)^{-(\frac{1}{4}-\sigma)},
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
\int_0^t & \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{i}\xi e^{\mathrm{i}\xi x} \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F}_1(\xi, s) d\xi \right\|_{L^2} ds \\
&= \int_0^t \left\| \mathrm{i}\xi \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F}_1(\xi, s) \right\|_{L^2} ds \\
&= \int_0^t \left(\int_{-\infty}^{\infty} \frac{|\xi|^2 e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} |\widehat{F}_1(\xi, s)|^2 d\xi \right)^{\frac{1}{2}} ds \\
&\leq \int_0^t \sup_{\xi \in R} |\widehat{F}_1(\xi, s)| \left(\int_{-\infty}^{\infty} \frac{|\xi|^2 e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} d\xi \right)^{\frac{1}{2}} ds \\
&\leq C[\varepsilon + (\varepsilon + M_1(T))^2] \int_0^t (1+s)^{-(1-\sigma)} (1+t-s)^{-\frac{3}{4}} ds \\
&\leq C[\varepsilon + (\varepsilon + M_1(T))^2] (1+t)^{1-(1-\sigma)-\frac{3}{4}} \\
&= C[\varepsilon + (\varepsilon + M_1(T))^2] (1+t)^{-(\frac{3}{4}-\sigma)},
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
\int_0^t & \left\| \tfrac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^2 e^{\mathrm{i}\xi x} \tfrac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F}_1(\xi,s) d\xi \right\|_{L^2} ds \\
&= \int_0^t \left\| (\mathrm{i}\xi)^2 \tfrac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F}_1(\xi,s) \right\|_{L^2} ds \\
&= \int_0^t \left(\int_{-\infty}^{\infty} \tfrac{|\xi|^4 e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2} |\widehat{F}_1(\xi,s)|^2 d\xi \right)^{\frac{1}{2}} ds \\
&\le \int_0^t \sup_{\xi\in R}((1+|\xi|)|\widehat{F}_1(\xi,s)|) \left(\int_{-\infty}^{\infty} \tfrac{|\xi|^4 e^{-2B(\xi)(t-s)}}{(1+\xi^2)(1+|\xi|)^4} d\xi \right)^{\frac{1}{2}} ds \qquad (3.23) \\
&\le C[\varepsilon + (\varepsilon + M_1(T))^2] \\
&\quad \times \int_0^t [(1+s)^{-(1-\sigma)} + (1+s)^{-(\frac{3}{2}-\sigma)}](1+t-s)^{-\frac{5}{4}} ds \\
&\le C[\varepsilon + (\varepsilon + M_1(T))^2] \int_0^t (1+s)^{-(1-\sigma)}(1+t-s)^{-\frac{5}{4}} ds \\
&\le C[\varepsilon + (\varepsilon + M_1(T))^2](1+t)^{-(1-\sigma)}.
\end{aligned}
$$

Thus, we proved $(3.16)_j$ for $j = 0, 1, 2$. □

Lemma 3.6 *Suppose that $v(x,t) \in X_2(0,T)$, then*

$$
\begin{aligned}
&\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^j e^{i\xi x} \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F}_2(\xi,s) d\xi \right\|_{L^2} ds \\
&\le C[\varepsilon + (\varepsilon + M_2(T))^3](1+t)^{-\frac{1+2j}{4}} \log(2+t) \qquad (3.24)_j
\end{aligned}
$$

for $j = 0, 1$, and

$$
\begin{aligned}
&\int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^2 e^{i\xi x} \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F}_2(\xi,s) d\xi \right\|_{L^2} ds \\
&\qquad \le C[\varepsilon + (\varepsilon + M_2(T))^3](1+t)^{-1}. \qquad (3.24)_2
\end{aligned}
$$

Proof. Let $v(x,t) \in X_2(0,T)$. Since

$$
\begin{aligned}
|F_2| &\le |\theta_{xt}| + \tfrac{1}{3}|\theta + v_x|^3 \\
&\le |\theta_{xt}| + \tfrac{1}{3}(|\theta|^3 + 3|\theta|^2|v_x| + 3|\theta||v_x|^2 + |v_x|^3)
\end{aligned} \qquad (3.25)
$$

$$
|F_{2x}| \le |\theta_{xxt}| + |\theta^2\theta_x| + 2|\theta\theta_x v_x| + |\theta_x v_x^2| + |\theta^2 v_{xx}| + 2|\theta v_x v_{xx}| + |v_x^2 v_{xx}|, \qquad (3.26)
$$

applying Lemma 3.1, we get

$$\begin{aligned}\sup_{\xi\in R}|\widehat{F}_2(\xi,s)| &\le \int_{-\infty}^{\infty}|F_2(x,s)|dx\\ &\le C\int_{-\infty}^{\infty}[|\theta_{xt}|+|\theta|^3+|\theta|^2|v_x|+|\theta||v_x|^2+|v_x|^3]dx\\ &\le C[\|\theta_{xt}\|_{L^1}+\|\theta\|_{L^2}^2\|\theta\|_{L^\infty}+\|\theta\|_{L^\infty}\|\theta\|_{L^2}\|v_x\|_{L^2}\\ &\quad+\|\theta\|_{L^\infty}\|v_x\|_{L^2}^2+\|v_x\|_{L^\infty}\|v_x\|_{L^2}^2]\\ &\le C[\varepsilon(1+s)^{-\frac{3}{2}}+\varepsilon^3(1+s)^{-1}+\varepsilon^2M_2(T)(1+s)^{-\frac{3}{2}}\log(2+s)\\ &\quad+\varepsilon M_2(T)^2(1+s)^{-\frac{7}{4}}\log^2(2+s)+M_2(T)^3(1+s)^{-\frac{19}{8}}(\log(2+s))^{\frac{5}{2}}]\\ &\le C[\varepsilon+(\varepsilon+M_2(T))^3](1+s)^{-1}.\end{aligned} \tag{3.27}$$

Similarly, we can prove

$$\begin{aligned}\sup_{\xi\in R}|\xi\widehat{F}_2(\xi,s)| &\le \int_{-\infty}^{\infty}|F_{2x}'(x,s)|dx\\ &\le C\int_{-\infty}^{\infty}[|\theta_{xxt}|+|\theta^2\theta_x|+|\theta\theta_xv_x|+|\theta_xv_x^2|\\ &\quad+|\theta^2v_{xx}|+|\theta v_xv_{xx}|+|v_x^2v_{xx}|]dx\\ &\le C[\|\theta_{xxt}\|_{L^1}+\|\theta\|_{L^2}^2\|\theta_x\|_{L^\infty}+\|\theta\|_{L^\infty}\|\theta_x\|_{L^2}\|v_x\|_{L^2}\\ &\quad+\|\theta_x\|_{L^\infty}\|v_x\|_{L^2}^2+\|\theta\|_{L^\infty}\|\theta\|_{L^2}\|v_{xx}\|_{L^2}\\ &\quad+\|\theta\|_{L^\infty}\|v_x\|_{L^2}\|v_{xx}\|_{L^2}+\|v_x\|_{L^\infty}\|v_x\|_{L^2}\|v_{xx}\|_{L^2}\\ &\le C[\varepsilon(1+s)^{-2}+\varepsilon^3(1+s)^{-\frac{3}{2}}\\ &\quad+\varepsilon^2M_2(T)(1+s)^{-\frac{7}{4}}\log(2+s)\\ &\quad+\varepsilon M_2(T)^2(1+s)^{-\frac{5}{2}}\log^2(2+s)+\varepsilon^2M_2(T)(1+s)^{-\frac{7}{4}}\\ &\quad+\varepsilon M_2(T)^2(1+s)^{-\frac{9}{4}}\log(2+t)\\ &\quad+M_2(T)^3(1+s)^{-\frac{21}{8}}(\log(2+t))^{\frac{3}{2}}]\\ &\le C[\varepsilon+(\varepsilon+M_2(T))^3](1+s)^{-3/2}.\end{aligned} \tag{3.28}$$

Therefore, making use of the Parseval's equality, and Lemmas 3.1, Lemma 3.2, we can prove

$$\begin{aligned}
&\int_0^t \Big\|\tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\xi x}\tfrac{e^{-A(\xi)(t-s)}}{1+\xi^2}\widehat{F}_2(\xi,s)d\xi\Big\|_{L^2}ds\\
&=\int_0^t \Big\|\tfrac{e^{-A(\xi)(t-s)}}{1+\xi^2}\widehat{F}_2(\xi,s)\Big\|_{L^2}ds\\
&=\int_0^t \Big(\int_{-\infty}^{\infty}\tfrac{e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2}|\widehat{F}_2(\xi,s)|^2d\xi\Big)^{\frac12}ds\\
&\le \int_0^t \sup_{\xi\in R}|\widehat{F}_2(\xi,s)|\Big(\int_{-\infty}^{\infty}\tfrac{e^{-2B(\xi)(t-s)}}{(1+\xi^2)^2}d\xi\Big)^{\frac12}ds\\
&\le C[\varepsilon+(\varepsilon+M_2(T))^3]\int_0^t(1+s)^{-1}(1+t-s)^{-\frac14}ds\\
&\le C[\varepsilon+(\varepsilon+M_2(T))^3](1+t)^{-\frac14}\log(2+t).
\end{aligned} \tag{3.29}$$

In a similar way, we can prove the higher order case

$$\begin{aligned}
&\int_0^t \Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{i}\xi e^{\mathrm{i}\xi x}\frac{e^{-A(\xi)(t-s)}}{1+\xi^2}\widehat{F}_2(\xi,s)d\xi\Big\|_{L^2}ds\\
&\le C[\varepsilon+(\varepsilon+M_2(T))^3](1+t)^{-\frac34}\log(2+t)
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
&\int_0^t \Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathrm{i}\xi)^2 e^{\mathrm{i}\xi x}\frac{e^{-A(\xi)(t-s)}}{1+\xi^2}\widehat{F}_2(\xi,s)d\xi\Big\|_{L^2}ds\\
&\qquad\le C[\varepsilon+(\varepsilon+M_2(T))^3](1+t)^{-1}.
\end{aligned} \tag{3.31}$$

Therefore, we have completed the proof of Lemma 3.6. □

Furthermore, we can prove that

$$\sup_{\xi\in R}|\widehat{F}_3(\xi,t)|\le\|F_3\|_{L^1}\le C[\varepsilon+(\varepsilon+M_3(T))^p](1+t)^{-\frac{p}{2}}, \tag{3.32}$$

and

$$\sup_{\xi\in R}|\xi\widehat{F}_3(\xi,t)|\le\|F'_{3x}\|_{L^1}\le C[\varepsilon+(\varepsilon+M_3(T))^p](1+t)^{-\frac{p+1}{2}}, \tag{3.33}$$

and note $p/2>1$, $(p+1)/2>1$ due to $p\ge3$, then, by using the Parseval's equality, and Lemmas 3.2 and 3.3, we can similarly prove the following lemma. We omit here the proof details.

Lemma 3.7 *Let $v(x,t)\in X_p(0,T)$ $(p\ge3)$, then*

$$\int_0^t \Big\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^j e^{i\xi x}\frac{e^{-A(\xi)(t-s)}}{1+\xi^2}\widehat{F}_p(\xi,s)d\xi\Big\|_{L^2}ds\le C[\varepsilon+(\varepsilon+M_p(T))^p](1+t)^{-\frac{1+2j}{4}} \tag{3.34}$$

for $j=0,1,2$.

Proof of Proposition 2.3 (A Priori Esitmetes). Let $v(x,t) \in X_p(0,T)$ be the unique solution of Eq. (3.4), $p \geq 1$. From (3.4) and (3.5), we obtain

$$\begin{aligned} \|\partial_x^j v(t)\|_{L^2} \leq & \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j e^{\mathrm{i}\xi x} e^{-A(\xi)t} v_0(\delta) d\xi \right\|_{L^2} \\ & + \int_0^t \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{i}\xi)^j e^{\mathrm{i}\xi x} \frac{e^{-A(\xi)(t-s)}}{1+\xi^2} \widehat{F}_p(\xi, s) d\xi \right\|_{L^2} ds \end{aligned} \tag{3.35}$$

for $j = 0, 1, 2$.

When $p = 1$, thanks to Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned} \|\partial_x^j v(t)\|_{L^2} \leq & \ C\|v_0\|_{W^{3,1}}(1+t)^{-\frac{1+2j}{4}} + C[\varepsilon + (\varepsilon + M_1(T))^2](1+t)^{-(\frac{1+2j}{4}-\sigma)} \\ \leq & \ C[\|v_0\|_{W^{3,1}} + \varepsilon + (\varepsilon + M_1(T))^2](1+t)^{-(\frac{1+2j}{4}-\sigma)} \end{aligned} \tag{3.36$_j$}$$

for $j = 0, 1$, and

$$\begin{aligned} \|v_{xx}(t)\|_{L^2} \leq & \ C\|v_0\|_{W^{3,1}}(1+t)^{-\frac{5}{4}} + C[\varepsilon + (\varepsilon + M_1(T))^2](1+t)^{-(1-\sigma)} \\ \leq & \ C[\|v_0\|_{W^{3,1}} + \varepsilon + (\varepsilon + M_1(T))^2](1+t)^{-(1-\sigma)}. \end{aligned} \tag{3.36$_2$}$$

We multiply $(3.36)_j$ by $(1+t)^{\frac{1+2j}{4}-\sigma}$ for $j = 0, 1$, and $(3.36)_2$ by $(1+t)$, respectively, we then add them to have

$$\begin{aligned} M_1(t) = & \ \sup_{0<t\leq T} \left\{ \sum_{j=0}^{1} (1+t)^{\frac{1+2j}{4}-\sigma} \|\partial_x^j v(t)\|_{L^2} + (1+t)^{1-\sigma} \|v_{xx}(t)\|_{L^2} \right\} \\ \leq & \ C[\|v_0\|_{W^{3,1}} + \varepsilon + (\varepsilon + M_1(T))^2], \end{aligned}$$

namely,

$$M_1(T)[1 - 2C\varepsilon - CM_1(T)] \leq C[\|v_0\|_{W^{3,1}} + \varepsilon + \varepsilon^2].$$

Now we choose δ_2 in Proposition 2.3 as

$$\delta_2 \leq \frac{1}{4C}$$

When $\varepsilon < \min\{1, \delta_2\}$, $M_1(T) < \delta_2$, we obtain

$$M_1(t) \leq 8C[\|v_0\|_{W^{3,1}} + \varepsilon] \quad \text{for all} \quad 0 \leq t \leq T.$$

That is,

$$\sum_{j=0}^{1} (1+t)^{\frac{1+2j}{4}-\sigma} \|\partial_x^j v(t)\|_{L^2} + (1+t)^{1-\sigma} \|v_{xx}(t)\|_{L^2} \leq 8C[\|v_0\|_{W^{3,1}} + \varepsilon]$$

for $\varepsilon < \min\{1, \delta_2\}$, $M_1(T) < \delta_2$, and $t \in [0, T]$. Thus we have proved the a priori estimates for Case $p = 1$.

When $p=2$, Lemma 3.4 and Lemma 3.6 give us

$$\begin{aligned}\|\partial_x^j v(t)\|_{L^2} &\le C\|v_0\|_{W^{3,1}}(1+t)^{-\frac{1+2j}{4}} + C[\varepsilon + (\varepsilon + M_2(T))^3](1+t)^{-\frac{1+2j}{4}}\log(2+t)\\ &\le C[\|v_0\|_{W^{3,1}} + \varepsilon + (\varepsilon + M_2(T))^3](1+t)^{-\frac{1+2j}{4}}\log(2+t),\end{aligned} \qquad (3.37)_j$$

for $j=0,1$, and

$$\begin{aligned}\|v_{xx}(t)\|_{L^2} &\le C\|v_0\|_{W^{3,1}}(1+t)^{-\frac{5}{4}} + C[\varepsilon + (\varepsilon + M_2(T))^3](1+t)^{-1}\\ &\le C[\|v_0\|_{W^{3,1}} + \varepsilon + (\varepsilon + M_2(T))^3](1+t)^{-1}.\end{aligned} \qquad (3.37)_2$$

Multiplying $(3.37)_j$ by $(1+t)^{\frac{1+2j}{4}}\log^{-1}(2+t)$ for $j=0,1$ and $(3.37)_2$ by $(1+t)$, respectively, and adding them, we obtain

$$M_2(T) \le C[\|v_0\|_{W^{3,1}} + \varepsilon + (\varepsilon + M_2(T))^3],$$

which implies that

$$M_2(T)[1 - 3C\varepsilon^2 - 3C\varepsilon M_2(T) - CM_2(T)^2] \le C[\|v_0\|_{W^{3,1}} + \varepsilon + \varepsilon^3].$$

Thus, there exists δ_2 satisfying

$$\delta_2 \le \frac{1}{\sqrt{8C}},$$

when $\varepsilon \le \min\{1, \delta_2\}$ and $M_2(T) < \delta_2$, then

$$M_2(t) \le 8C[\|v_0\|_{W^{3,1}} + \varepsilon + \varepsilon^3] \le 16C[\|v_0\|_{W^{3,1}} + \varepsilon] \quad \text{for all} \ \ 0 \le t \le T.$$

That is,

$$\sum_{j=0}^{1}(1+t)^{\frac{1+2j}{4}}\log^{-1}(2+t)\|\partial_x^j v(t)\|_{L^2} + (1+t)\|v_{xx}(t)\|_{L^2} \le C[\|v_0\|_{W^{3,1}} + \varepsilon], \quad t \in [0,T].$$

Thus, the proof for case $p=2$ is complete.

Finally, when $p \ge 3$, in the same way, making use of Lemma 3.4 and Lemma 3.7, we can prove the *a priori* estimates (2.1). Here we omit its details. □

4 Numerical Computations

In this section, we introduce our numerical results, which should be another positive answers to our theoretical results–Theorem 1.1. Our numerical method carried out here is the explicit finite difference method. For the consideration of the Cauchy problem, we adopt a suitable IBVP for a sufficiently large domain $\Omega \subset \boldsymbol{R}$ with zero Neumann boundary instead of the original IVP.

We denote $\theta(x,t)$ and $u(x,t)$ the solutions of the Burgers solution (1.3) (diffusion wave's equation) and the BBM-Burgers equation (1.1), respectively, and choose the initial data $\theta_0(x)$ and $u_0(x)$ as

$$\theta_0(x) = u_0(x) = \frac{1}{1+x^4}.$$

Thus,

$$\int_{-\infty}^{\infty} \theta_0(x)dx = \int_{-\infty}^{\infty} u_0(x)dx = \int_{-\infty}^{\infty} \frac{1}{1+x^4}dx > 0$$

but

$$\int_{-\infty}^{\infty} [\theta_0(x) - u_0(x)]dx = 0.$$

Now we are going to divide to three cases to carry out our numerical computations. Firstly, we treat Case 3: $p=3$.

Let

$$F_1(t) := \max_{x\in \boldsymbol{R}} \frac{|\theta(x,t)|}{(1+t)^{-\frac{1}{2}}} \quad \text{and} \quad F_2(t) := \max_{x\in \boldsymbol{R}} \frac{|u(x,t)|}{(1+t)^{-\frac{1}{2}}},$$

according to the previous theory, we believe that the ratios $F_1(t)$ and $F_2(t)$ should be constants as time t goes to infinity. Indeed, our numerical result shows us that it is really true when time t is large enough, see Figure 1 below.

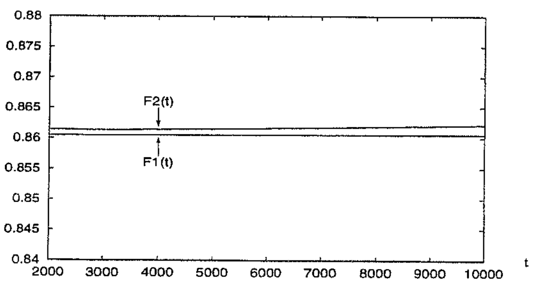

Figure 1: Case 3. $p=3$: Decay rates of $u(x,t)$ and $\theta(x,t)$

By use of the above data, we check further the behavior of the difference between two solutions $\theta(x,t)$ and $u(x,t)$. Let

$$F(t) := \frac{\|u-\theta\|_{L^\infty(\boldsymbol{R})}}{(1+t)^{-1}},$$

our numerical calculation Figure 2 presents that $F(t)$ seems also to be a constant line when t is large enough. This means that, the convergence rate for the solution $u(x,t)$ to the corresponding diffusion wave $\theta(x,t)$ is $(1+t)^{-1}$, namely, $\|u-\theta\|_{L^\infty(\boldsymbol{R})} = C(1+t)^{-1}$ for some positive constant C as $t \geq 1$.

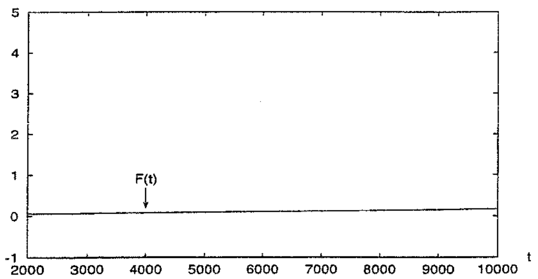

Figure 2: Case 3. $p = 3$: Convergence rate of $u(x,t)$ to $\theta(x,t)$

Secondly, we treat Case 2: $p = 2$. We are going to show the numerical simulations for the ratios

$$G_1(t) := \max_{x\in\boldsymbol{R}} \frac{|\theta(x,t)|}{(1+t)^{-\frac{1}{2}}} \quad \text{and} \quad G_2(t) := \max_{x\in\boldsymbol{R}} \frac{|u(x,t)|}{(1+t)^{-\frac{1}{2}}}$$

as follows.

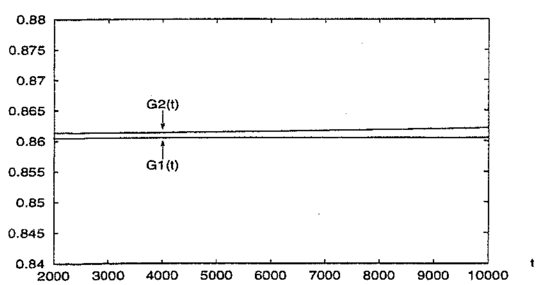

Figure 3: Case 2. $p = 2$: Decay rates of $u(x,t)$ and $\theta(x,t)$

Put

$$G(t) := \frac{\|u-\theta\|_{L^\infty(\boldsymbol{R})}}{(1+t)^{-\frac{7}{8}}\sqrt{\log(2+t)}},$$

the numerical result further shows that $G(t)$ seems to be also a constant line as t is large, see Figure 4 below, which illustrates that $\|u-\theta\|_{L^\infty(\boldsymbol{R})}$ decays fast just as the function $(1+t)^{-\frac{7}{8}}\sqrt{\log(2+t)}$ does.

Finally, we treat Case 1: $p = 1$. As before done, we let

$$H_1(t) := \max_{x\in\boldsymbol{R}} \frac{|\theta(x,t)|}{(1+t)^{-\frac{1}{2}}} \quad \text{and} \quad H_2(t) := \max_{x\in\boldsymbol{R}} \frac{|u(x,t)|}{(1+t)^{-\frac{1}{2}}}.$$

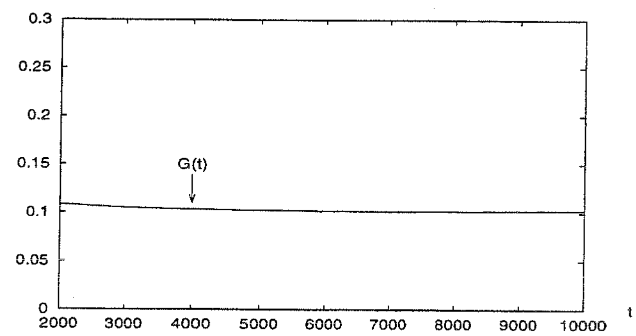

Figure 4: Case 2. $p = 2$: Convergence rate of $u(x,t)$ to $\theta(x,t)$

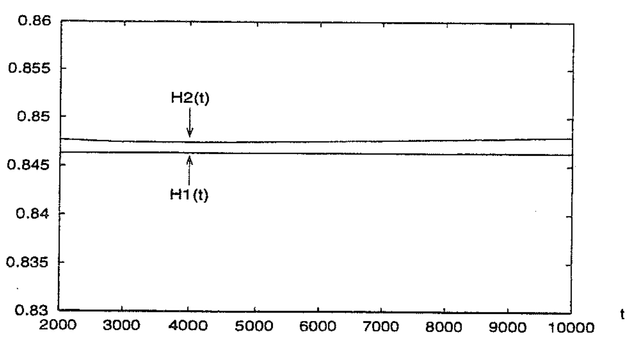

Figure 5: Case 1. $p = 1$: Decay rates of $u(x,t)$ and $\theta(x,t)$

Our numerical calculation makes the following figure.

Furthermore, let

$$H(t) := \frac{\|u-\theta\|_{L^\infty(\boldsymbol{R})}}{(1+t)^{-\frac{7}{8}+\sigma}},$$

we compute it numerically for $\sigma = 0.05$, 0.1 and 0.15, see Figure 6.

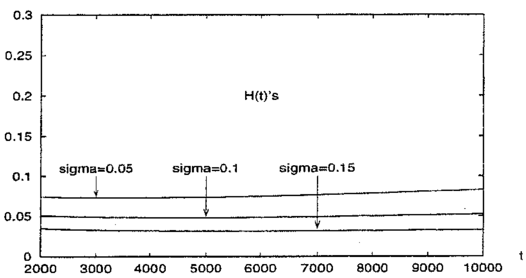

Figure 6: Case 1. $p = 1$: Convergence rate of $u(x,t)$ to $\theta(x,t)$

All of these show that $H(t)$ for each $\sigma = 0.05$, 0.1 and 0.15 are almost the constant lines for the large t. These represent also that $\|u-\theta\|_{L^\infty(\boldsymbol{R})}$ decays fast just as the function $(1+t)^{-\frac{7}{8}+\sigma}$ does.

Therefore, we conclude that, due to these numerical simulations, it seems our convergence rates in Theorem 1.1 are sharp.

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