Taylor Reaminder, Taylor Series

Last time: approximations are useless if we don't know how much error we are making!

Taylor's Theorem, or: Error in Taylor Approximations

Taylor's theorem gives an unequivocable upper bound for the error in any Taylor approximation. First, suppose we have the nth Taylor approximation for $f(x)$ centered at c:

$$P_n(x) = a_0 + a_1(x-c) + \cdots + a_n(x-c)^n,$$

with $a_k = \frac{f^{(k)}(c)}{k!}$ as we have seen.

Then we want to look at the error, or the difference between f and P_n. We call this the remainder, $R_n(x) = f(x) - P_n(x)$. We look at the absolute value of the remainder, and the theorem says it necessarily satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1}$$

where M is an upper bound for $|f^{(n+1)}|$ on the interval between x and the center c.

Notice how the expression $\frac{M_{(n+1)}}{(n+1)!}|x-c|^{n+1}$ looks like the next term in the Taylor approximation, but is not quite the next term! How does it differ?

Let's try an example:

1. (a) Use an appropriate degree 2 Taylor polynomial to approximate $e^{-0.1}$.

(b) Use Taylor's Remainder Theorem to give an upper bound on the error of your approximation.

2. (a) Use an appropriate degree 2 Taylor polynomial to approximate $\sqrt{26}$.

(b) Use Taylor's Remainder Theorem to give an upper bound on the error of your approximation.

(c) Redo your work, now to approximate (and find an upper bound on the error of your approximation) $\sqrt{23}$.

3. So how could we do better at approximating any of these previous quantities: $e^{-0.1}$, $\sqrt{26}$, etc?

4. What is the Taylor series expansion of $\cos x$ centered at 0? Write it in both sigma and $\cdots$ notation.

5. What is the Taylor series expansion of $\sin x$ centered at 0? Write it in both sigma and $\cdots$ notation.

6. Write an "infinite polynomial" representation of:

(a) $\sin x^3$.

(b) $\displaystyle\int \sin x^3 \, dx$. (Note: The antiderivatives of $\sin x^3$ are not elementary functions!)

7. Find the degree 100 Taylor polynomial approximation of $\sin x^3$.

8. Let $f(x) = \sin x^3$. What is $f^{(100)}(0)$? $f^{(51)}(0)$?