Taylor approximation 2

1. If you want to find the degree n Taylor polynomial approximation $a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$ for $f(x)$ centered at c, write a formula for the coefficient a_k.

2. Find the degree 15 Taylor polynomial approximation for $f(x) = \cos x$ centered at 0.

3. Rewrite the following sums in $\cdots$ notation. (This means: Write down at least the first 3 terms $+\cdots+$ the last term.)

(a) $\displaystyle\sum_{k=1}^{100} \frac{1}{2k}$.

(b) $\displaystyle\sum_{k=3}^{15} \frac{(-1)^k}{k}$.

(c) $\displaystyle\sum_{k=0}^{25} \frac{x^{2k+1}}{k+1}$.

(d) $\displaystyle\sum_{k=1}^{26} \frac{x^{2k-1}}{k}$.

4. Rewrite the following sums in sigma notation.

(a) $\dfrac{1}{3^3} + \dfrac{1}{4^3} + \dfrac{1}{5^3} + \dfrac{1}{6^3} + \cdots + \dfrac{1}{100^3}$.

(b) $\sqrt{2} - \sqrt{4} + \sqrt{6} - \sqrt{8} + \sqrt{10} - \cdots - \sqrt{120}$.

(c) $1 + x^3 + x^6 + x^9 + x^{12} + x^{15} + \cdots + x^{30}$.

(d) $x - x^3 + x^5 - x^7 + x^9 - \cdots + x^{101}$.

(e) $x - \dfrac{x^3}{2} + \dfrac{x^5}{3} - \dfrac{x^7}{4} + \cdots - \dfrac{x^{31}}{16}$.

5. (a) Rewrite your answer to 2 in sigma notation.

(b) Find the degree 25 Taylor polynomial approximation for $f(x) = \cos x$ centered at 0. Write it in both sigma and $\cdots$ notation.

(c) Use the degree 25 Taylor polynomial approximation to get an approximation for $\cos 0.1$.

But approximations are useless if we don't know how much error we are making! Recall from last time we saw in our table that the error seemed to be roughly the size of the next term to be added to the sum. This is *basically* true under *some* conditions, and is called Taylor's Theorem.

Taylor's Theorem, or: Error in Taylor Approximations

Taylor's theorem gives an unequivocable upper bound for the error in any Taylor approximation. First, suppose we have the nth Taylor approximation for $f(x)$ centered at c:

$$P_n(x) = a_0 + a_1(x-c) + \cdots + a_n(x-c)^n,$$

with $a_k = \frac{f^{(k)}(c)}{k!}$ as we have seen.

Then we want to look at the error, or the difference between f and P_n. We call this the remainder, $R_n(x) = f(x) - P_n(x)$. We look at the absolute value of the remainder, and the theorem says it necessarily satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1}$$

where M is an upper bound for $|f^{(n+1)}|$ on the interval between x and the center c.

Notice how the expression $\frac{M_{(n+1)}}{(n+1)!}|x-c|^{n+1}$ looks like the next term in the Taylor approximation, but is not quite the next term! How does it differ?