MATH 726 — AUTOMORPHIC REPRESENTATIONS
COURSE NOTES

ABSTRACT. These are notes from a course at McGill on automorphic representations, given by Jayce Getz in the winter term of 2010. The notes were typeset collectively by Francesc Castella, Andrew Fiori and Cameron Franc. The authors thank Brian Conrad for pointing out many errors in an earlier version.

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Introduction

The goal of this course is to introduce and study *automorphic representations*. Given a global field F and a reductive algebraic group G over F, then an automorphic representation of G is a $(\mathfrak{g}, K) \times G(\mathbf{A}_F^\infty)$-module which is isomorphic with a subquotient of $L^2(G(F)\backslash G(\mathbf{A}_F))$. We will begin the course by introducing all of these objects.

We thank Brian Conrad for many useful corrections and comments. The abundance of errors which remain are due entirely to the typesetters.

1. Lecture 1: Background on adele rings

1.1. **Adeles.** We begin by recalling the definition of the arithmetic objects of interest in this course, global fields. They can be defined axiomatically, but we take a more pedestrian approach. For more information consult chapter 5 of [48].

Definition 1.1. A *global field* F is a field which is a finite extension of $\mathbf{Q}$ or of $\mathbf{F}_q(t)$ for some prime power $q = p^r$. Global fields over $\mathbf{Q}$ are called *number fields* while global fields over $\mathbf{F}_q(t)$ are called *function fields*.

Definition 1.2. A *place* of a global field F is an equivalence class of valuations, where two valuations are said to be equivalent if they induce the same topology on F.

The places of a global field F fall into two categories: the finite and infinite places. If F is a number field then the finite places correspond with prime ideals of its ring of integers $\mathcal{O}_F$. The infinite primes of a number field are induced by the absolute value on $\mathbf{C}$ via the distinct field embeddings $F \hookrightarrow \mathbf{C}$. If $F = \mathbf{F}_q(t)$ is a function field then places are obtained by completing $\mathbf{F}_q[t]$ or $\mathbf{F}_q[1/t]$ at a prime ideal. If ν is an additive valuation of F then we write $|\cdot|_\nu$ for the corresponding multiplicative valuation. Recall that these are related by the formula

$$|x|_\nu = \alpha^{\nu(x)}$$

for a choice of α a positive real number with $0 < \alpha < 1$. We will usually take $\alpha = 1/q$ where q is the size of the *residue field* $k(\nu) = \mathcal{O}_{F_\nu}/\mathfrak{m}_{F_\nu}$ of F at ν. Here F_ν denotes the completion of F at ν,

$$\mathcal{O}_{F_\nu} = \{x \in F_\nu \mid |x|_\nu \leq 1\}$$

denotes the *ring of integers* in F_ν and

$$\mathfrak{m}_{F_\nu} = \{x \in F_\nu \mid |x|_\nu < 1\}$$

denotes the unique *maximal ideal* of $\mathcal{O}_{F_\nu}$. We will often write $\mathcal{O}_\nu$ and $\mathfrak{m}_\nu$ for $\mathcal{O}_{F_\nu}$ and $\mathfrak{m}_{F_\nu}$, respectively.

Example 1.3. If $F = \mathbf{Q}$ and $p \in \mathbf{Z}$ is a finite prime, then completing $\mathbf{Q}$ at the p-adic absolute value gives the local field $\mathbf{Q}_p$. Its ring of integers is $\mathbf{Z}_p$ and the maximal ideal is $p\mathbf{Z}_p$. The residue field is $\mathbf{Z}_p/p\mathbf{Z}_p \cong \mathbf{Z}/p\mathbf{Z} \cong \mathbf{F}_p$.

Definition 1.4. Let F be a global field. The ring of *adeles* of F, denoted $\mathbf{A}_F$, is the restricted direct product of the completions F_ν with respect to the rings of integers $\mathcal{O}_\nu$:

$$\mathbf{A}_F = \left\{ (x_\nu) \in \prod_\nu F_\nu \;\middle|\; x_\nu \in \mathcal{O}_\nu \text{ for all but finitely many places } \nu \right\}.$$

The restricted product is usually denoted by a prime:

$$\mathbf{A}_F = \prod_\nu{}' F_\nu.$$

Note that the adeles are a subring of the full product $\prod_\nu F_\nu$. If S is a finite set of places of F then we write

$$\mathbf{A}_F^S = \prod_{\nu \notin S}{}' F_\nu, \quad \mathbf{A}_{F,S} = \prod_{\nu \in S} F_\nu$$

We endow $\mathbf{A}_F$ with the *restricted product topology*, or the *adelic topology*, where a neighbourhood base at 0 is given by all sets of the form $\prod_\nu N_\nu$ where N_ν is a neighbourhood of 0 in F_ν for every ν, and in fact $N_\nu = \mathcal{O}_\nu$ for all but finitely many places ν. This is not the same as the topology induced on $\mathbf{A}_F$ from the product $\prod_\nu F_\nu$. While $\prod_\nu F_\nu$ is not locally compact, for $\mathbf{A}_F$ one has the following:

Proposition 1.5. *The adeles $\mathbf{A}_F$ of a global field F are a locally compact hausdorff topological ring.*

Proof. We argue that $\mathbf{A}_F$ is locally compact and leave the other details to the reader. For this note that for any finite set of places S, the subset

$$\prod_{\nu\in S} F_\nu \times \prod_{\nu\notin S} \mathcal{O}_\nu$$

is an open subring of $\mathbf{A}_F$ for which the induced topology coincides with the product topology. The above subring is thus locally compact. Every $x \in \mathbf{A}_F$ is contained in some such subring, which shows that $\mathbf{A}_F$ is locally compact. $\square$

There is a natural embedding $F \hookrightarrow \mathbf{A}_F$ into the diagonal elements of $\mathbf{A}_F$.

Lemma 1.6. *The induced subspace topology on F arising from the embedding $F \hookrightarrow \mathbf{A}_F$ is the discrete topology.*

Proof. Let $x \in F^\times$. For each finite place ν of F let $n_\nu = \nu(x)$, so that $x \in \mathfrak{m}_\nu^{n_\nu}$ but $x \notin \mathfrak{m}_\nu^{n_\nu+1}$ for all ν. Note that $n_\nu = 0$ for all but finitely many places. For each infinite place ν let $U_\nu \subseteq F_\nu$ be the open ball of radius $\prod_{\nu<\infty} |x|_\nu^{-1}$ about x. Consider the open subset of $\mathbf{A}_F$ defined by

$$\mathcal{O} = \prod_{\nu|\infty} U_\nu \times \prod_{\nu<\infty} \mathfrak{m}_\nu^{n_\nu}.$$

Of course $x \in \mathcal{O}$ by construction; suppose $y \in F$ is also contained in $\mathcal{O}$. Recall that the product formula from algebraic number theory says that for any global field F and any $z \in F^\times$,

$$\prod_\nu |z|_\nu = 1.$$

Apply this to $x - y$; note that $|x-y|_\nu \leq |x|_\nu$ for all finite places ν. Thus

$$\prod_\nu |x-y|_\nu \leq \prod_{\nu<\infty} |x|_\nu \times \prod_{\nu|\infty} |x-y|_\nu < 1$$

since $y \in U_\nu$. The product formula thus shows that we must have $x - y = 0$, and hence $F \cap \mathcal{O} = \{x\}$. This shows that F obtains the discrete topology from $\mathbf{A}_F$. $\square$

We often identify F with its image in $\mathbf{A}_F$.

Theorem 1.7 (Approximation)**.** *For every global field F, one has a decomposition*

$$\mathbf{A}_F = F_\infty + \prod_{\nu\nmid\infty} \mathcal{O}_\nu + F.$$

Proof. See Theorem 5-8 of [48]. $\square$

Claim 1.8. *For every global field F,*

$$\left(F_\infty + \prod_{\nu\nmid\infty} \mathcal{O}_\nu\right) \cap F = \mathcal{O}_F.$$

Proof. One inclusion is obvious. For the other, if $x \in F$ satisfies $x \in \mathcal{O}_\nu$ for all finite places ν then $x\mathcal{O}_F$ is a proper ideal of $\mathcal{O}_F$, and not just fractional, since

$$x\mathcal{O}_F = \prod_{\nu<\infty} \mathfrak{m}_\nu^{v(x)}.$$

Thus $x \in \mathcal{O}_F$, which concludes the proof. $\square$

Remark 1.9. The fact that $\mathbf{A}_F$ is a topological ring for the adelic topology relies on the fact that the local rings $\mathcal{O}_\nu$ are one dimensional. In higher dimensional settings, say for function fields of algebraic surfaces, one must be more creative when defining appropriate analogues of the adeles.

1.2. **Adelic points of affine schemes.** This section presents a result of Brian Conrad on topologizing points of schemes of finite type over a topological ring R. The note [18] can be found on Conrad's webpage.

Theorem 1.10. *Let R be a topological ring and let $S = \operatorname{Spec}(R)$. Let X be an affine scheme of finite type over S. Then there exists a unique way to topologize $X(S)$ such that:*

(1) *the topology is functorial in X; this means that if $X \to Y$ is a morphism of affine schemes of finite type over S, then the induced map on points $X(S) \to Y(S)$ is continuous;*
(2) *the topology is compatible with fibre products; this means that if $X \to Z$ and $Y \to Z$ are morphisms of affine schemes, all of finite type over S, then the topology on $X \times_Z Y(S)$ is exactly the fibre product topology;*
(3) *closed immersions of schemes $X \hookrightarrow Y$ correspond to topological embeddings $X(S) \hookrightarrow Y(S)$;*
(4) *if $X = \operatorname{Spec}(R[T])$ then $X(S)$ is homeomorphic with R under the natural identification $X(S) \cong R$.*

Explicitely, if $A = \Gamma(X, \mathcal{O}_X)$ then $X(R) = \operatorname{Hom}_{R-\mathbf{alg}}(A, R)$ can be embedded in the product R^A. Give $X(R)$ the topology induced by the product topology on R^A.

If R is Hausdorff or locally compact, then so is $X(S)$.

Proof. See Conrad's paper for the proof. □

1.3. **Group schemes.** For a nice introduction to affine group schemes, consult Waterhouse's book [58]. Fix a commutative ring R.

Definition 1.11. An *affine group scheme* over R is a functor

$$R - \mathbf{alg} \to \mathbf{Group}$$

representable by an R-algebra.

Example 1.12. The additive group $\mathbf{G}_a$ is the functor assigning to each R-algebra T its additive group, $\mathbf{G}_a(T) = (T, +)$. It is representable by the polynomial algebra $R[X]$, as an R-algebra homomorphism $R[X] \to T$ is determined by the image of X. As this can be any element of T, we see that naturally $\mathbf{G}_a(T) = T$. It is easily seen to be a group homomorphism for T under addition.

Example 1.13. The multiplicative group $\mathbf{G}_m$ is the functor assigning to each each R-algebra T its multiplicative group, $\mathbf{G}_m(T) = T^\times$. It is a simple exercise to see that this is represented by the algebra $R[X, Y]/(XY - 1)$.

Example 1.14. The general linear group GL_n for $n \geq 1$ is the functor taking $T \mapsto \mathrm{GL}_n(T)$. It is an affine group scheme represented by the R-algebra $R[X_{i,j} : 1 \leq i, j \leq n][Y]/(\det(X_{i,j}) \cdot Y - 1)$. Note that $\mathrm{GL}_1 = \mathbf{G}_m$.

2. Lecture 2: Algebraic Groups

We first recall the following basic definitions:

Definition 2.1. An algebraic group G is said to be *linear* if there exists a faithful embedding $\phi : G \hookrightarrow \mathrm{GL}_n$ as a closed subscheme.

Theorem 2.2. *An algebriac group is linear if and only if it is affine.*

Proof. See [10, I.1.10]. □

Definition 2.3. An element $x \in M_n(\overline{k})$ is said to be:

- *semi-simple* if there exists $g \in \mathrm{GL}_n(\overline{k})$ such that $g^{-1}xg$ is diagonal
- *nilpotent* if there exists $n \in \mathbf{N}$ such that $x^n = 0$
- *unipotent* if $(x - \mathrm{id})$ is nilpotent

For an arbitrary linear algebraic group $\phi : G \hookrightarrow \mathrm{GL}_n$ we say that an element $g \in G$ is *semi-simple*, (resp *nilpotent*, resp *unipotent*) if $\phi(g)$ is so.

Remark 2.4. We remark that the properties above are all representation theoretic properties which are independant of the choice of continuous representation of G (this essentially follows from the following theorem).

Theorem 2.5. *(Jordan decomposition) Let G be a linear algebraic group. Given $x \in G(\overline{k})$ there exists $x_s, x_u \in G(\overline{k})$ such that x_s is semi-simple, x_u is unipotent, $x = x_s x_u = x_u x_s$, and this decomposition is unique.*

Proof. See [10, I.4.4]. □

We should point out that, in the theorem above, even though x may be a k point, neither x_s nor x_u need be.

Definition 2.6. Let G be an algebraic group. The *unipotent radical* $R_u(G)$ of G is the maximal connected unipotent normal subgroup of $G(\overline{k})$. The (*solvable*) *radical* is the maximal connected solvable normal subgroup.

We remark that since a unipotent subgroup is always solvable we always have $R_u(G) \subseteq R(G)$.

Remark 2.7. It is crucial that these radicals are only defined over $\overline{k}$. If one uses the same definitions over imperfect fields, then one obtains the incorrect notions of unipotent and solvable radicals. For the correct notions in such cases, consult the book [19].

Definition 2.8. An algebraic group G is said to be *reductive* if $R_u(G) = \{\mathrm{id}\}$ and *semi-simple* if $R(G) = \{\mathrm{id}\}$.

Example 2.9.

- GL_n is reductive but not semi-simple since its center is normal.
- SL_n is semi-simple (which implies reductive)
- More generally if G is reductive then G^{der} the derived subgroup is semi-simple and $G = G^{der} Z_G$. In particular if G is reductive then $R(G) = Z_G$.
- The subgroup of upper triangular matrices are not reductive (as they are solvable). We remark that unipotent groups are always upper-triangularizable (as groups) [10, I.4.8].

2.1. **Lie Algebras.** A basic question that arises is that given H, G reductive algebraic groups, we would like to classify maps $H \to G$. For example with GL_n this is just the representation theory of H. One approach to this question is the Cartan-Weil theory of Lie Algebras and their representations.

Definition 2.10. Let R be a Ring, a Lie Algebra (over R) is an R-Module $\mathfrak{g}$ together with a pairing $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the following:

(1) $[\cdot,\cdot]$ is bilinear.
(2) $[x, x] = 0$ for all $x \in \mathfrak{g}$.
(3) $[\cdot,\cdot]$ satisfies the Jacobi-identity, that is $[[x,y],z]+[[y,z],x]+[[z,x],y] = 0$ for all $x,y,z \in \mathfrak{g}$

Morphisms of Lie Algebras are simply R-module maps preserving $[\cdot,\cdot]$.

Remark 2.11. if R' is an R-algebra then $\mathfrak{g} \otimes_R R'$ can be given a Lie algebra structure.

A basic result in the theory of algebraic groups is that there exists a functor

$$\mathrm{Lie} : LAG_{/k} \to Lie - Alg_{/k}$$

defined by:

$$\mathrm{Lie}(G) = \ker(G(k[t]/t^2) \to G(k))$$

where the map $G(k[t]/t^2) \to G(k)$ is induced by the map $k[t]/t^2 \to k$. We will define the bracket operation shortly.

Example 2.12. The lie algebra of GL_n. The kernel of the map $G(k[t]/t^2) \to G(k)$ is easily seen to be $\mathrm{id} + tA$ where $A \in M_n$ thus we have that $\mathfrak{gl}_n = \mathrm{Lie}(\mathrm{GL}_n) \simeq M_n$ it shall turn out that $[X, Y] = XY - YX$.

An alternate description of the Lie functor is based on the following notion:

Definition 2.13. Let A be a k-algebra and M an A-module. An M-valued *k-derivation* is a k-linear map $D : A \to M$ such that

$$D(fg) = D(f)g + fD(g).$$

When $M = A$, we write $\mathrm{Der}_k(A)$ for the k-module of k-derivations $D\colon A \to A$.

The condition to be a derivation implies: (1) $D(1) = 0$, as one sees by taking $f = g = 1$, and (2) $D(c) = 0$ for all $c \in k$, which follows by (1) since D is k-linear.

We claim there is a connection between k-algebra maps $\alpha : A \to k[t]/t^2$ and derivations. Indeed, write $\alpha(f) = \alpha_0(f) + \alpha_1(f)t$ then we will have that $\alpha_0(fg) = \alpha_0(f)\alpha_0(g)$ and $\alpha_1(fg) = \alpha_0(f)\alpha_1(g) + \alpha_1(f)\alpha_0(g)$. So that if we use α_0 to make k into an A-module then α_1 will be a k-derivation.

Claim 2.14. *The map $\mathrm{Lie}(G) \to Der_k k[G]$ defined by taking $1 + tB \mapsto B$ and extending this point derivation to a left G-invariant derivation of $k[G]$, yields an isomorphism of $\mathrm{Lie}(G)$ with the space of left G-invariant derivations of $k[G]$.*

Proof. An element of $\mathrm{Lie}(G)$ is precisely a map α as above, and one can show that such an α_1 extends to a left invariant derivation of $k[G]$. Conversely, given such a derivation $D\colon k[G] \to k[G]$, one can compose with the identity $k[G] \to k$ of the group to obtain a point derivation at the identity. One can use this to get back to the Lie algebra and show that $\mathrm{Lie}(G)$ is isomorphic to the collection of left invariant derivations of $k[g]$.

We define the bracket operation on $\mathrm{Lie}(G)$ to be that induced by this morphism through the above construction. ☐

Remark 2.15. This discussion should be rewritten, but we don't have time right now. Any readers should look in a better source, such as in the Appendix of the wonderful book [19, 7.1–7.5].

Remark 2.16. One can naively compute $Lie(G)$ as the collection of all $X \in Lie(\mathrm{GL}_n)$ such that $1 + X \in G$

Example 2.17.
- GL_n has lie algebra $\mathfrak{gl}_n = M_n$ with bracket $[X, Y] = XY - YX$
- SL_n has lie algebra $\mathfrak{sl}_n = \{X \in M_n | Tr(X) = 0\}$ with bracket as above.
- $\mathrm{SO}_{2n} = \{g \in \mathrm{GL}_n \, | g^t S g = S\}$ where S is the symmetric matrix $\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$ has lie algebra $\mathfrak{so}_{2n} = \{X \in M_n | X^t S + SX = 0\}$. In this example one must assume char$(k) \neq 2$.

2.2. **Restriction of Scalars.** For an algebraic group G_R defined over R a k-algebra we would like to define the Weil restriction of scalars. As a group scheme we define for any k-algebra R':

$$Res_{R/k}(G_R)(R') = G(R \otimes_k R')$$

In general one needs to show that this is in fact a representable functor. If we restrict to R which are sufficiently simple k algebras (for example fields) this can be shown concretely via explicitly using the equations that define R as a k algebra. We can use these equations to rewrite the equations that defined G. In particular letting R-act on R as a k-vector space we get a map $R \hookrightarrow M_n(k)$ and so we can rewrite the equations for G into matrix equations which can be expanded in terms of matrix entries. This set of equations will define the restriction of scalars as an algebraic group over k. This construction is sufficient whenever R is finite dimensional over k and the representation above is faithful.

Example 2.18. If L/k is a finite separable extension of fields then over $\overline{k}$ one has:

$$Res_{L/k}(G)_{\overline{k}} \simeq \prod_{\tau} G_\tau$$

Where τ are the embeddings of L/k into $\overline{k}$ the "descent data" that allows the recovery of a scheme over k is the action of $\mathrm{Gal}(\overline{k}/k)$ on the τ.

Example 2.19. Let $L = k(\sqrt{(}d))$ then taking the regular representation of L acting on L with basis $\{1, \sqrt{d}\}$ we see that $Res_{L/k}(\mathbf{G}_m) = \left\{ \begin{pmatrix} a & db \\ b & a \end{pmatrix} | a^b - db^2 \neq 0 \right\}$.

For a good reference see for example [12].

2.3. **Tori.**

Definition 2.20. An *algebraic torus* is a Linear Algebraic Group T such that $T_{\overline{k}} \simeq \mathbf{G}_m^n$ for some n.

Definition 2.21. A *character* of an algebraic group G is an element of $X^*(G) = \mathrm{Hom}_{\overline{k}}(G, \mathbf{G}_m)$. A *co-character* or one parameter subgroup is an element of $X_*(G) = \mathrm{Hom}_{\overline{k}}(\mathbf{G}_m, G)$.

Remark 2.22. One may decorate $X^*(G)_{\overline{k}}$ or $X^*(G)_k$ or similarly to specify that the characters are defined over a particular field.

We can alternatively characterize algebraic Tori via the following:

Claim 2.23. *Suppose that k is separably closed. Then the following are equivalent:*

- *T is an algebraic torus.*
- *T is connected and diagonalizable (for one and hence all faithful continous representations).*
- *$k[T] = span_k(X^*(T))$.*

[10, III.8.2]

Theorem 2.24. *There exists a contravariant equivalence of Categories between the category of algebraic tori defined over k and finite dimensional $\mathbf{Z}$-torsion free $\mathbf{Z}[\mathrm{Gal}(\overline{k}/k)]$-Modules.*

Remark 2.25. The proof uses the fact that any such group must split over a finite separable extension of k. One then uses the results of [10, III.8.3-8.4] plus descent to conclude the result.

Example 2.26.

- $\mathrm{SO}_2 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a^2 + b^2 = 1 \right\}$

Over any field containing a square root of -1 we can diagonalize this via:

$$\frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} a-bi & 0 \\ 0 & a+bi \end{pmatrix}$$

- Let L/k be any (separable) field extension then: $Res_{L/k}(\mathbf{G}_m)$ is an algebraic torus. Moreover one can show that: $X^*(Res_{L/k}(\mathbf{G}_m)) \simeq \bigoplus_\tau \mathbf{Z}_\tau$ where the summation runs over τ the embeddings of L/k into an algebraic closure of k with natural galois action. In particular this example illustrates the connection between the "descent data" for etale algebras and that for the tori coming from their multiplicative group.
- Even though $\{x \in L | N_{L/k}(x) = 1\}$ is not an algebraic group (over L) we still have:
$$Res_{L/k}(x \in L | N_{L/k}(x) = 1) = \{x \in Res_{L/k}(\mathbf{G}_m) | \det(x) = 1\}$$
defines an algebraic torus over k. This includes the example of SO_2 above where one could take $k = \mathbf{Q}$ and $L = \mathbf{Q}(i)$. In that particular case we see that complex conjugation acts on $X^*(\mathrm{SO}_2)$ as -1.

In the above examples we see that the isomorphism with $\mathbf{G}_m^n$ is not always defined over k. As such we introduce the following definitions:

Definition 2.27. An algebraic torus T is said to be *split* (over k) if $T \simeq \mathbf{G}_m^n$ over k or if equivalently $X^*(T)_k = X^*(T)_{\overline{k}}$. Respectively, T is said to be *anisotropic* if $X^*(T)_k = \{\mathrm{id}\}$.

2.4. **Maximal tori in reductive groups.** The next goal shall be to explain how some of the representation theory of G can be reduced to the representation theory of a maximal torus.

Claim 2.28. *Every reductive group over k has a maximal torus which is defined over k.* [53, CH2 3.1.1]

All maximal tori in $G(\overline{k})$ are conjugate by $G(\overline{k})$. [10, IV.11.3]

For the remainder of lecture G is a connected reductive group. Let $T \subset G$ be a maximal torus.

Definition 2.29. The *Weyl Group* $W(G,T)$ is $N_G(T)/Z_G(T)$.

Remark 2.30. We first remark that if T is defined over k then so too are $N_G(T)$ and $Z_G(T)$. We remark further that one should try to consider $W(G,T)$ as a group scheme.

Example 2.31.
- For the maximal torus consisting of diagonal matricies in GL_n we have that $W(G,T) \cong S_n$ which acts on T by permuting the diagonal.
- For the maximal torus corresponding to $Res_{F/k}(\mathbf{G}_m)$, where F/k is separable of degree n we again have that (over $\overline{k}$) $W(G,T)$ acts as S_n by permuting the n embeddings of F/k. We remark however that this action need not be defined over k.

Our next goal shall be to associate to pairs $(T \subset G)/k$ a "root datum" $\Psi(G,T) = (X, V, \Phi, \Phi^\vee)$ that will characterize G.

3. Lecture 3: The Definition

First a remark on Weyl groups. Let $T \subset \mathrm{GL}_n$ be a maximal torus over $\mathbf{Q}$. We have a decomposition $T = T_{\mathrm{ani}} \cdot T_{\mathrm{split}}$ of T into the product of its maximal anisotropic part and its maximal split part; note that this product is not direct in general, as the factors may have some nontrivial, but finite, intersection. When T is split, i.e. $T = T_{\mathrm{split}}$, then $W(\mathrm{GL}_n, T) = N_T(\mathrm{GL}_n)/C_T(\mathrm{GL}_n)$ is isomorphic to $\mathfrak{S}_n$ (over $\mathbf{Q}$). In the opposite extreme, if $T = T_{\mathrm{ani}} \cdot Z_{\mathrm{GL}_n}$, then $T = \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m$, for a field extension $F/\mathbf{Q}$ of degree n, and $W(\mathrm{GL}_n, T)$ is isomorphic to $\mathrm{Aut}(F/\mathbf{Q})$.

We will assume throughout this lecture that $G_{/k}$ is a connected reductive (linear algebraic) group over a field k.

3.1. **Root data.** Let $\mathfrak{g}$ denote the Lie algebra of G, and consider the *adjoint representation* of G on $\mathfrak{g}$:

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

(For example, for $G = \mathrm{GL}_{n/k}$ this is the usual action of GL_n on M_n by conjugation.) Assume that $T \subset G$ is a maximal split torus, i.e. that G is split over k. Then $\mathrm{Ad}(T)$ consists of commuting semisimple elements, and therefore the action of T on $\mathfrak{g}$ is diagonalizable. For a character $\alpha \in X^*(T)$, let $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \mathrm{Ad}(t)X = \alpha(t)X \text{ for all } t \in T(k)\}$.

Definition 3.1. The nonzero $\alpha \in X^*(T)$ such that $\mathfrak{g}_\alpha \neq 0$ are called the *roots* of G relative to T. We let $\Phi(G,T)$ be the (finite) set of all such roots α, and call the corresponding $\mathfrak{g}_\alpha$ *root spaces*.

Claim 3.2. *Let $T \subset G$ be as above, and let $\mathfrak{t} = \mathrm{Lie}(T)$. Then*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha.$$

Furthermore, each of the root spaces is one-dimensional (cf. [53] *Cor.8.1.2.). (Does it remain true if G is only assumed to be semisimple?)*

Let V be the real vector space $\langle \Phi \rangle \otimes_{\mathbf{Z}} \mathbf{R}$, where $\langle \Phi \rangle \subset X^*(T)$ denote the ($\mathbf{Z}$-linear) span of $\Phi = \Phi(G,T)$. Then the pair (Φ, V) is a *root system*, according to the following:

Definition 3.3. Let V be a finite dimensional $\mathbf{R}$-vector space, and Φ a subset of V. We say that (Φ, V) is a *root system* if the following three conditions are satisfied:

(R1) Φ is finite, does not contain 0, and spans V;
(R2) For each $\alpha \in \Phi$ there exists a reflection s_α relative to α (i.e. an involution s_α of V with $s_\alpha(\alpha) = -\alpha$ and restricting to the identity on a subspace of V of codimension 1) such that $s_\alpha(\Phi) = \Phi$;
(R3) For every $\alpha, \beta \in \Phi$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.

A root system (Φ, V) is said to be of *rank* $\dim_{\mathbf{R}} V$, and to be *reduced* if for each $\alpha \in \Phi$, $\pm\alpha$ are the only multiples of α in Φ. The *Weyl group* of (Φ, V) is $W(\Phi, V) := \{s_\alpha \mid \alpha \in \Phi\} \subset \mathrm{GL}(V)$.

Let (Φ, V) be the root system associated with $T \subset G$. (We note that it is reduced, and that $W(\Phi, V) \cong W(G,T)$.) There exists a pairing $(\ ,\ ) : V \times V \to \mathbf{C}$ for which the elements in the Weyl group become orthogonal transformations. Then if $\alpha \in \Phi$, there exists a unique $\alpha^\vee \in X_*(T)$ such that $\langle -, \alpha^\vee \rangle := \alpha^\vee(-) = \frac{2(-,\alpha)}{(\alpha,\alpha)}$ as maps $X^*(T) \to \mathbf{C}$. Let $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$, and $V^\vee := \langle \Phi^\vee \rangle \otimes_{\mathbf{Z}} \mathbf{R}$.

Claim 3.4. *The pair $(\Phi^\vee, V^\vee)$ is a root system.*

A fundamental result to be stated below is that the quadruple $\Psi = (X^*(T), X_*(T), \Phi, \Phi^\vee)$ attached to $T \subset G$ will contain enough information to characterize G, at least over $\overline{k}$.

Definition 3.5. A *root datum* is a quadruple $(X, Y, \Phi, \Phi^\vee)$ consisting of a pair of free abelian groups X, Y with a perfect pairing $\langle\ ,\ \rangle : X \times Y \to \mathbf{Z}$, together with finite subsets $\Phi \subset X$, $\Phi^\vee \subset Y$ in 1-to-1 correspondence ($\Phi \ni \alpha \leftrightarrow \alpha^\vee \in \Phi^\vee$) such that

- $\langle \alpha, \alpha^\vee \rangle = 2$;
- If for each $\alpha \in \Phi$, we let $s_\alpha : X \to X$ be defined by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$, then $s_\alpha(\Phi) \subset \Phi$, and the group $\langle s_\alpha \mid \alpha \in \Phi \rangle$ generated by $\{s_\alpha\}$ is finite.

We say that a root datum is *reduced* if $\alpha \in \Phi$ only if $2\alpha \notin \Phi$.

An isomorphism of group data, $(X, Y, \Phi, \Phi^\vee) \xrightarrow{\sim} (X', Y', \Phi', (\Phi')^\vee)$, is a group isomorphism $X \xrightarrow{\sim} X'$ inducing dual isomorphisms sending Φ to Φ' and $\Phi^\vee$ to $(\Phi')^\vee$, respectively.

Theorem 3.6 (Chevalley, Demazure)**.** *Assume $k = \overline{k}$. The map*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{connected reductive groups}\\ \textit{over } k\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{reduced root data}\end{array}\right\}$$

induced by $G \mapsto \Psi(G,T) := (X^*(T), X_*(T), \Phi, \Phi^\vee)$ is bijective.

Remark 3.7. Our original attributions for the above thereom were incorrect. Brian Conrad corrected us as follows: “Demazure introduced the notion of root datum so as to systematically keep track of a nontrivial central torus in the theory, but over an algebraically closed field all of the nontrivial content for the existence and isomorphism parts of the story is in the semisimple case, which is entirely due to Chevalley [17]. Demazure’s contribution in [21, Expose XXII] was to solve the Existence and Isomorphism problems over **Z** (and so over any scheme). Actually, Chevalley did make constructions of everything over **Z**, but without an intrinsic characterization of what he was doing (and without an Isomorphism Theorem) – this was the initial motivation for Demazure’s work, to figure out the intrinsic significance of Chevalley’s constructon over **Z**.”

Example 3.8. $G = \mathrm{GL}_n$. The group of diagonal matrices $T(R) := \{\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix} \mid t_i \in R^\times\}$ is a maximal torus in G. The groups of characters and of cocharacters of T are both isomorphic to $\mathbf{Z}^n$ via $(k_1,\dots,k_n) \mapsto (\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix} \mapsto t_1^{k_1}\cdots t_n^{k_n})$ and $(k_1,\dots,k_n) \mapsto (t \mapsto \begin{pmatrix} t^{k_1} & & \\ & \ddots & \\ & & t^{k_n}\end{pmatrix})$, respectively. Note that with these identifications, the natural pairing $\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbf{Z}$ corresponds to the standard “inner product” in $\mathbf{Z}^n$. The roots of G relative to T are the characters e_{ij} : $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n\end{pmatrix} \mapsto t_i t_j^{-1}$ for every pair of integers $(i,j) \in \{1,\dots,n\}^2$ with $i \neq j$, and the corresponding root spaces $\mathfrak{gl}_{n,e_{ij}}$ are the linear span of the $n\times n$ matrix with all entries zero except the (i,j)-th component. The coroot $e_{ij}^\vee$ associated with e_{ij} is $t \mapsto \mathrm{diag}(1,\dots,1,t,1,\dots,1,t,1,\dots,1)$, where all but the i-th and j-th entries along the diagonal of the latter matrix are equal to 1.

Example 3.9. $G = \mathrm{Sp}_{2n}$. These are the split *symplectic groups*, whose R-valued points are given by $G(R) = \{g \in \mathrm{GL}_{2n}(R) \mid g^t J_{n,n} g = J_{n,n}\}$, where $J_{n,n}$ is the block matrix $\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$, with J_n being the $n\times n$ matrix $\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$. We discuss Sp_4. Its Lie algebra is $\mathfrak{sp}_4 = \{X \in \mathfrak{gl}_4 \mid X^t J_{2,2} + J_{2,2}X = 0\}$. The group of diagonal matrices $T(R) = \{x(t_1,t_2) := \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1}\end{pmatrix} \mid t_1,t_2 \in R^\times\}$ is a maximal torus in $G = \mathrm{Sp}_4$. Consider the characters $e_1 : x(t_1,t_2) \mapsto t_1$ and $e_2 : x(t_1,t_2) \mapsto t_2$. Then the set of roots of G relative to T is $\Phi(G,T) = \{\pm(e_1-e_2), \pm(e_1+e_2), \pm 2(e_1+e_2), \pm 2e_2\}$. The corresponding root spaces are easily computed. For example: $\mathfrak{sp}_{4,e_1-e_2} = \{\begin{pmatrix} A & \\ & A'\end{pmatrix} \mid A = \begin{pmatrix} & v \\ & \end{pmatrix}, A' = \begin{pmatrix} & -v \\ & \end{pmatrix}\}$, $\mathfrak{sp}_{4,2e_2} = \{\begin{pmatrix} & B \\ & \end{pmatrix} \mid B = \begin{pmatrix} & b \\ & \end{pmatrix}\}$, $\mathfrak{sp}_{4,e_1+e_2} = \{\begin{pmatrix} & B \\ & \end{pmatrix} \mid B = \begin{pmatrix} & \\ c & \end{pmatrix}\}$, etc. The coroots are given by $(ae_1+be_2)^\vee(t) = \begin{pmatrix} t^a & & & \\ & t^b & & \\ & & t^{-b} & \\ & & & t^{-a}\end{pmatrix}$. We thus have $(e_1-e_2)^\vee(t) = \begin{pmatrix} t & & & \\ & t^{-1} & & \\ & & t & \\ & & & t^{-1}\end{pmatrix}$, $(2e_2)^\vee(t) = \begin{pmatrix} 1 & & & \\ & t^2 & & \\ & & t^{-2} & \\ & & & 1\end{pmatrix}$, etc.

Remark 3.10. There is a complete classification of all the possible reduced irreducible root systems. This is one of the main outcomes of the Weyl-Cartan theory. The exhaustive list is A_ℓ ($\ell \geq 1$), B_ℓ ($\ell \geq 1$), C_ℓ ($\ell \geq 3$) and D_ℓ ($\ell \geq 4$), corresponding to $\mathrm{SL}_{\ell+1}$, $\mathrm{SO}_{2\ell+1}$, $\mathrm{Sp}_{2\ell}$ and $\mathrm{SO}_{2\ell}$, respectively, and the exceptional E_6, E_7, E_8, F_4 and G_2.

3.2. **Borel subgroups.** Recall that G is throughout assumed to be a connected reductive group, unless otherwise indicated.

Definition 3.11. A subgroup scheme $B \subset G$ is said to be a *Borel subgroup* if it is a maximal connected solvable subgroup. A closed subgroups $P \subset G$ is said to be a *parabolic subgroup* if it contains a Borel subgroup.

Claim 3.12. *A closed subgroup $P \subset G$ is parabolic if and only if G/P is projective. For example, $\mathrm{SL}_2/B \cong \mathbf{P}^1$, where $B = \{\left(\begin{smallmatrix} x & * \\ & x^{-1}\end{smallmatrix}\right)\}$, as one checks easily via $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) B \mapsto [a : c]$.*

We can always write a parabolic P as $P = M \cdot N$, with M reductive and N unipotent. Thus, for GL_n, the conjugacy classes of parabolic subgroups (over $\overline{k}$) correspond to the possible partitions of n, the partition $n = 1 + 1 + \cdots + 1$ corresponding to the conjugacy class of a Borel.

Remark 3.13. Borel subgroups always exist over $\overline{k}$, but not necessarily over k itself. For example, consider $G(R) = (B \otimes_k R)^\times$, where B is a division algebra over k.

Definition 3.14. A reductive group G is said to be *split* if there exists a maximal split torus $T \subset G$ (over k); it is said to be *quasi-split* if it contains a Borel subgroup.

Note that G is split only if it is quasi-split, but that the converse is not true, as evidenced by the following:

Example 3.15. Take $G = \mathrm{U}(1,1)$ and consider $\mathbf{C}/\mathbf{R}$ (or simply a quadratic field extension). Then $G(R) = \{g \in \mathrm{GL}_2(\mathbf{C} \otimes_{\mathbf{R}} R) \;:\; \overline{g}^t \begin{pmatrix} & -1 \\ -1 & \end{pmatrix} g = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}\}$, and $\{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \in G\} \subset G$ is a Borel. Thus G is quasi-split, but it is not split (over $\mathbf{R}$).

3.3. **Admissible representations.** Let G be a reductive group over a blobal field F. Assume first that F is actually a number field. Then $G(\mathbf{R} \otimes_{\mathbf{Q}} F)$ is a real reductive Lie group, and we let $K_\infty \subset G(\mathbf{R} \otimes_{\mathbf{Q}} F)$ be a maximal compact subgroup. (For example, for $G = \mathrm{GL}_2$, $K_\infty = \mathrm{O}_2(\mathbf{R})$.) Let $\mathcal{H}_\infty := \mathcal{H}(G(\mathbf{R} \otimes_{\mathbf{Q}} F), K_\infty)$ be the convolution algebra of distributions of $G(\mathbf{R} \otimes_{\mathbf{Q}} F)$ supported on K_∞.

Definition 3.16. A *fundamental idempotent* in $\mathcal{H}_\infty$ is an element of the form

$$\xi_\sigma = d(\sigma)^{-1} \chi_\sigma \mathrm{d}K_\infty,$$

where $\sigma : K_\infty \to \mathrm{Aut}(V)$ is a representation of degree $d(\sigma) < \infty$, χ_σ is its character, and $\mathrm{d}K_\infty$ denotes a Haar measure giving unit volume to K_∞.

The convolution of $f \in \mathcal{H}_\infty$ with a fundamental idempotent $\xi_\sigma \in \mathcal{H}_\infty$ is given by the formula

$$f * \xi_\sigma = \int_{K_\infty} f(\kappa) \xi_\sigma(\kappa) d(\sigma)^{-1} \mathrm{d}K_\infty.$$

In the following we let F be a general global field, with the convention that $G(\mathbf{A}_F^\infty) = G(\mathbf{A}_F)$ if F happens to be a function field. Then we have a *Hecke algebra* defined as:

$$C_c^\infty(G(\mathbf{A}_F)) := \varinjlim_{S \not\supseteq S_\infty} C_c^\infty(G(F_S)) \otimes_{v \notin S_\infty} \xi_{G(\mathcal{O}_v)},$$

where $\xi_{G(\mathcal{O}_v)} = \mathbf{1}_{G(\mathcal{O}_v)} \frac{1}{\mathrm{vol}(G(\mathcal{O}_v))}$, and the superscript "$\infty$" indicates smoothness, that is, local constancy in this context.

Definition 3.17. A *fundamental idempotent* in $\mathcal{H}^\infty$ is an idempotent of the form

$$\xi_{K^\infty} = \mathbf{1}_{K^\infty} \frac{1}{\mathrm{vol}(K_\infty)}$$

for some compact open subgroup $K^\infty \subset G(\mathbf{A}_F^\infty)$. A *fundamental idempotent* in $\mathcal{H} := \mathcal{H}_\infty \otimes \mathcal{H}^\infty$ is an idempotent of the form $\xi_\sigma \otimes \xi_{K^\infty}$.

Definition 3.18. A representation (π_v, V_v) of $C_c^\infty(G(F_v))$ is *admissible* if for all fundamental idempotents ξ, $\dim(\pi_v(\xi)V_v)$ is finite. (Note that this is a fairly strong finiteness condition.) A representation (π, V) of $\mathcal{H}$ is *admissible* if for all fundamental idempotents ξ, $\dim(\pi(\xi)V)$ is finite.

Denote by A_G the identity component of the real points of the greatest $\mathbf{Q}$-split torus in $\mathrm{Res}_{F/\mathbf{Q}}(Z_G)$. (For example, for $G = \mathrm{GL}_{2/\mathbf{Q}}$, $A_G = \mathbf{R}^\times_{>0}$.) Consider the space $\mathrm{L}^2(G(F)A_G\backslash G(\mathbf{A}_F))$, which carries a natural action R of $G(\mathbf{A}_F)$ by right translation. Here, we take a Haar measure $\mathrm{d}g$ on $G(F)$, and we extend it to $G(\mathbf{A}_F)$, which in turn induce a measure on $G(F)A_G\backslash G(\mathbf{A}_F)$; the convergence condition is then with respect to the norm associated with the pairing

$$(f_1, f_2) = \int_{G(F)A_G\backslash G(\mathbf{A}_F)} f_1(g)\overline{f_2(g)}\mathrm{d}g.$$

Remark 3.19. It is an important result due to Harish-Chandra, that if $\varphi \in \mathrm{L}^2(G(F)A_G\backslash G(\mathbf{A}_F))$, then $\overline{R(G(\mathbf{A}_F))\varphi}$ is admissible (in the sense that its K*-finite vectors* are admissible, a notion that will be defined later on).

Definition 3.20. An *automorphic representation* of $G(\mathbf{A}_F)$ is an admissible representation of $\mathcal{H}$ which is a subquotient of $\mathrm{L}^2(G(F)A_G\backslash G(\mathbf{A}_F))$.

4. Lecture 4: Hecke algebras, etc.

Last time we finished discussing root data. Recall that to every reductive group G we associated a tuple of data

$$G \mapsto \Psi = (X^*(T), X_*(T), \Phi, \Phi^\wedge)$$

and this determined the reductive group.

4.1. Hecke algebras.

From here on, G will always be a connected reductive group over a global field F. Last time we introduced the Hecke algebra

$$\mathcal{H} = \mathcal{H}(G(F \otimes_\mathbf{Q} \mathbf{R})K_\infty) \otimes_\mathbf{C} C_c^\infty(G(\mathbf{A}_F^\infty)).$$

The space

$$L^2(G(F)A_G\backslash G(\mathbf{A}_F))$$

carries an action of $G(\mathbf{A}_F)$; here A_G is a subgroup of the center of $G(\mathbf{A}_F)$ and the quotient $G(F)A_G\backslash G(\mathbf{A}_F)$ has finite volume.[1]

Consider $C_c^\infty(G(\mathbf{A}_F^\infty))$, which by definition is the space of smooth compactly supported and complex valued functions on $G(\mathbf{A}_F^\infty)$. Here *smooth* just means locally constant. Any element $f \in C_c^\infty(G(\mathbf{A}_F^\infty))$ can be expressed as a finite linear combination of characteristic functions [2]

$$f = \sum_i c_i 1_{K^\infty a_i K^\infty}$$

for $K^\infty \subseteq G(\mathbf{A}_F^\infty)$ a compact open subgroup, $a_i \in G(\mathbf{A}_F)$ and $c_i \in \mathbf{C}$.

Example 4.1. When $G = \mathrm{GL}_1$ then examples of compact open subgroups of $\mathrm{GL}_1(\mathbf{A}_\mathbf{Q}^\infty)$ are given by sets of the form

$$K_S \prod_{p\notin S} \mathbf{Z}_p^\times$$

for S a finite set of finite primes and K_S any compact open subgroup of $\prod_{p\in S} \mathbf{Q}_p^\times$. This is clear from the definition of the adelic topology, and since each $\mathbf{Z}_p^\times$ is compact open.

[1] What is this A_G?

[2] Prove this, or at least say how one might prove it

Example 4.2. For $G = \mathrm{GL}_2$ an example is given by $\prod_{p\not|\infty} \mathrm{GL}_2(\mathbf{Z}_p)$. Conjugates of this will also be maximal compact open subgroups. A nonmaximal compact open is given by the kernel of the natural reduction map

$$\prod_{p\not|\infty} \mathrm{GL}_2(\mathbf{Z}_p) \to \prod_{p\not|\infty} \mathrm{GL}_2(\mathbf{F}_p).$$

All maximal compact opens of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\infty})$ are conjugate to $\prod_p \mathrm{GL}_2(\mathbf{Z}_p)$, c.f. [50], Chapter IV, Appendix 1. This is also true for GL_n in general. For a quasi-split group, however, there can be several conjugacy classes. For example, in Sp_4 there are two conjugacy classes. If the group is split over an unramified extension, then out of all the possible conjugacy classes, there exists a canonical one: the class consisting of *hyperspecial* maximal compact subgroups. These are characterized by having the same type of special fibre as the full group G; more colloquially, when you reduce mod p, you "get the same thing". See J. Tits article in the Corvallis proceedings [56]; also see the preprints section of J.K Yu's webpage for [60].

When $a \in G(\mathbf{A}_F^{\infty})$, then

$$1_{K^\infty a K^\infty} = 1_{K_S a_S K_S} \otimes 1_{K^S}$$ [3]

for some finite set of finite places S. This reduces the study of Hecke operators away from ∞ to the study of the local Hecke algebra

$$C_c^\infty(G(F_\nu)),$$

the space of smooth compactly supported functions for some finite place ν of F.

Let $n = 2$ and we'll study the *spherical Hecke algebra*:

$$C_c^\infty(\mathrm{GL}_2(\mathbf{Q}_p) /\!/ \mathrm{GL}_2(\mathbf{Z}_p)).$$

Here the double slash means that these functions are invariant under the left and right actions of $\mathrm{GL}_2(\mathbf{Z}_p)$. Examples of functions in the spherical Hecke algebra are given by characteristic functions of compact open subgroups. Let

$$1_{(n,d)} = 1_{\mathrm{GL}_2(\mathbf{Z}_p)\begin{pmatrix} p^n & 0 \\ 0 & p^d \end{pmatrix}\mathrm{GL}_2(\mathbf{Z}_p)}.$$

As n and d vary, these span the spherical Hecke algebra. [4]

For general n the spherical Hecke algebra is defined the same way and a basis is given by

$$\{1_{\mathrm{GL}_n(\mathbf{Z}_p)\lambda(p)\,\mathrm{GL}_n(\mathbf{Z}_p)}\}_{\lambda=(\lambda_1,\dots,\lambda_n),\lambda_1\geq\cdots\geq\lambda_n}$$

where

$$(\lambda_1,\dots,\lambda_n)(p) = \mathrm{diag}(p^{\lambda_1},\dots,p^{\lambda_n}).$$

The Smith normal form for matrices over $\mathbf{Q}_p$, from the theory of elementary divisors, gives the decomposition

$$\mathrm{GL}_n(\mathbf{Q}_p) = \mathrm{GL}_n(\mathbf{Z}_p) T(\mathbf{Q}_p)\,\mathrm{GL}_n(\mathbf{Z}_p)$$

and it follows from this that the set above is a basis for the spherical Hecke algebra.

Multiplication in $C_c^\infty(G(F_\nu))$. We define the *convolution*

$$f_1 * f_2(g) = \int_{G(F_\nu)} f_1(x) f_2(x^{-1}g)\,dx.$$

Fix $K \subseteq G(F_\nu)$ a compact open. Then for $\gamma \in G(F_\nu)$ we write $1_\gamma = 1_{K\gamma K}$. Note that

$$1_\alpha * 1_\beta = \sum_{K\gamma K \in K\backslash G(F_\nu)/K} c_{\alpha,\beta,\gamma} 1_\gamma,$$

[3] Give more detail here

[4] Proof or reference!

where the $c_{\alpha,\beta,\gamma}$ are defined as follows: put $K_\alpha = \alpha K \alpha^{-1} \cap K$, which is compact and open. It is thus of finite index in K. So we can write

$$K = \coprod_i x_i K_\alpha$$

for some finite number of $x_i \in K$. Similarly write

$$K = \coprod_j y_j \beta K_\beta$$

Then $c_{\alpha,\beta,\gamma}$ is the number of pairs (i,j) such that $\gamma x_i \alpha y_j \beta \in K$. For this see Chapter 3, section 1 of Shimura's classic [51].

Example 4.3. In this example we will work out the connection between this definition of the Hecke algebra and the classical definition.[5]

Show $1_{(1,0)} * 1_{(1,0)} = 1_{(2,0)} * 1_{(1,1)} * q1_{(1,1)}$[6].

4.2. **A bit of real representation theory.** Let (π, V) be a representation of $G(F_\infty)$ where V is assumed to be a Hilbert space, but π is not assumed to be unitary. Recall that there exists an exponential map

$$\exp\colon \operatorname{Lie}(G(F_\infty)) \to G(F_\infty).$$

Example 4.4. For GL_n, the Lie algebra $\mathfrak{gl}_n$ is the collection of $n \times n$ matrices. The exponential is simply the matrix exponential in this case.

Given $\phi \in V$ we write

$$\pi(X)\phi = \frac{d}{dt}\pi(\exp(tX))\phi|_{t=0}$$

whenever this makes sense. We will sometimes simply write $X\phi$ for $\pi(X)\phi$. We say that a vector $\phi \in V$ is C^1 if for all $X \in \mathfrak{g}$, the derivative $\pi(X)\phi$ is defined and the map $X \mapsto \pi(X)\phi$ is continuous. We say $\phi \in V$ is C^k if ϕ is C^1 and $X\phi$ is C^{k-1} for all $X \in \mathfrak{g}$. It is C^∞ if it is C^k for all $k \geq 1$.

Definition 4.5. A vector $\phi \in V$ is said to be *smooth* if ϕ is C^∞. Set $V^\infty \subseteq V$ to be the subspace of smooth vectors.

Note that we *can* differentiate in V^∞.

Lemma 4.6. *The space V^∞ is invariant under $G(F_\infty)$.*

Proof. Let $g \in G(F_\infty)$ and let $X \in \mathfrak{g}$. Then

$$\begin{aligned} X(\pi(g)\phi) &= \lim_{t\to 0}\frac{1}{t}(\pi(\exp(tX)g)\phi - \pi(g)\phi) \\ &= \pi(g)\lim_{t\to 0}\frac{1}{t}(\pi(\exp(t\operatorname{Ad}(g^{-1})X)\phi - \phi) \end{aligned}$$

where $\operatorname{Ad}(g^{-1})X = g^{-1}Xg$. The limit exists if ϕ is C^1. This implies that $\pi(g)\phi$ is C^1. One shows that $\pi(g)\phi$ is C^k for all k if ϕ is so by induction. □

Lemma 4.7. *Let (π, V) be a Hilbert space representation of $G(F_\infty)$. Then the action of $\mathfrak{g}$ defined above is a Lie algebra representation.*

[5] Finish me

[6] add a proof!!

Proof. We need to show that
$$X(Y\phi) - Y(X\phi) = [X, Y]\phi.$$
It is sufficient to show that
$$\langle X(Y\phi) - Y(X\phi), f\rangle = \langle [X, Y]\phi, f\rangle$$
for all $f \in V$. Fix such an f and let $L\colon V^\infty \to C^\infty(G)$ be defined by $L\phi(g) = \langle \pi(g)\phi, f\rangle$. One needs to check that this is indeed smooth. Now we must show that
$$(dX \circ L)\phi(g) = ((L \circ X)\phi(g).$$
For this we compute:
$$\begin{aligned} \frac{d}{dt}(L\phi)(g\exp(tX))|_{t=0} &= \frac{d}{dt}\langle \pi(g)\pi(\exp(tX))\phi, f\rangle|_{t=0} \\ &= \langle \pi(g)X\phi, f\rangle \\ &= (L \circ X)\phi(g). \end{aligned}$$
Since we are assuming that ϕ is smooth, we see that this is as well. So we moved to $C^\infty(G)$ and there we have a representation. □

The point so far is that on this space of smooth vectors V^∞ we have a representation of the Lie algebra. Note that so far we don't even know if V^∞ is nonzero; it had best be large for this representation to be useful!

How big is V^∞? If $f \in C_c^\infty(G(F_\infty))$ then define
$$\pi(f)\phi = \int_{G(F_\infty)} f(g)\pi(g)\phi dg.$$

Proposition 4.8. (1) *If f is as above, and $\phi \in V$ then $\pi(f)\phi \in V^\infty$.*
(2) *The space V^∞ is dense in V.*

Proof. For the first part we begin by trying to find an expression for $X\pi(f)$ that makes sense.
$$\begin{aligned} X\pi(f)\phi &= \frac{d}{dt}\pi(\exp(tX))\pi(f)\phi|_{t=0} \\ &= \frac{d}{dt}\int_{G(F_\infty)} f(g)\pi(\exp(tX)g)\phi dg|_{t=0} \\ &= \frac{d}{dt}\int_{G(F_\infty)} \phi(\exp(-tX)g)\pi(g)\phi dg|_{t=0} \\ &= \int_{G(F_\infty)} f_X(g)\pi(g)\phi dg \end{aligned}$$
where $f_X(g) = (d/dt)f(\exp(-tXg)|_{t=0}$. As we know this is smooth, it follows that $X\pi(f)\phi$ is as well.

For the second claim let $\varepsilon > 0$. The map $G \times V \to V$ given by $(g, \phi) \mapsto \pi(g)\phi$ is continuous. This implies that there exists a neighbourhood $U \subseteq G(F_\infty)$ of the identity such that $|\pi(g)\phi - \phi| < \varepsilon$ for all $g \in U$. Take $f \in C_c^\infty(G(F_\infty))$ to be positive valued with support in U and such that
$$\int_{G(F_\infty)} f(g)dg = 1.$$
Then
$$\begin{aligned} |\pi(f)\phi - \phi| &= \left|\int_{G(F_\infty))} f(g)(\pi(g)\phi - \phi)dg\right| \\ &\leq \int_{G(F_\infty)} f(g)\left|\pi(g)\phi - \phi\right| dg \leq \varepsilon \end{aligned}$$
which implies that $\pi(f)\phi$ is as close to ϕ as we wish. Hence V^∞ is dense. □

Note that we followed Bump [14] above, 2nd chapter.

Denote the smooth vectors in $L^2(G(F)A_G\backslash G(\mathbf{A}_F))$ by $\mathcal{A}^\infty(G)$. Let $K_\infty \subseteq G(F_\infty)$ be a maximal compact subset.

Definition 4.9. A vector $\phi \in V$ as above is said to be K_∞-*finite* if

$$\dim_{\mathbf{C}} \operatorname{span}(\pi(k)\phi \mid k \in K_\infty) < \infty.$$

5. Lecture 5:

5.1. **Haar Measure.** If G is a locally compact group (for example GL_n) then there exists a positive regular borel measure $d_L g$ on G that is left invarient under the action of G. That is we have:

$$\int_G f(xg)d_L gL = \int_G g(g)d_L g \qquad \forall x \in G.$$

Moreover, this measure is unique up to scalars.

There is also a right invariant measure $d_R g = d_L(g^{-1})$.

If there exists a constant C such that $d_R g = C d_L g$ then G is said to be *unimodular*.

Example 5.1. For example all abelian groups, reductive groups and unipotent groups are unimodular.

An example of a group which is not unimodular is:

$$B = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} x^{\frac{1}{2}} & xy^{\frac{1}{2}} \\ 0 & y^{\frac{1}{2}} \end{pmatrix} \in \mathrm{GL}_n(\mathbf{R})^+ \right\}$$

Where we have $d_L g = \frac{dxdydu}{y^2 u}$ and $d_R g = \frac{dxdydu}{yu}$.

5.2. **Back to Automorphic forms.** Recall that $G(F_\infty)$ gave us a representation on a hilbert space V and that we had that V^∞ is dense in V. Moreover this gave us a representation of:

$$\mathfrak{g} = \operatorname{Lie} G(F_\infty)$$

$K_\infty \subseteq G(F_\infty)$ is a maximal compact subgroup.

Lemma 5.2. *Let (π, V) be a hilbert space representation of $G(F_\infty)$ then there exists a hermitian inner product $\langle -, - \rangle : V \times V \to \mathbf{C}$ which gives the same topology on V but with respect to which $\pi|_{K_\infty}$ is unitary. That is $\langle \pi(k)v, \pi(k)u \rangle = \langle v, u \rangle \; \forall k \in K_\infty$.*

Proof. Let $(\cdot, \cdot)$ denote the original hilbert space pairing. Define:

$$\langle v, w \rangle = \int_K (\pi(k)v, \pi(k)w)dk$$

By construction it is K-invariant so we need only check the claim about the topology. The maps $K \to \mathbf{C}$ given by $k \mapsto (\pi(k)v, \pi(k)v)$ where $v \neq 0$ form a family of non-vanishing continous functions. Thus $|\pi(k)v|$ is a bounded family of functions which implies by uniform boundedness that the operator norm is bounded. In particular there is some C such that $|\pi(k)v| < C\,|v|$. We can likewise find a similar bound for $\pi(k^{-1})$ and so $C^{-1}\,|v| \leq |\pi(k)v| \leq C\,|v|$. From this we find that $|v|^2_{new} = \int_K (\pi(k)v, \pi(k)v)dk$ satisfies:

$$C^{-2}Vol(K)\,|V| < |v|^2_{new} < C^2 Vol(K)\,|v|$$

Which completes the result. □

Definition 5.3. If π_1 and π_2 are two representations of G an *intertwining operator* is a continuous linear map $L : V_1 \to V_2$ such that:

$$\pi_2(g) \circ L = L \circ \pi_1(g)$$

We say that such a map L is G-equivariant.

Proposition 5.4. *Suppose K is compact and (π_1, V_1) and (π_2, V_2) are two representations with π_2 unitary. If there exist matrix coefficients f_1, f_2 for π_1, π_2 respectively that are not orthogonal in $L^2(K)$ then there exists a non-trivial intertwining operator $L : V_1 \to V_2$.*

Proof. [7] Let $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$ be such that:

$$\int_K (\pi_1(k)x_1, y_1)_{V_1} \overline{(\pi_2(k)x_2, y_2)_{V_2}} dk \neq 0$$

Let $L(v) = \int_K (\pi_1(k)v, y_1)_{V_1} \pi_2(k^{-1})(y_2) dk$, this integral makes sense as V_2 is complete. Moreover the above is equivalent to $L \neq 0$, since we have:

$$\begin{aligned}
(L(x_1), x_2)_{V_2} &= (\int_K (\pi_1(k)x_1, y_1)_{V_1} \pi_2(k^{-1}) y_2 dk, x_2)_{V_2} \\
&= \int_K (\pi_1(k)x_1, y_1)_{V_1} (\pi_2(k^{-1})y_2, x_2)_{V_2} dk && \text{pairing is linear and smooth} \\
&= \int_K (\pi_1(k)x_1, y_1)_{V_1} \overline{(\pi_2(k)x_2, y_2)_{V_2}} dk && V_2 \text{ is unitary and pairing is hermitian} \\
&\neq 0.
\end{aligned}$$

Moreover we have:

$$\pi_2(g) \circ L(V) = \int_K (\pi_1(k)v, y_1)_{V_1} (\pi_2(gk^{-1})(y_2))_{V_2} dk$$

and if we change perform the change of variables $k \mapsto kg$ this becomes $L \circ \pi_1(g)$ and so this is indeed a non-trivial intertwining operator. ☐

Theorem 5.5 (Peter-Weyl Theorem)**.** *Let $K \subset \mathrm{GL}_n(\mathbf{C})$ be compact then:*

1. *The matrix coefficients of finite dimensional unitary representations of K are dense in $C(K)$ and $L^p(K)$ for all $1 \leq p \leq \infty$.*
2. *Any Irreducible unitary representation of K is finite dimensional*
3. *Let (π, V) be a unitary representation of K, then V decomposes into a hilbert space direct sum of irreducible unitary subrepresentations.*

Remark 5.6. What this says is that all the representation theory for K is contained in $L^2(K)$ and consequently a common strategy to understand representations of G is to look at the restrictions to compact subgroups.

Proof. We may regard K as a subset of $M_n(\mathbf{R})$ for some n. We shall further identify this with $\mathbf{R}^{n^2}$. We call a function on K polynomial if it is the restiction of a polynomial in $\mathbf{R}^{n^2}$ to K.

Observe that every polynomial function is a matrix coefficient for some finite dimensional representation. Indeed, let $r \in \mathbf{Z}_{>0}$ and let (ρ, R) be the representation of K on the polynomials of degree less than r via right multiplication (so $\rho(g)f(x) = f(xg)$). By the lemma we can find a hermitian metric on R making this representation unitary and then by using that every bounded linear functional on a hilbert space comes from an inner product we have that there exists some $f_0 \in R$ such that $f(1) = \langle f, f_0 \rangle$ where f is the polynomial we wish to represent as a matrix

[7] do the subscripts in this proof look...bad?

coefficient. We then have that $f(g) = \rho(g)f(1) = (\rho(g)f, f_0)$. This implies claim 1 by using that $C(K)$ is dense in L^p and using Stone-Weirstrauss for the density of polynomials.

Now claim that if (π, V) is a non-zero unitary representation of K then V admits a non-zero finite dimensional invariant subspace. Let ψ be a non-zero matrix coeficient ($\psi = (\pi(g)x, x)$). We can aproximate ψ by polynomials. This implies there exists a polynomial not orthogonal to ψ in $L^2(K)$. This in tern implies the existance of a non-zero intertwinning map $L : R \to V$ which implies we have a subrepresentation. This completes claim 2.

For claim 3 we use Zorn's Lemma to construct a maximal subspace which is a direct sum of finite subrepresentations. It then follows from 2 that this is the whole space as the orthogonal complement has finite subrepresentations. ☐

Now, let (π, V) be an admissible representation of $G(F_\infty)$ on a hilbert space then without loss of generality suppose that (π, V) is unitary. For each equivalence class of irreducible representations σ of K_∞ we write:

$$V(\sigma) = \{v \in V | \langle \pi|_{K_V} v\rangle \simeq \sigma\}$$

This is the *σ-isotypic* subspace.

The definition of an admissible representation implies $\dim(V(\sigma)) < \infty$. We claim (and will eventually show) that $V = \oplus V(\sigma)$ where this is a hilbert space direct sum.

Define $V_{fin} = \oplus V(\sigma)$ to be the algebraic direct sum. This is the space of K_∞ finite vectors. The above claim is then that V is essentially the completion of V_{fin}.

Proposition 5.7. *Let $\mathfrak{k} = \mathrm{Lie}(K_\infty)$ then the following are equivalent:*

(1) *$v \in V$ is K_∞ finite.*
(2) *$\langle \pi(k)v | k \in K_\infty\rangle$ is finite dimensional*
(3) *$\langle \pi(x)v | x \in \mathfrak{k}\rangle$ is finite dimensional*

Exercise [14, $2.4]

Proposition 5.8. *Let (π, V) be an admissible hilbert space representation of $G(F^\infty)$ then K^∞-finite vectors are smooth ($V_{fin} \subset V_\infty$) moreover they form a dense subset which is $\mathfrak{g}$ invariant.*

Definition 5.9. A $(\mathfrak{g}, K_\infty)$-module is a vector space V with a representation of π of $\mathfrak{g}$ and K_∞ which satisfy the following:

(1) $V = \oplus V_i$ is the countable algebraic direct sum with V_i finite dimensional K_∞-invariant vector spaces;
(2) for $X \in \mathfrak{k}$ and $v \in V$ we have:

$$\pi(X)v = Xv = \frac{d}{dt}\exp(tX)v|_{t=0} = \lim_{t\to 0}\frac{1}{t}(\pi(\exp(tX)v) - v)$$

(3) for $g \in K_\infty$ and $X \in \mathfrak{g}$ we have $\pi(g)\pi(X)\pi(g^{-1})v = \pi(\mathrm{Ad}(g)X)v$.

We say moreover that $(\mathfrak{g}, K_\infty)$-module is *admissible* if the V_i have distinct K_∞-types.

A representation of $G(F_\infty)$ as in the proposition gives a $(\mathfrak{g}, K_\infty)$-module.

6. Lecture 6: An Alternative Definition

6.1. Finite vectors.

Proposition 6.1. *Let (π, V) be an admissible Hilbert space representation of $G(F_\infty)$. Then:*

- *The K_∞-finite vectors are smooth (even analytic!);*
- *$V_{\mathrm{fin}} \subset V$ is dense and invariant under the action of $\mathfrak{g}$.*

Thus an admissible Hilbert space representation of $G(F_\infty)$ yields a $(\mathfrak{g}, K_\infty)$-module.

We say that two Hilbert space representations are *infinitessimaly equivalent* if their underlying $(\mathfrak{g}, K_\infty)$-modules are isomorphic. The notion of infinitessimal equivalence is strictly finer than that of equivalence (cf.[14] Exer. 2.6.1).

Proof. We can assume that $\pi|_{K_\infty}$ is unitary. Write $V_0 := V_\infty \cap V_{\text{fin}}$.

(1) *V_0 is dense in V:* Let U be a neighborhood of 1 in $G(F_\infty)$ and let $\varepsilon > 0$. For the ease of notation, we write simply K and G for K_∞ and $G(F_\infty)$, respectively. Suppose that f is a positive smooth function on G with support in KU, such that $\int_G f(g)\mathrm{d}(g) = 1$ and $\int_{G\smallsetminus U} |f(g)|\mathrm{d}g < \varepsilon$. If U and ε are sufficiently small, then $\pi(f)\phi$ will be near ϕ for all ϕ in V, and furthermore all $\pi(f)\phi$ will be K-finite after the following construction of an f as above: Let $U_1 \subset G$ and $W \subset K$ be neighborhoods of 1 such that $WU_1 \subset U$, and let f_1 be a positive smooth function supported in U_1 such that $\int_G f_1(g)\mathrm{d}g = 1$. By the Peter-Weyl theorem, there exists a matrix coefficient f_0 of a finite dimensional representation of K such that $\int_K f_0(k)\mathrm{d}k = 1$ and $\int_{K\smallsetminus W} |f_0(k)|\mathrm{d}k < \varepsilon$. Then we take f to be given by $f(g) := \int_K f_0(k) f_1(k^{-1}g)\mathrm{d}k$. Clearly, f has support contained in $KU_1 \subset KU$, and note that since $WU_1 \subset U$ and $\mathrm{Supp}(f_1) \subset U_1$, we have that $k \notin W$ if $g \in G \smallsetminus U$ is such that $f_1(k^{-1}g) \neq 0$ (since otherwise would have $g = k(k^{-1}g) \in WU_1 \subset U$). Therefore:

$$\begin{aligned}
\int_{G\smallsetminus U} |f(g)|\mathrm{d}g &\leq \int_{G\smallsetminus U}\int_K |f_0(k)|\cdot|f_1(k^{-1}g)|\mathrm{d}k\mathrm{d}g \\
&= \int_{G\smallsetminus U}\int_{K\smallsetminus W} |f_0(k)|\cdot|f_1(k^{-1}g)|\mathrm{d}k\mathrm{d}g \\
&\leq \int_{K\smallsetminus W} |f_0(k)| \int_G f_1(k^{-1}g)\mathrm{d}g\mathrm{d}k \\
&= \int_{K\smallsetminus W} f_0(k)\mathrm{d}k \\
&< \varepsilon.
\end{aligned}$$

Now, $\pi(f)\phi$ is K-finite: Let (ρ, R) be a finite dimensional unitary representation of which f_0 is a matrix coefficient: $f_0(k) = \langle \rho(k)\xi, \zeta\rangle$, for certain $\xi, \zeta \in R$. Then if $k_1 \in K_1$:

$$\begin{aligned}
f_1(k_1^{-1}g) &= \int_K f_0(k) f_1(k^{-1}k_1^{-1}g)\mathrm{d}k \\
&= \int_K \langle \rho(k)\xi, \zeta\rangle f_1(k^{-1}k_1^{-1}g)\mathrm{d}k \\
&= \int_K \langle \rho(k^{-1})\rho(k)\xi, \zeta\rangle f_1(k^{-1}g)\mathrm{d}k \\
&= \int_K \langle \rho(k)\xi, \rho(k)\zeta\rangle f_1(k^{-1}g)\mathrm{d}k.
\end{aligned}$$

Therefore, the linear span of the functions $f(k_1^{-1}g)$, is contained in the linear span of the functions $g \mapsto \int_K \langle \rho(k)\xi, \zeta\rangle f_1(k^{-1}g)\mathrm{d}k$, which is finite dimensional. Thus the space spanned by the functions $\pi(k_1)\pi(f)\phi = \int_G f(g)\pi(k_1 g)\mathrm{d}g = \int_G f(k_1^{-1}g)\pi(g)\mathrm{d}g$ is finite dimensional, so $\pi(f)\phi \in V_{\text{fin}}$, and the $\pi(f)\phi$ are smooth vectors by a previous proposition. It follows that V_0 is dense in V_∞, which is dense in V.

(2) $V_{\text{fin}} \subset V_\infty$: First observe that V_0 is K_∞-invariant, since V_∞ is. Let σ be an irreducible unitary representation of K_∞. Then $V_0(\sigma)$ is contained in $V(\sigma)$, since V_{fin} is an algebraic direct sum of the $V(\sigma)$; so it suffices to show that $V_0(\sigma) = V(\sigma)$. Let $\phi \in V(\sigma)$ be in the orthogonal complement of $V_0(\sigma)$; note that $V(\sigma)$ is finite dimensional by admissibility,

and so $V_0(\sigma)$ has a well-defined orthogonal complement (this is the only part of the proof where admissibility is used). Then ϕ is orthogonal to all of V_0, because it is orthogonal to $V(\tau)$ for every $\tau \neq \sigma$ (Exercise). Therefore $\phi = 0$, since V_0 is dense.

Finally, we want to show that V_{fin} is invariant under $\mathfrak{g}$. Let $\phi \in V_{\text{fin}}$, and R be the span of ϕ under K_∞, and $R_1 := \langle Y\phi \mid Y \in \mathfrak{g} \text{ and } \phi \in R\rangle$, which is clearly finite dimensional.

Claim 6.2. *R_1 is invariant under R.*

Indeed, if $X \in \mathfrak{k}$, and $Y\phi \in R$, then $X(Y\phi) = [X,Y]\phi + Y(X\phi)$, which is an element in R_1. Therefore the elements of R_1 are K_∞-finite, and hence $Y\phi$ is K_∞-finite for all $Y \in \mathfrak{g}$.

□

Remark 6.3. For $\mathrm{GL}_2(\mathbf{R})$, we will be able to classify all $(\mathfrak{g}, K_\infty)$-modules, and can hope to determine all those that arise from unitary representations. This will give restrictions to what can occur in $\mathrm{L}^2(G(F)A_G\backslash G(\mathbf{A}_F))$, which is our ultimate object of study.

6.2. **The Hecke algebra at the infinite places.** Recall that $\mathcal{H}_\infty$ denotes the algebra of distributions on $G(F_\infty)$ with (compact) support contained in K_∞.

Claim 6.4.
- *$\mathcal{H}_K \otimes_{U(\mathfrak{k}_\mathbf{C})} U(\mathfrak{g}_\mathbf{C}) \xrightarrow{\sim} \mathcal{H}_\infty$, where $U(\mathfrak{k}_\mathbf{C})$ and $U(\mathfrak{g}_\mathbf{C})$ denote the universal enveloping algebra of the complexification of the Lie algebras of K_∞ and G, respectively.*
- *Every smooth (in a sense that we will have no need to specify) $\mathcal{H}_\infty$-module is a $(\mathfrak{g}, K_\infty)$-module in a natural manner.*
- *There is an equivalence between $(\mathfrak{g}, K_\infty)$-modules and smooth modules over $\mathcal{H}_\infty$.*

Recall that if $\varphi \in \mathrm{L}^2(G(F)A_G\backslash G(\mathbf{A}_F))$, then $\overline{R(G(\mathbf{A}_F))\varphi}$ is a $G(F_\infty)$-representation, and it is a result due to Harish-Chandra that the K_∞-vectors in this space form an admissible $\mathcal{H}_\infty$-module. Therefore, the K_∞-vectors in $\overline{R(G(\mathbf{A}_F))\varphi}$ form an admissible $(\mathfrak{g}, K_\infty)$-module, and with this we can alternatively define:

Definition 6.5. An *automorphic representation* of G is an admissible $(\mathfrak{g}, K_\infty) \times G(\mathbf{A}_F^\infty)$-module that is a subquotient of $\mathrm{L}^2(G(F)A_G\backslash G(\mathbf{A}_F))$.

6.3. **Approximation.** Let G be a connected linear algebraic group (so we do not assume G to be necessarily reductive). A reference for this section is the paper [9] by A. Borel. Recall that $K^\times$ embeds into $\mathbf{A}_F^\times$ diagonally as a discrete subspace, with non-compact quotient, i.e. $GL_1(F)\backslash GL_1(\mathbf{A}_F)$ is non-compact, and actually it has infinite volume. An analogous phenomenon occurs for other groups, and this motivates the following definition:

$$G(\mathbf{A}_F)^1 := \bigcap_{\chi \in X^*(G)} \ker(\|\cdot\|_F \circ \chi : G(\mathbf{A}_F) \to \mathbf{R}_{>0}^\times).$$

Note that $G(F)$ is contained $G(\mathbf{A}_F)^1$ in virtue of the product formula.

Theorem 6.6 (Borel)**.**
- *$G(\mathbf{A}_F)^1$ is unimodular.*
- *$G(F)\backslash G(\mathbf{A}_F)^1$ has finite volume with respect to the Haar measure.*
- *The $G(F)\backslash G(\mathbf{A}_F)^1$ is compact (or equivalently, every unipotent element of $G(F)$ belongs to the radical; note that this is satisfied for $G(R) = (B \otimes R)^\times$ for a division algebra B, but not for $G = \mathrm{GL}_2$, for example. This corresponds to the fact that classical modular curves are non-compact among Shimura curves.)*

Let $K^\infty \subset G(\mathbf{A}_F^\infty)$ be a compact open subgroup, and let J be a set of representatives for $G(F) \times G(F_\infty)K^\infty$ that intersect $G(\mathbf{A}_F)^1$.

- *Then $G(\mathbf{A}_F)^1 = \bigcup_J G(F) \times (G(F_\infty)K^\infty \cap G(\mathbf{A}_F)^1)$, and moreover, J is finite.*

We can rephrase the last statement in the theorem as follows. We have the decomposition $G(\mathbf{A}_F) = A_G \times G(\mathbf{A}_F)^1$, where $A_G = Z_G(F \otimes_{\mathbf{Q}} \mathbf{R})^+$, the sign "+" meaning that we are taking the identity component in the real topology, that is the positive elements for each real embedding of F. Thus $A_G\backslash G(\mathbf{A}_F) = \bigcup_J G(F).x.A_G\backslash G(F_\infty)K^\infty$. We can assume that $x_\infty = 1$ for all $x \in J$, and then this gives a homeomorphism:

$$\begin{aligned} G(F)A_G\backslash G(\mathbf{A}_F)/K^\infty &\xrightarrow{=} \coprod_J \Gamma_x(K^\infty)\backslash A_G/G(F_\infty) \\ \Gamma_x(K^\infty) &\mapsto \Gamma_x(K^\infty).g, \end{aligned}$$

where $\Gamma_x(K^\infty) := G(F) \cap x.A_G\backslash G(F_\infty)K^\infty.x^{-1}$.

Example 6.7. Consider $G = \mathrm{SL}_{2/\mathbf{Q}}$. Then $K^\infty = \mathrm{SL}_2(\hat{\mathbf{Z}})$ is a compact open subgroup, where $\hat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$ is the profinite completion of $\mathbf{Z}$. Then by the latter statement we can identify:

$$\mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R}) = \mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A}_{\mathbf{Q}})/\mathrm{SL}_2(\hat{\mathbf{Z}}).$$

Also $K_0(N)$, defined as the completion of $\Gamma_0(N)$ in $\mathrm{SL}_2(\hat{\mathbf{Z}})$ is an example of a compact open subgroup (which equals $\prod_{p\nmid N} \mathrm{SL}_2(\mathbf{Z}_p) \times \prod_{q|N}\{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \mod q\mathbf{Z}_q\}$). Then the above gives:

$$\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbf{R}) = \mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A}_{\mathbf{Q}})/K_0(N).$$

Thus if ϕ is a complex function on the double coset space $\mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A}_{\mathbf{Q}})/K_0(N)$, then it gives rise to a complex function on the quotient $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbf{R})$, and viceversa.

We obtain a third example taking $K_\infty = \mathrm{SO}_2(\mathbf{R})$. Then:

$$\Gamma_0(N)\backslash\mathfrak{H} \xrightarrow{\sim} \mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A}_{\mathbf{Q}})/K_0(N)K_\infty = \mathrm{SL}_2(\mathbf{Q})\backslash\mathfrak{H} \times \mathrm{SL}_2(\mathbf{A}_{\mathbf{Q}})/K_0(N).$$

7. Lecture 7: Not yet titled

Last time we discussed a bijection

$$G(F)A_G\backslash G(\mathbf{A}_F)/K_\infty \to \coprod_\Gamma \Gamma A_G\backslash G(F_\infty).$$

Automorphic representations live on the left side. Today we're going to define spaces of automorphic forms. These will give $(\mathfrak{g}, K_\infty)$-modules and automorphic representations.

Definition 7.1. A *norm* $\|\cdot\|$ on $G(F_\infty)$ is a function of the form

$$\|g\| = \mathrm{tr}(\sigma(g)^*\sigma(g))^{1/2}$$

where $\sigma\colon G(F_\infty) \to \mathrm{GL}(E)$ is a finite dimensional representation with finite kernel such that $\sigma|_{K_\infty}$ is unitary. Here $*$ denotes the adjoint for the Hilbert space structure.

Definition 7.2. A function $\phi\colon G(F_\infty) \to \mathbf{C}$ is said to be *slowly increasing* if there exists a norm $\|\cdot\|$, a constant C and a positive integer r such that

$$|f(x)| \leq C\|x\|^r$$

for all $x \in G(F_\infty)$.

Note somewhat paradoxically that a rapidly decreasing function is also slowly increasing.

Definition 7.3. Let $\Gamma \subseteq G(\mathbf{Q})$ be an arithmetic subgroup. A function $\phi\colon G(F_\infty) \to \mathbf{C}$ is an *automorphic form* if

(1) ϕ is smooth;
(2) ϕ is slowly increasing;
(3) $\phi(\gamma x) = \phi(x)$ for all $x \in G(F_\infty)$, $\gamma \in \Gamma$;

(4) there exists an elementary idempotent $\xi \in \mathcal{H}_\infty$ such that $\phi * \xi = \phi$. (This says ϕ has a particular K-type and is K-finite);
(5) there exists an ideal $J \subseteq Z(\mathfrak{g})$ of finite codimension such that $f * X = 0$ for all $X \in J$.

Remark 7.4. Above $Z(\mathfrak{g})$ is the center of the universal enveloping algebra $U(\mathfrak{g})$, where $\mathfrak{g} = \mathrm{Lie}_{\mathbf{R}}\, G(F_\infty) \otimes \mathbf{C}$.

We denote the space of automorphic forms, with notations as above, by

$$\mathcal{A}(\Gamma, \xi, J).$$

We will also put

$$\mathcal{A}(\Gamma, J) = \bigcup_{\xi} \mathcal{A}(\Gamma, \xi, J),$$

which is a union of $(\mathfrak{g}, K_\infty)$-modules.

7.1. **Automorphic forms on adele groups.**

Definition 7.5. An automorphic form on $G(\mathbf{A}_F)$, of type ξ, J, is a function $\phi\colon G(\mathbf{A}_F) \to \mathbf{C}$ such that

(1) $\phi(\gamma x) = \phi(x)$ for all $\gamma \in G(F)$ and for all $x \in G(\mathbf{A}_F)$;
(2) There exists a fundamental idempotent ξ such that $f * \xi = f$;
(3) there exists an ideal $J \subseteq Z(\mathfrak{g})$ such that $\phi * X = 0$ for all $X \in J$;
(4) for all $y \in G(\mathbf{A}_F)$, $x \mapsto \phi(xy)$ is slowly increasing and smooth.
(5) $\phi(ag) = \phi(g)$ for all $g \in A_G$ and $g \in G(\mathbf{A}_F$.

Should be pretty clear that we have an isomorphism as follows: if $\xi = \xi_\infty \otimes \xi_{K^\infty}$ then

$$\mathcal{A}(\xi_\infty \otimes \xi_{K^\infty}) \to \bigcup_x \mathcal{A}(\Gamma_x(K^\infty), \xi_\infty, J).$$

The map is given by $\phi \mapsto (x \mapsto \phi(cx))$. Harrish-Chandra proved the fundamental result that these spaces are finite dimensional.

Definition 7.6. An automorphic form ϕ is said to be *cuspidal* if

$$\int_{N(\mathbf{A}_F} \phi(ng) dn = 0$$

for all parabolic subgroups $P \subseteq G$ with Levi decomposition $P = MN$, and for all $g \in G(\mathbf{A}_F)$. Let $\mathcal{A}^0(\xi, J) \subseteq \mathcal{A}(\xi, J)$ denote the subspace of cuspical automorphic forms.

Remark 7.7. If you weaken this definition to hold for almost all $g \in G(\mathbf{A}_F)$ then you obtain a cuspidal subspace denoted

$$L^2_0(G(F)A_G \backslash G(\mathbf{A}_F)) \subseteq L^2(G(F)A_G \backslash G(\mathbf{A}_F)).$$

Remark 7.8. This subspace is presereved by $\mathfrak{g}$.

Definition 7.9. A *cuspidal automorphic representation* is an automorphic representation equivalent with a subrepresentation of $L^2_0(G(F)A_G \backslash G(\mathbf{A}_F))$.

Note that we do mean subrepresentation above, and not just subquotient as we had for automorphic representations.

As ξ and J vary, we have

$$\bigcup_{\xi, J} \mathcal{A}^0(\xi, J) \subseteq L^2_0(G(F)A_G \backslash G(\mathbf{A}_F))$$

is dense.

7.2. **Classical to automorphic forms.** Let $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ be a congruence subgroup. Then recall the following definition

Definition 7.10. The space of *weight k modular forms* for Γ is the space $M_k(\Gamma)$ of functions $f\colon \mathfrak{H} \to \mathbf{C}$ satisfying the following conditions:

(1) $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$;
(2) f is holomorphic;
(3) f extends holomorphically to the cusps.[8]

We have maps

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbf{R}) & & \\ \downarrow & \searrow & \\ \mathrm{GL}_2(\mathbf{R})/A_G O_2(\mathbf{R}) = \mathfrak{H} & \longrightarrow & \mathbf{C} \end{array}$$

where here A_G is the collection of matrices $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ with $r > 0$.

Definition 7.11. Set $j(g,z) = \det(g)^{-1/2}(cz+d)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \in \mathrm{GL}_2(\mathbf{R})^+$. This is called an *automorphy factor*.

Set $\phi(g) = j(g,i)^{-k} f(gi)\colon\ \mathrm{GL}_2(\mathbf{R})^+ \to \mathbf{C}$ wher g acts on i by fractional linear transformation. Then one easily computes that

(1) $\phi(\gamma g) = \phi(g)$ for all $g \in \mathrm{GL}_2(\mathbf{R})^+$ and $\gamma \in \Gamma$;
(2) $\phi(g w_\theta) = e^{2\pi i k\theta}\phi(g)$ for $w_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

These immediately imply that ϕ is K_∞-finite, Γ-invariant and A_G-invariant. For the time being let $\mathfrak{g} = \mathfrak{gl}_2 \otimes_{\mathbf{R}} \mathbf{C}$ and $U(\mathfrak{g})$, $Z(\mathfrak{g})$ are the universal enveloping algebra and its center, as usual.

Fact. Can write $Z(\mathfrak{g}) = \langle \mathbf{C}\Delta, Z\rangle$ where $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the *Casimir operator* Δ is

$$\Delta = (1/4)(H^2 + 2XY + 2YX)$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(See section 2.5 of Bump). In terms of coordinates, if we write

$$g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

then

$$\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2) - iky\partial/\partial x.$$

One computes that $\Delta\phi = (1/4)(k^2-1)\phi$ for ϕ as above. So ϕ is annihilated by

$$J = \langle \Delta - (1/4)(k^2-1)\,\mathrm{id}\rangle.$$

[8] See [ref] for further explanation on this point.

Assume $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ now, for simplicity.

$$\begin{array}{ccc}
\mathrm{GL}_2(\mathbf{Q})A_G\backslash \mathrm{GL}_2(\mathbf{A_Q}) & & \\
\downarrow & \searrow^{\phi} & \\
\mathrm{GL}_2(\mathbf{Q})A_G\backslash \mathrm{GL}_2(\mathbf{A_Q})/\mathrm{SL}_2(\widehat{\mathbf{Z}}) & & \\
\downarrow & & \\
\Gamma A_G\backslash \mathrm{GL}_2(\mathbf{R}) & \xrightarrow{\quad f \quad} & \mathbf{C}.
\end{array}$$

The ϕ above is the automorphic form attached to f. It is of type $(\xi_k \otimes \xi_{\mathrm{SL}_2(\widehat{\mathbf{Z}})})$ where k is the reprsentation $\theta \mapsto e^{ik\theta}$ and f (?) is as above.

Now we ask: which $(\mathfrak{g}, K_\infty)$-module do we obtain?

Suppose $k \geq 2$. Then the discrete series D_k of weight k is representation π_K on the module

$$V = \bigoplus_{|l|\geq k, l\equiv k \pmod 2} \mathbf{C}v_l$$

where

(1) $\pi_K(w_\theta)v_l = e^{il\theta}v_l\pi_k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}v_l = v_{-l}$
(2) $\pi_k(X)v_l = (1/2)((k+l)v_{l+2})$
(3) $\pi_k(Y)v_l = (1/2)((k-l)v_{l-2})$
(4) $\pi_k(y)v_k = 0$, $\pi_k(X)v_{-k} = 0$,
(5) $\pi_k(\Delta) = \frac{k(k-2)}{4}$

There is another family of representations and for GL_2, these are all of the representations.

Principal series. Let $\lambda, \mu \in \mathbf{C}$ and suppose $\lambda \neq (k/2)(1-k/2)$ with $k \equiv \varepsilon \pmod 2$. Write $\lambda = s(1-s)$ and $s = (1/2)(s_1 - s_2 + 1)$ where $\mu = s_1 + s_2$. Then define

$$H^\infty = \left\{ f \in C^\infty(\mathrm{GL}_2(\mathbf{R})) \;\middle|\; \begin{array}{ll} f\left(\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} g\right) & = y_1^{s_1+1/2}y_2^{s_2+1/2}f(g) \text{ and} \\ f\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g\right) & = (-1)^\varepsilon f(g) \end{array} \right\}.$$

This is also known as the *induction from the central character* of type μ_1, μ_2. Here Δ acts by $-\lambda/4$, Z acts by μ. The k types are all the integers congruent to ε mod 2.

7.3. **Factorization theorem.** You'll often see people decompose an automorphic representation

$$\pi = \otimes_v \pi_v.$$

We're going to study what this means and why one has such a decomposition. Let $\{W_v \mid v \in \Xi\}$ be a family of vector spaces and let $\Xi_0 \subseteq \Xi$ be a finite subcollection. Think of these as the infinite places. For each $v \in \Xi - \Xi_0$ let $x_v \in W_v$. For $\Xi_0 \subseteq S \subseteq \Xi$ finite let $V_S = \otimes_{v\in S}W_v$ and if $S \subseteq S'$ then let

$$f_s \colon W_S \to W_{S'}$$

be defined by

$$\otimes_{v\in S}w_v \mapsto \otimes_{v\in S}w_v \otimes \bigotimes_{v\in S'-S} w_v.$$

Then set

$$W = \varinjlim_{S} W_S = \otimes' W_v.$$

That is, W is the collection of sequences $(w_v)_{v\in S} \subseteq \otimes_x W_v$ such that $w_v = x_v$ for almost all v.

Given $B_v\colon W_v \to W_v$ such that $B_v x_v = x_v$ for almost all $v \in \Xi$, then this gives a map $B = \otimes_v B_v\colon W \to W$ defined by

$$B(\otimes w_v) = \otimes B w_v$$

Given algebras $\{A_v, v \in \Xi\}$ and idempotents ρ_v in A_v for $v \notin \Xi$, then

$$A = {\bigotimes_v}' A_v$$

is an algebra with respect to the ρ_v.

If W_v is an A_v-module for all $v \in \Xi$ such that $\rho_v x_v = x_v$ for almost all v, then $\otimes'_{x_v} W_v$ is an A-module.

Remark 7.12. The isomorphism class of W (as an A-module, say) depends on the $\{X_v\}$ but if X_v and X'_v are proportional, the two resulting modules are isomorphic.

Example 7.13. The ring $\mathbf{C}[X_1, X_2, \ldots] = \otimes_{\rho_i} \mathbf{C}[X_i]$ with ρ_i the identity in $\mathbf{C}[X_i]$.

Example 7.14. One has a decomposition of the convolution algebra

$$C_c^\infty(G(\mathbf{A}_F^\infty)) = \otimes'_{\rho_v} C_c^\infty(G(F_v))$$

where $\rho_v = (1/\operatorname{Vol}(K))1_K$ where $K_v = G(\mathcal{O}_{F_v})$.

8. Lecture 8: Factorization of representations

Definition 8.1. We shall denote be $C_c^\infty(G(F_\nu) /\!/ K_\nu)$ the smooth compactly supported functions which are invariant on both the left and right under K_ν.

The basic example of this construction for us is $C_c^\infty(G(\mathbf{A}_F^\infty)) = \otimes'_{v\neq\infty} C_c^\infty(G(F_\nu))$ where we take $e_\nu = 1_K$, $vol(K_\nu) = 1$ and $K_\nu = G(O_\nu)$ the hyperspecial subgroups.

Assume that W_ν is an admissible $G(F_\nu)$-module for all $v \in \Xi$ and assume further that $\dim(W_\nu^{K_\nu}) = 1$ for almost all v. Let $W = \otimes_\nu W_\nu$ with respect to any choice of elements $e_\nu \in W_\nu$. Then the isomorphism class of W as a C_c^∞-module does not depend on this choice of e_ν.

Definition 8.2. We say that a C_c^∞-module W is *factorizeable* if we can write $W = \otimes'_\nu W_\nu$ where $\dim(W_\nu^{K_\nu}) = 1$ for almost all ν.

Note that in this case W is admissible and irreducible if and only if the W_ν are for all ν.

Theorem 8.3 (Flath). *Suppose $C_c^\infty(G(F_\nu) /\!/ K)$ is commutative for almost all ν then every admissible irreducible representation W of $C(\mathbf{A}_F^\infty)$ is factorizable. Morevore, the isomorphism class of the W_ν is determined by W and in particular $\dim(W_\nu^{K_\nu}) = 1$ for almost all ν.*

Definition 8.4. We say that an admissible representation W of $G(\mathbf{A}_F)$ is *factorizable* if $W \simeq W_\infty \otimes W^\infty$ where W^∞ is factorizable and W_∞ is an admissible $(\mathfrak{g}, K_\infty)$-module.

Corollary 8.5. *Every admissible representation W of $G(\mathbf{A}_F)$ is factorizable.*

(The extra input for the proof of this is that $C_c^\infty(G(F_\nu) /\!/ K_\nu)$ is commutative for almost every ν)

Definition 8.6. A vector $\phi_\nu \in W_\nu^{K_\nu}$ is said to be *spherical*.

$C_C^\infty(G(F_\nu) /\!/ K_\nu)$ is called the *spherical* or *unramified* Hecke Algebra provided G/F_ν is unramified (is quasi-split and splits over an unramified extension) and K_ν is hyperspecial.

Theorem 8.7. *Let G_1, G_2 be locally compact totally disconnected groups and let $G = G_1 \times G_2$ then:*

(1) *if π_i is an admissible irreducible representation of G_i then $\pi_1 \otimes \pi_2$ is an admissible irreducible representation of G.*
(2) *if π is an admissible irreducible representation of G then there exists π_i admissible irreducible representations of G_i such that $\pi \simeq \pi_1 \otimes \pi_2$ and moreover the isomorphism classes of the π_i are uniquely determined.*

Recall a representation is smooth if and only if the stabilizers of vectors are open and this is if and only if $V = \cup_K V^K$ for $K \subset G$ compact open.

Proposition 8.8 (Irreducibility Criterion)**.** *A smooth G-module W is irreducible if and only if W^K is an irreducible $C_c^\infty(G /\!/ K)$-module for all $K \subset G$ compact and open.*

Note that $C_c^\infty(G /\!/ K) = e_K C_c^\infty(G) e_K$ where $e_K = \frac{1}{vol(K)} 1_K$.

Proof of Irreduciblity Criterion. Suppose $W = W_1 \oplus W_2$ under C_c^∞. Then the same is true for some $K \subset G$ by smoothness. If $W^K = W_1^K \oplus W_2^K$ for some K then we have $C_c^\infty(G) W_1^K \neq W$ because $(C_c^\infty(G) W_1)^K = W_1^K$. ☐

Corollary 8.9. *Let $K \subset G$ be a compact open such that $C_c^\infty(G /\!/ K)$ is commutative and let W be an admissible G-module. If W is irreducible then $\dim(W^K) = 1$.*

Proof of 8.7. We have the following facts:
(1) $C_c^\infty(G_1 \times G_2) \simeq C_c^\infty(G_1) \times C_c^\infty(G_2)$
(2) $C_c^\infty(G_1 \times G_2 /\!/ K_1 \times K_2) \simeq C_c^\infty(G_1 /\!/ K_1) \times C_c^\infty(G_2 /\!/ K_2)$
(3) $(W_1 \otimes W_2)^{K_1 \times K_2} \simeq W_1^{K_1} \otimes W_2^{K_2}$

These claims would imply part (1) of the theorem.

Conversetly, let W be an admissible G-module. Choose $K = K_1 \times K_2$ such that $W^K \neq 0$ (this is possible by smoothness). Then since W^K is finite dimension there exists vector spaces W_i and an isomorphism of $C_c^\infty(G /\!/ K)$ modules $W^K \to W_1^{K_1} \otimes W_2^{K_2}$. From this we attain (via uniqueness of this decomposition) a compatible family of decompositions over all $K' \subset K$. We then let $W_i = \mathrm{indlim} W_i^{K_i}$ then by the irreducibility criterion W_i is irreducible and admissible.

(We leave as an exercise the proof that this decomposition is unique[9]) ☐

Proof of Flath's Theorem. The arguement proceeds as in the previous case. The additional input is the fact that $\dim(W_\nu^K) = 1$ for almost all v implies uniqueness of the product up to isomorphism. ☐

So, given an Automorphic representation π of $C(\mathbf{A}_F)$ we automatically obtain a family π_∞, π_ν where π_∞ is a $(\mathfrak{g}, K_\infty)$-module and $\pi^\infty = \otimes'_\nu \pi_\nu$ is a $G(\mathbf{A}_F)$-module with $\pi = \pi^\infty \otimes \pi_\infty$. ¿From this we see that to understand π it suffices to understand π_ν for all ν. In the case where ν is infinite this leads to $(\mathfrak{g}, K_\infty)$-modules. When ν is finite we have $\dim(\pi_\nu^{K_\nu}) = 1$ for almost all ν and so we desire to investigate representations of reductive groups over non-archimedian local fields with this property.

Definition 8.10. Let G be unramified over F_ν and $K_\nu \subset G(F_\nu)$ be hyperspecial. If $\dim(\pi_\nu^{K_\nu}) \neq 0$ then π_ν is said to be *ramified* or *spherical*.

We now return to the issue of showing that $C_c^\infty(G /\!/ K)$ is abelian in the cases that interest us.

Definition 8.11. Suppose $H \subset G$ is a closed algebraic subgroup then (G, H) is a *Gelfond pair* if for all irreducible admissible representations V of G we have that $\dim \mathrm{Hom}_H(V, \mathbf{C}) \cdot \dim \mathrm{Hom}_H(V^\vee, \mathbf{C}) \leq 1$.

[9] do it?

Proposition 8.12. *If $H \subset G$ is compact then (G, H) is a Gelfond pair if and only if $C_c^\infty(G /\!/ H)$ is commutative.*

Proof. From $V \to V^H$ we obtain:

$$\{\text{reps } V \text{ generated by } V^H\} \leftrightarrow \{\text{reps of } C_c^\infty(G /\!/ H)\}$$

It follows that $C_c^\infty(G /\!/ H)$ is Gelfond if and only if all representations have dimension 1. □

How to construct Gelfond pairs?

$D'(G) =$ linear dual of $C_c^\infty(G)$. We have that $G \times G$ acts on $D'(G)$ via its left and right actions on functions of $C_c^\infty(G)$.

Lemma 8.13 (Gelfond's Lemma)**.** *Assume there exists an involution I of G which stabilizes H and acts trivially on $D'(G)^{H\times H}$ then (G, H) is a Gelfond pair.*

This implies for example that with $H = \mathrm{GL}_n(\mathcal{O}_\nu)$ and $G = \mathrm{GL}_n(F_\nu)$ that by taking for I the matrix transpose we will have that (G, H) is a Gelfond pair.

9. Lecture 9: Unramified representations

(PRELIMINARY VERSION lacking completion and revision)

9.1. Gelfand's lemma.

Recall that if $K \subset G = G(F_v)$ is hyperspecial, then $C_c^\infty(G /\!/ K)$ is commutative.

Proposition 9.1. *If $K \subset G$ is a compact open subgroup, then (G, K) is a Gelfand pair if and only if the algebra $C_c^\infty(G /\!/ K)$ is commutative.*

Proof. Indeed, the assignment $V \mapsto V^K$ yields an equivalence of categories between representations V of G generated by V^K and representations of $C_c^\infty(G /\!/ K)$. But $C_c^\infty(G /\!/ K)$ is commutative if and only if all its irreducible representations are one-dimensional. The claim follows. □

Denote by $D'(G)$ the linear dual of $C_c^\infty(G)$. The group G acts on $D'(G)$ by left and right multiplication, so $G \times G$ acts on $D'(G)$.

Proposition 9.2 (Gelfand's Lemma)**.** *Assume there exists an anti-involution ι of G which stabilizes K and fixes $D'(G)^{K\times K}$. Then (G, K) is a Gelfand pair.*

Proof. Let V be an irreducible representation of G, and let $\ell : V \to \mathbf{C}$ and $m : V^\wedge \to \mathbf{C}$ be nonzero invariant linear forms. Define linear maps $F_\ell : C_c^\infty(G) \to V^\wedge$ and $F_m : C_c^\infty(G) \to V$ by $F_\ell(f)(v) = \int_G f(g)\ell(gv)\mathrm{d}g$ and $F_m(f)(v^\wedge) = \int_G f(g)m(gv^\wedge)\mathrm{d}g$, respectively. Since $V^\wedge$ and V are irreducible, by Schur's lemma these maps are determined up to scaling by their kernels. We consider the composite map:

$$B : C_c^\infty(G) \times C_c^\infty(G) \xrightarrow{F_m \otimes F_\ell} V \times V^\wedge \xrightarrow{\langle\,,\,\rangle} \mathbf{C},$$

where $\langle\ ,\ \rangle$ is a G-invariant pairing. Note that $B(f_1, f_2) = m(F_\ell(f_1 * f_2))$. Then B can be viewed as a distribution on $G \times G$ right invariant under $K \times K$ and left invariant under G (embedded diagonally). For $f \in C_c^\infty(G)$, define $\tilde{f} := f(\iota(g^{-1}))$. Since $f \mapsto m(F_\ell(f))$ is bi-K-invariant, it is fixed by ι. Thus $m(F_\ell(f)) = m(F_\ell(\tilde{f}))$. We now take $f = f_1 * f_2$, and we have $\tilde{f} = \tilde{f}_1 * \tilde{f}_2$, since ι is an involution. Thus we see that $B(f_1, f_2) = B(\tilde{f}_2, \tilde{f}_1)$, so the left kernel of B determines the right kernel of B. Hence m determines $\ker(F_\ell)$, and therefore determines ℓ up to scaling. But since m was arbitrary, we must have $\dim_K \mathrm{Hom}(V, \mathbf{C}) \leq 1$. And similarly we see that $\dim_K \mathrm{Hom}(V^\wedge, \mathbf{C}) \leq 1$. □

Remark 9.3. Determining when a compact open subgroup gives a Gelfand pair is a problem of interest, given the recurrent strategy of studying representations of G via the study of suitable restrictions. Anyway, when $K \subset G$ is hyperspecial, $C_c^\infty(G \,/\!/\, K)$ is commutative.

9.2. **Unramified representations.**

Definition 9.4. An admissible irreducible representation (π_v, V_{π_v}) of $G(F_v)$ is called *unramified* if G_{F_v} is quasi-split or split over an unramified extension, and $V_{\pi_v}^K \neq 0$, where $K \subset G(F_v)$ is a hyperspecial subgroup (or said another way, V_{π_v} contains a non-zero vector fixed by a hyperspecial subgroup).

Remark 9.5. When G_{F_v} is quasi-split or split over an unramified extension, we also say that it is *unramified*. If G is defined over a global field F, then G_{F_v} is unramified for almost every v (cf. [56]).

There is a nice way to parametrize (irreducible admissible) unramified representations. This is done via the so-called unramified *L-parameters*, which naturally leads to the local Langlands conjecture, which will occupy us for the next few lectures.

For now, assume G_{F_v} is a connected reductive group over a nonarchimedean local field, $P \subset G$ is a parabolic subgroup, $M \subset P$ is its Levi subgroup, and N its unipotent radical, so that we have $P = MN$. Let (σ, V_σ) be a smooth irreducible representation of $M(F_v)$.

The *induced representation* $(\mathrm{ind}_P^G \sigma, V)$ is the smooth representation of $G(F_v)$ on the space $V = \{f : G(F_v) \to V_\sigma \mid f(mng) = \sigma(m) f(g) \text{ for all } g \in G(F_v), m \in M(F_v), n \in N(F_v)\}$. The action of the group $G(F_v)$ is given by right translation: $(\mathrm{ind}_P^G \sigma(g)) f(g_1) = f(g_1 g)$. The formation of induction from a parabolic gives a functor

$$\mathrm{ind}_P^G : \mathrm{SmRep} M(F_v) \rightsquigarrow \mathrm{SmRep} G(F_v),$$

which is also called *Jacquet functor* in some literatures. But we note that as defined, it does not necessarily preserve unitaricity.

Let A be a maximal split torus in G, $M = C_A(G)$ its centralizer in G, and P a minimal parabolic containing M. We have a map $\mathrm{ord}_M : M(F_v) \to X_*(M)$ with the defining property that $\langle \mathrm{ord}_M(m), \chi \rangle = v(\chi(m))$. Denote by $\Lambda(M) \subset X_*(M)$ the image of ord_M and let $M(F_v)^o$ be the group fitting into the exact sequence:

$$1 \to M(F_v)^o \to M(F_v) \to \Lambda(M) \to 1. \tag{9.2.0.1}$$

(Note that by definition, $m \in M(F_v)^o$ if and only if $\lambda(m) \in \mathcal{O}_{F_v}^\times$ for all $\lambda \in X^*(M)$.)

Remark 9.6. One can identify the $M(F_v)^o$ with the $\mathcal{O}_{F_v}$-valued points of some scheme.

Definition 9.7. A quasi-character $\chi : M(F_v) \to \mathbf{C}$ is said to be *unramified* if $\chi|_{M(F_v)^o}$ is trivial.

Definition 9.8 (Unramified principal series)**.** Consider the modular character $\delta : M(F_v) \to \mathbf{C}$ given by $\delta(m) = |\det(\mathrm{Ad}(m))|_v$ for $m \in M(F_v)$. Let χ be an unramified character of $G(F_v)$. The *unramified principal series* if the pair $(I(\chi), V_\chi)$ with V_χ the space consisting of the locally constant functions $f : G(F_v) \to \mathbf{C}$ such that $f(mng) = \delta(m)^{1/2} \chi(m) f(g)$ for all $m \in M(F_v), n \in N(F_v), g \in G(F_v)$, and the action $I(\chi)$ of $G(F_v)$ on V_χ given by $I(\chi)(g).f(x) = f(xg)$ for $f \in V_\chi$, $g, x \in G(F_v)$.

Remark 9.9. As defined, the unramified principal series will *not* necessarily be irreducible.

Theorem 9.10.
- *For χ unramified, $I(\chi)$ is admissible.*
- *For χ unitary, $I(\chi)$ is pre-unitary (i.e. V_χ can be endowed with an hermitian metric so that the action of $G(F_v)$ is given by isometries).*

Proof. Consider the *Iwasawa decomposition* $G(F_v) = KP(F_v)$, where $K = G(\mathcal{O}_{F_v})$. (Note that this has not been discussed in these lectures so far.) Since $G(F_v)/P(F_v)$ is compact, the first point follows. For the second, define a surjective linear map $P_\chi : C_c^\infty(G(F_v)) \to V_\chi$ given by $f \mapsto \int_{M(F_v)} \int_{N(F_v)} \delta(m)^{1/2}(m)\chi^{-1}(m)f(mg)\mathrm{d}m\mathrm{d}n$. It intertwines the action on $C_c^\infty(G(F_v))$ by right translation with the action of $I(\chi)$. Define a linear map of $V\delta^{1/2}$ by $J(P\delta^{1/2}f) = \int_{G(F_v)} f(g)\mathrm{d}g$. For $f \in V_\chi$ with χ unitary, $\bar{f}_1, f_1 \in I(\delta^{1/2})$. Defining $\langle f_1, f_2\rangle := J(\bar{f}_1 f_2)$, we see that V_χ becomes unitary. □

Consider now the following concrete instance of the above construction. Let $\chi_1, \chi_2 : \mathbf{Q}_p^\times \to \mathbf{C}^\times$ be characters (non-necessarily unitary). Then $I(\chi_1, \chi_2)$ is the space of locally constant functions $f : \mathrm{GL}_2(\mathbf{Q}_p) \to \mathbf{C}$ with $f(\left(\begin{smallmatrix} a_1 & * \\ & a_2 \end{smallmatrix}\right) g) = \chi_1(a_1)\chi_2(a_2)\frac{|a_1|_v^{1/2}}{|a_2|_v^{1/2}} f(g)$. Note that χ_1, χ_2 are unramified if and only if they restrict trivially to $\mathbf{Z}_p^\times$, in which case they are completely determined by the value at a uniformizer ϖ, thus by the complex number s_i such that $\chi_i(\varpi) = q^{s_i}$ $(i = 1, 2)$. This connects with the theme of *Satake parameters*, a topic that will be discussed in a future lecture.

Consider $W = N(A)(F_v)/M(F_v)$. It acts on $X_*(M)$ and leaves $X_*(A) \subset X_*(M)$ and $\Lambda(M)$ invariant. For $w \in W$, define the character χ^w by $\chi^w(m) := \chi(x_w^{-1} m x_w)$, where x_w represents w in $V(A)$. We say that χ is *regular* if $\chi^w = \chi$ only when $w = 1$. The proof of this theorem is postponed to a future lecture.

Theorem 9.11. *Let $\chi = (\chi_1, \chi_2)$ be a pair of characters of $F_v^\times$. Then the following hold:*

1. *$I(\chi)$ is irreducible if and only if $\chi_1\chi_2^{-1} \neq |\ |_v^{\pm}$.*
2. *$I(\chi)$ and $I(\chi^w)$ are isomorphic for all $w \in W$, and these account for all the possible isomorphisms among the $I(\chi)$.*
3. *Every unramified representation is isomorphic to a unique subquotient of a unique $I(\chi)$.*

Remark 9.12. We note that the third point in the above result is a general phenomenon in the local theory: every unramified representation appears as a unique subquotient of an induced representation.

Recall the exact sequence (9.2.0.1), and consider the complex torus $\hat{T} := \mathrm{Spec}(\mathbf{C}(\Lambda(M)))$. We have $\Lambda(M) = X^*(\hat{T})$ by definition, and $\hat{T}(\mathbf{C}) = \mathrm{Hom}(\Lambda(M), \mathbf{C})$. Let X^o be the group of unramified characters of M. We have a morphism $X^o \to \hat{T}(\mathbf{C})$ given by $\chi_t \mapsto t$. Let $t \in \hat{T}(\mathbf{C})$ and $\varphi \in X_*(M) = X^*(\hat{T})$. Note that $\chi_t(\varphi(\varpi)) = \varphi(t)$. (For example for GL_1, $\hat{T}(\mathbf{C}) = \mathbf{C}^\times$ and $\chi_c(\varpi) = |\varpi|^s$.) The torus $\hat{T}$ is a maximal torus in $\hat{G}$, the dual reductive group of G (assuming G split for simplicity, since only under this assumption the notion of dual has been discussed). The elements $t \in \hat{T}(\mathbf{C})$ define semi-simple conjugacy classes in $\hat{G}(\mathbf{C})$. We thus obtain a bijection:

$$\left\{\begin{array}{c}\text{semi-simple} \\ \text{conjugacy classes} \\ \text{in } \hat{G}(\mathbf{C})\end{array}\right\} \xleftarrow{1:1} \left\{\begin{array}{c}\text{isomorphism classes} \\ \text{of unramified representations} \\ \text{of } G(F_v)\end{array}\right\}.$$

10. Lecture 10: Statement of the Langlands conjectures and functoriality

Today we plan to state the Langland's conjectures in a reasonably precise manner.

(Here is a problem Jayce stated that he needs for a paper: **Problem:** Let $H \subseteq A_n$ for $n \geq 5$ be a maximal solvable subgroup. Prove that there exists $\tau \in A_n$ with $|\tau|$ coprime to 6, such that A_n is generated by H and τ. Same thing for exceptional simple groups?)

Now back to the course.

Let G be a connected reductive group which is unramified over a local field F_v. Then last time we set up a bijection

$$\{\text{semisimple conjugacy class in } \widehat{G}\} \longleftrightarrow \{\text{isom. classes of unramified reps of } G(F_v)\}.$$

10.1. **Weil group.** Recall that if if we have a continuous homomorphism $G_F \to \mathrm{GL}_n(\mathbf{C})$, where G_F is the absolute Galois group of a global field F, then it necessarily has finite image. On the other hand, there are many continuous homomorphisms $G_F \to \mathrm{GL}_n(\mathbf{Q}_l)$ with infinite image. For example, if E is an elliptic curve without CM, then the Tate module gives such a representation for almost every l.

In order to sidestep these limitations for global fields, Weil introduced the following notion.

Definition 10.1. Let F be a local or global field. Then a *Weil group* for F is a tuple (W_F, ϕ, Γ_E), where E/F is finite, W_F is a group,

$$\phi\colon W_F \to \mathrm{Gal}(\overline{F}/F)$$

is a homomorphism with dense image. We require this to satisfy certain properties: For E/F finite, let $W_E = \phi^{-1}(\mathrm{Gal}(\overline{F}/E))$. Then ϕ induces a bijection

$$W_F/W_E \to \mathrm{Gal}(\overline{F}/F)/\,\mathrm{Gal}(\overline{F}/E) \cong \mathrm{Hom}(E, \overline{F}).$$

This is a homomorphism if E/F is normal.

If F is local put $C_E = E^\times$ and if global then put $C_E = E^\times\backslash\mathbf{A}_E^\times$. Then (there exist?) maps

$$r_E\colon C_E \to W_E^{ab}$$

subject to the compatibility conditions

$$C_E \overset{r_E}{\to} W_E^{ab} \overset{\phi}{\to} \mathrm{Gal}(\overline{F}/E)^{ab}$$

is the reciprocity map of class field theory and for $w \in W_F$, $\sigma = \phi(w) \in G_F$, for every E the diagram

$$\begin{array}{ccc} C_E & \xrightarrow{r_E} & W_E^{ab} \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ C_{E^\sigma} & \xrightarrow[r_{E^\sigma}]{} & W_{E^\sigma}^{ab} \end{array}$$

commutes. Thidly for $E' \subseteq E$

$$\begin{array}{ccc} C_{E'} & \longrightarrow & W_{E'}^{ab} \\ \downarrow & & \downarrow \\ C_E & \xrightarrow[r_{E'}]{} & W_E^{ab} \end{array}$$

commutes. Fourthly the map

$$W_F \to \varprojlim W_{E/F}$$

is an isomorphism, where $W_{E/F} = W_F/W_E^c$.

This definition is taken from Tate's article in the Corvalis proceedings. If the Weil group exists, it is unique up to isomorphism. It is cooked up so that Hecke characters correspond with characters of W_F.

Example 10.2. Let F be a local field and for all finite extensions E/F let k_E be the residue field of E and q_E the cardinality of k_E. Put $\overline{k} = \bigcup_E k_E$. Then in this case W_F is the dense subgroup of $\mathrm{Gal}(\overline{F}/F$ generated by the $\sigma \in \mathrm{Gal}(\overline{F}/F)$ such that on $\overline{k}$, σ acts as $x \mapsto x^{q_E^n}$ for some $n \in \mathbf{Z}$. Then $r_E(a)$ acts as $x \mapsto x^{|a|}$ on $\overline{k}$.

Example 10.3. For $F = \mathbf{C}$, then $W_F = \mathbf{C}^\times$, ϕ is the trivial map and $r_F = \mathrm{id}$.

Example 10.4. For $F = \mathbf{R}$, then $W_F = \overline{F}^\times \cup j\overline{F}^\times$ where $j^2 = -1$ and $jcj^{-1} = \overline{c}$. Here ϕ takes $\overline{F}^\times$ to 1 and $j\overline{F}^\times$ to the nontrivial element of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$.

For global fields don't have a nice intrinsic description like the above two examples. Unfortunately the Weil group is not big enough. For example, if an elliptic curve has semistable reduction at a prime, then the corresponding Galois representation is not accounted for by the Weil group. This motivates the following

Definition 10.5. The *Weil-Deligne group* W_F' is the group scheme over F defined as follows: by definition $W_F = \varprojlim_{J \text{ closed subgroup}} W_F/J$ is a projective limit of discrete groups. Then W_F' is an extension of W_F by the additive group $\mathbf{G}_a$. Let $\mathbf{G}_a$ act by

$$wxw^{-1} = \|w\|x,$$

that is, we have an exact sequence

$$1 \to \mathbf{G}_a \to W_F' \to W_F \to 1$$

of group scheme over F. Concretely, the points of W_F' over some field E/F are given by

$$W_F'(E) = \{(a, w) \in E \times W_F\}$$

and for two points we have $(a_1, w_1)(a_2, w_2) = (a_1 + \|w_1\|a_2, w_1w_2)$.

Remark 10.6. What is the norm appearing above? We always have a norm map

$$\|\cdot\| \colon C_E \to \mathbf{C}^\times,$$

so we obtain one on W_E via the identification $W_E^{ab} \cong C_E^{ab}$.

A *representation* of W_E' is a map $\phi\colon W_F' \times_F E \to \mathrm{GL}(V)$ with V and E-vector space. Suppose now that F is a number field

Definition 10.7. Let E be a field of characteristic zero. A *representation* of W_F' over E is a pair (ρ, N) such that $\rho\colon W_F \to \mathrm{GL}(V)$ is a homomorphism whose kernel is contained in an open subgroup of the inertia group, plus a nilpotent endomorphism N of V such that

$$\rho(w)N\rho(w)^{-1} = \|w\|N$$

for $w \in W_F$.

Example 10.8. For GL_2, take ρ to be the trivial representation and $N = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$; this corresponds to the semistable reduction case of an elliptic curve.

10.2. **Semisimplification.** Let $\rho' = (\rho, N)$ be a representation of W_F over E and let $v\colon W_F \to \mathbf{Z}$ be the valuation induced by our norm, where $\|w\| = q^{-v(w)}$ defines v. Then there exists a unique unipotent automorphism u of V such that u commutes with N and $\rho(W_F)$, such that $\exp(aN)\rho(w)u^{-v(w)}$ is a semisimple automorphism of V for all $x \in E$ and $w \in W_F$.

Definition 10.9. Then $\rho_{ss}' = (\rho u^{-v}, N)$ is the Frobenius semisimplification (Tate calls it the ϕ-semisimplification in his Corvallis article). We say that ρ' is *Frobenius semisimple* if and only if $\rho' = \rho_{ss}'$.

10.3. **Local Langlands conjecture.** This conjecture states the following:

Conjecture 10.10 (Local Langlands)**.** *There existsa bijection between Frobenius semisimple representations of W_F of rank n and irreducible admissible representations of* $\mathrm{GL}_n(F)$.

Even when $N = 1$ this is a nontrivial statement. This was proved by Harris-Taylor first, then a simplified proof was found by Henniart.

Let F be local nonarchimedean.

Definition 10.11. A map $\rho\colon W_F' \to \widehat{G}$ is said to be *unramified* if $\rho|_I$ is trivial, where I is the inertia group, and $\rho|_{\mathbf{G}_a}$ is also trivial.

In this case $\rho\colon W_F' \to \widehat{\mathbf{G}}$ is determined by $\rho(\mathrm{Frob}_v)$. Assuming that G is split, we get a bijection between isomorphism classes of semisimple unramified representations $\rho\colon W_F' \to \widehat{G}$ and unramified admissible representations of G. (This is an important point; one should understand this well to understand the modern theory).

10.4. **Langlands functoriality.** Recall that to a connected reductive algebraic group G over $\mathbf{C}$, we associated a root datum $\Psi(G,T) = (X^*(T), \Phi, X_*(T), \Phi^\vee)$. We choose a basis Δ of Φ and obtain a dual basis $\Delta^\vee \subseteq \Phi^\vee$, which is equivalent to the choice of a Borel subgroup containing T. Why? For each $\alpha \in \Phi$, there exists a unique homomorphism $\exp_\alpha\colon \mathfrak{g}_\alpha \to G$ such that $t\exp_\alpha(x)t^{-1} = \exp(\alpha(t)x)$ and $\mathrm{Lie}(\exp_\alpha) = (\mathfrak{g}_\alpha \to \mathfrak{g})$.

For example if $G = \mathrm{GL}_n$ and $\alpha - (\alpha_{ij})$ then

$$\exp_\alpha(x) = \sum_{n\geq 0} \frac{(xE_{ij})^n}{n!}$$

Put $U_\alpha = \mathrm{im}(?)$. Then $B = (T, U_\alpha)$ for some choice of basis and conversely, given a Borel there exists a unique basis such that this is true.

Put $\Psi_0(G,T) = (X^*, \Delta, X_*(T), \Delta^\vee)$, then there exists an exact sequence

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(\Psi_0(G,T)).$$

Assume that F is local or global. Given $\sigma \in \mathrm{Gal}(\overline{F}/F)$, there exists $g \in G(\overline{F})$ such that

$$gT^\sigma g^{-1} = T, \quad gB^\sigma g^{-1} = B$$

so that σ induces an automorphism of $\Psi_0(G,T)$.

Let $u_G\colon\ \mathrm{Gal}(\overline{F}/F) \to \Psi_0(G,T)$ and note that

$$\mathrm{Aut}(\Psi_0(G,T)) \cong \mathrm{Aut}(\Psi_0(\widehat{G}, \widehat{T})).$$

Choose a section

$$\mathrm{Aut}(\Phi_0^\vee) \to \mathrm{Aut}(\widehat{G})$$

(via a choice of pinning of (G,T,B)[10]). Then we obtain a map

$$\mu\colon\ \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}(\widehat{G}).$$

Definition 10.12. Put ${}^LG_F = \widehat{G} \times\!\| \,\mathrm{Gal}(\overline{F}/F)$ where the semidirect product is defined with respec to this action of the Galois group.

Definition 10.13. If G_1 and G_2 are two F-groups, then an *L-map*

$${}^LG_1 \to {}^LG_2$$

is a homomorphism such that it commutes with the maps to W_F, and is it complex analytic on $\widehat{G}_1 \to \widehat{G}_2$.

[10]Add more details

Given ${}^LG_{F_v} \to {}^LG_F$, then we can compose with the map $\phi\colon W_{F_v} \to {}^LG_{F_v}$ any map $f\colon {}^LG_1 \to {}^LG_2$ to obtain another map?[11]

Langlands functoriality. There exists a partition of the automorphic representations of $G(\mathbf{A}_F)$ into disjoint sets Π, called *L-packes*, such that the following is true:

Let $f\colon {}^LG_1 \to {}^LG_2$ be an L-map. Then f induces a map of L-packets such that if $\pi_1 \in \Pi$ and $\pi_2 \in \Pi$ and $\pi_1 = \pi_1(\phi)$ for some unramified representation, then $\pi_2 = \pi(f \circ \phi)$ (Note well that one needs to assume Local Langlands for G_1 and G_2 holds).

11. Lecture 11: L-Functions and Classification of Spherical Representations

11.1. L-functions.

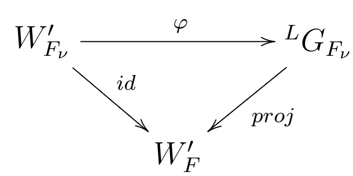
The primary reference for this section is Borel's article in Corvallis [5].

As usual let G_F be a reductive group over a global field. In this section we will typically fix a place ν of F and work with G_{F_ν}. We will in general require that G_F is quasi-split however it would generally suffice to have G_{F_ν} quasi-split.

Definition 11.1. Consider a map:

$$\varphi : W'_{F_\nu} \mapsto {}^LG_{F_\nu}$$

Which is a continuous homorphism where the following diagram is commutative:



Where we moreover requre that $\varphi(w)$ is semi-simple for all $w \in W'_{F_\nu}$ Such a map is known as an *L-map*.

We now wish to construct L-functions associated to such φ. Let V be an n-dimensional $\mathbf{C}$-vector space. Let $r : {}^LG_{F_\nu} \to \mathrm{GL}(V)$ be a continous complex-analytic homomorphism. We wish to associate to the composition $r \circ \varphi : W'_{F_\nu} \to \mathrm{GL}(V)$ an L-function.

Suppose $r \circ \phi$ is defined by (ϕ, N) then set $V_N^I = (ker(N))^I$ (where $I \subset W'_{F_\nu}$ is innertia.) We then define the local L-function to be:

$$L(s, r \circ \varphi) = det(1 - \mathrm{Frob}_v\, q^s|_{V_N^I})^{-1}$$

We now wish to define the associated ε factors. Suppose we are in the case of $n = 1$ then we have that $\exists!\varepsilon(s, \chi, \phi, dx) \in \mathbf{C}$ which depends on the Harr measure dx of F_ν^* the character $\psi : F_\nu^* \to \mathbf{C}$ and has the property that for all $f \in C^\infty(F_\nu^*)$ we have:

$$\int \frac{\hat{f}(x)\,|x|_\nu^{1+s}\,\chi^{-1}(x)}{L(1-s,\chi^{-1})}\,\frac{dx}{|x|_\nu} = \varepsilon(s,\chi,\phi,dx)\int \frac{f(x)\,|x|_\nu^{s}\,\chi(x)}{L(s,\chi)}\,\frac{dx}{|x|_\nu}$$

[12] More generally (for arbitrary n) we have only an existance result by Delign and Langlands [13] that is inductive which is to say that $\varepsilon(s, Ind_\sigma) = \varepsilon(s, \sigma)$.

Remark 11.2. We remark further that it the L-function construction is also independant of induction.

[11] What should this say?

[12] check "Fourier Analysis on Number Fields" for correct statement

[13] Reference?

Suppose now that under the Langlands correspondance we have that:

$$\mathrm{L-packet}\{\pi_\nu\} \leftrightarrow \mathrm{L-parameter}\varphi : \mathrm{W}'_{\mathrm{F}_\nu} \to {}^{\mathrm{L}}\mathrm{G}_{\mathrm{F}_\nu}$$

Then we define $L(s, \pi_\nu, r) = L(s, r \circ \varphi)$ and $\varepsilon(s, \pi_\nu, r, \psi, dx) = \varepsilon(s, r \circ \varphi, \psi, dx)$ for each $\pi_\nu \in \pi_\nu$.

An important part of the Langlands program is to attempt to define L-functions and ε factors representation theoretically. One would then want to show that those constructions give the same results as the above. There are two key approaches to this they are:

- Rankin-Selberg Theory : Jacquet, Praletski, Shapiro, Shalika
  For a reference see Cogdell [14]
- Eisenstein Series Method : Langlands, Shahidi
  For a reference see Kim, Shahidi [15]

Having constructed local L-functions, we now wish to package these into a global L-function.

If we can factor a global representation $\pi = \bigotimes_\nu \pi_n u$ then we define: $L(s, \pi, r) = \prod_\nu L(s, \pi_\nu, r)$ and $\varepsilon(s, \pi, r) = \prod_\nu \varepsilon(s, \pi_\nu, r, dx, \psi)$. Where we normalize dx so that $dx(F\backslash \mathbf{A}_F) = 1$ and ψ trivial on F^*.

Conjecture 11.3.
- *$L(s, \pi, r)$ is meromorphic as a function of s.*
- *$L(s, \pi, r) = \varepsilon(s, \pi, r)L(1 - s, \pi^\vee, r^\vee)$.*

These conjectures are known only in limited cases such as for $G = \mathrm{GL}_n \times \mathrm{GL}_m$ in the particular case where r is the natural map to GL_{nm}. In this case we define $L(s, \pi_1 \times \pi_2) = L(s, \pi_1 \times \pi_2, r)$ and we have the result that this is holomorphic in s unless $\pi_1 \simeq \pi_2^\vee$ in which case we have a simple pole of order 1 at $s = 1$ [16]

Example 11.4. The unramified case of $G = \mathrm{GL}_n$.

The map $\varphi_\nu : W_{F_\nu} \to {}^L\mathrm{GL}_n$ an unramified L-parameter gives us the Langlands class:

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

The (a_i) are called the *Satake parameters* and correspond to the characters $w_i \mapsto |w_i|^{a_i}$. Supposing this is regular we then have that:

$$L(s, \varphi_\nu) = \prod_i (1 - q_\nu^{-a_i - s})^{-1}$$

$\pi_v(\varphi_v) = Ind(X)$ where $X : T(F_\nu) \to \mathbf{C}^*$ is given by $X(f_1, \ldots, f_n) = \prod_i |f_i|_\nu^{a_i}$.

What then happens for the case $L(s, I(X_1) \times I(X_2))$ where $X_1 \leftrightarrow (a_1, \ldots, a_m)$ and $X_2 \leftrightarrow (b_1, \ldots, b_n)$? We have that:

$$L(s, I(X_1) \times (X_2)) = \prod_j \prod_i (1 - q_\nu^{-s - a_i - b_j})^{-1}$$

Theorem 11.5 (JS). *If π_1, π_2 are cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ such that $\pi_{1\nu} \simeq \pi_{2\nu}$ for almost all ν then $\pi_1 \simeq \pi_2$.*

(This is strong multiplicity 1 which is an analog of Chebatorov density theorem.)

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[14] References?

[15] References?

[16] References? RS-CS-Cogdell

11.2. **Classifying Spherical Representations.** The reference for this section is Laumon's book "Cohomology of Drinfeld Modular Varieties" [41].

The major goal os this section is the following theorem:

Theorem 11.6. *Let π_ν be a shpherical representation of $G(F_\nu)$ then $\exists!$ unramified principle series representation $I(X)$ such that π_ν is a subquotient of $I(X)$. Moreover, each principle series has a unique spherical subquotient.*

We continue to have G_{F_ν} a quasi-split group with hyperspecial subroup K_ν.

For simplicity we will be working concretely with $G = \mathrm{GL}_n$ though most of the statements hold in an analogous way more generally.

Let $B \subset G$ be the standard Borel subgroup of upper triangular matricies. Let $T \subset B$ be the standard diagonal torus. Let $\Delta = \{\varepsilon_i - \varepsilon_{i+1} | 1 \leq i \leq n-1\}$ be the set of simple roots of T.

There exists a correspondance:

$$\{I \subset \Delta\} \leftrightarrow \{P | B \subset P \text{ Pparabolic}\}$$

Where to I we associate the parabolic whose lie algebra contains the simple roots of I. This correspondance can be constructed explicitly by viewing I as a partition of n and using the correspondance between partitions and parabolics. We shall denote by P_I the parabolic associated to a subset $I \subset \Delta$

Let $P_I = M_I N_I$ be the Levi-decomposition, so that M_I is Levi and N_I is unipotent. Let $W_I \subset W \simeq S_n$ be the subgroup generated by the $(i, i+1)$ where $\varepsilon_i + \varepsilon_{i+1} \in I$ that is W_I corresponds to the subgroup of the Weil group for T generated by P_I.

Example 11.7. We have that:

- $W_\emptyset = \{1\}$, $d_\emptyset = (1, \ldots, 1)$ and $P_\emptyset = B$.
- $W_\Delta = W$, $d_\Delta = (d)$ and $P_\Delta = G$.

We shall denote by $Z_I = Z_{M_I}$ the center of M_I. Let $P_J^I = P_J \cap M_I$ be the piece of the parabolic P_J inside P_I.

Claim 11.8. *Let P be a parabolic subgroup of M_I then its associated levi subgroup is M_J for some $J \subset I$ and if $P = M_J N$ we have:*

1. $N(F_\nu) \subset P(F_\nu) \subset M_I(F_\nu)$
2. $N(F_\nu) \triangleleft P(F_\nu)$
3. *Any compact subset of $N(F_\nu)$ is contained in a compact subgroup and thus $N(F_\nu)$ is unimodular.*
4. $M_I(F_\nu) = P(F_\nu) M_I(\mathcal{O}_{F_\nu})$
5. *dn is the Haar measure on N then $dpnp^{-1} = \delta_p(p)dn$ where $\delta_p : P(F_\nu) \to \mathbf{R}_>$ is a modular character.*

We will denote by $Rep_s(M_I) = \mathrm{Category of smooth reps of M_I(F_\nu)}$. We shall denote by $Rep_a(M_I)$ the subcategory of admissible representations.

Where M_J, M_I, P are as above we shall use the notation:

$$Ind_{M_J,P}^{M_I} : Rep_s(M_J) \to Rep_s(M_I)$$

for induction and

$$Res_{M_I}^{M_J,P} : Rep_s(M_I) \to Rep_s(M_J)$$

for restriction.

More concretely:

$$Ind(\sigma) = \text{`locally constant functions'} \phi : \mathrm{M_I(F_\nu)} \to \mathrm{W such that} \phi(\mathrm{m_I n m_I}) = \delta_{\mathrm{p}}^{1/2}(\mathrm{m_J})\sigma(\mathrm{m_J})\phi(\mathrm{m_I})$$

and

$$Res(\pi) = V/V(N) \mathrm{where V(N)} = \rangle \pi(\mathrm{n})\mathrm{v} - \mathrm{v} | \mathrm{v} \in \mathrm{Vn} \in \mathrm{N}(\mathrm{F}_\nu) \langle$$

The action is $\delta_p^{-1/2} \otimes \pi|_P : P(F_\nu) \to \mathrm{Aut}(V)$.

We will write $Ind_J^I = Ind_{M_J, P_I^J}^{M_I}$ and $Res_J^I = Res_{M_I}^{M_J, P_I^J}$.

Definition 11.9. The *contragradient* $\vee : Rep_s(M_I) \to Rep_s(M_I)$ is the functor which takes $V^\vee = \mathrm{smooth functions in}\,\mathrm{Hom(V}, \mathbf{C})$ where the action is $\pi(m)\tilde{v} = \tilde{v}(\pi^{-1}(m))$.

Proposition 11.10.
- *Induction and Restriction are exact.*
- *Res_J^I is left adjoing to Ind_J^I.*
- *Induction commutes with $\vee$.*
- *Induction maps the subcategory of admissible to admissible.*
- *For $K \subset J \subset I$ we have $Ind_K^I = Ind_J^I \circ Ind_K^J$ and $Res_I^K = Res_J^K \circ Res_I^J$.*

Theorem 11.11 (Jacquet)**.** *Restriction maps admissible to admissible.*

For a parabolic P denote by $\tilde{P}$ the opposite parabolic (for GL_n that is the transpose of the group).

Theorem 11.12 (Casselman)**.** *For each $\pi \in Rep_s(M_J)$ there is a canonical isomorphism:*

$$Res_{M_I}^{M_J, \tilde{P}}(\pi^\vee) \simeq (Res_I^J)(\pi)^\vee$$

We now introduce some further notation that we shall need later:
$Z_J^I(1) = \{z_j \in Z_J(F_\nu) | \, |\alpha(z_j)| \leq 1 \; \forall \alpha \in I \setminus J\}$
Note that: $Z_J^I(1) = Z_J(F_\nu) \cap Z_\emptyset^I(1)$.
For $\epsilon \in (0,1]$ set $Z_{\emptyset,J}^I(\epsilon) = \{z \in Z_\emptyset^I(1) | \, |\alpha(z)| \leq \epsilon \; \forall \alpha \in I - J\}$
$Z_J^I(\epsilon) = Z_J(F_\nu) \cap Z_{\emptyset,J}^I(\epsilon)$
$M_I(F_\nu)^1 = \{m = (g_1, \ldots, g_s) \in M_I(F) | \, |det(g_i)| = 1 \; \forall j\}$

Definition 11.13. A *super cuspidal* (resp *quasi-cuspidal* representation of $M_I(F_\nu)$ is an admissible (resp smooth) representation such that all matrix coefficents are compactly supported modulo $Z_I(F_\nu)$.

Theorem 11.14 (Jacquet)**.** *A smooth representation π of $M_I(F_\nu)$ is quasi-cuspidal if and only if $Res_I^J(\pi) = 0$ for all $J \subsetneq I$*

Remark 11.15. As a consequence of the above theorem one often calls restriction a 'Jacquet Functor'.

For the time being we shall assume this theorem.

Corollary 11.16. *Let π be a smooth irreducible representation of $M_J(F_\nu)$ then:*

(1) *There exists $J \subset I$, a supercuspidal representation σ of $M_J(F_\nu)$ and an embedding $\pi \hookrightarrow Ind_J^I(\sigma)$.*
(2) *π is admissible.*

Proof. The second assertion follows from the first together with the statement that induction preserves admissibility. We proceed to prove the first assertion.

Choose $J \subset I$ such that $Res_I^J(\pi) \neq 0$ and $Res_I^K(\pi) = 0$ for all $K \subset J$. We then have that $Res_I^J(\pi)$ is quasi-cuspidal. If σ is an irreducible quotient of $Res_I^J(\pi) \in Rep_s(M_J(F_\nu))$ then σ is quasi-cuspidal.

Claim 11.17. *Quasi-cuspidal and Irreducible implies super-cuspidal.*

By adjointness we have that $Hom_{M_I}(Res_I^J(\pi), \sigma) \neq 0$ implies $Him_{M_I}(\pi, Ind_J^I(\sigma)) \neq 0$ so by irreducibility of σ any homomorphism π is an embedding, thus we need only show the eqistance of at least one smooth subquotient. This will occur if and only if the $C_c^\infty(M_J(f_\nu))$-module $V/V(N_J^I)$ has at least one irreducible quotient. Indeed, by Zorn's lemma this is true if the module is finitely generated. Which, is the case because V is generated by a single element as a $C_c^\infty(M_I(F_\nu))$-module. Thus, $\pi(M_I(\mathcal{O}_{F_\nu}))$ is a finite set by smoothness and using that $M_I(F_\nu) = P_J^I M_I(\mathcal{O}_{F_\nu})$ the image of this finite set in $V/V(N_J^I(F_\nu))$ generates it.

To Be continued ...

□

12. Lecture 12: More on supercuspidal representations

As always, G is a connected reductive group. In this section we again take it to be defined over a nonarchimedean field F_v. Today we'll prove that quasicuspidal and irreducible implies supercuspidal, which explains why the condition of being quasicuspidal is rarely seen in the literature. It is typically only used in proofs.

Given an irreducible smooth representation π, there exists a parabolic $P = MN$ and a supercuspidal representation σ on $M(F_v)$ such that π is a subquotient of $\mathrm{Ind}_M^G(\sigma)$. Hence π is admissible by properties of induction.

12.1. **The philosophy of cusp forms.** This is due to Harrish-Chandra. It states informally that the representations of G are built out of "cuspidal" representations of Levi subgroups $M \subseteq P \subseteq G$ via induction. The same is true for finite and adelic groups! One shows that one can obtain everything in this way via the theory of Eisenstein series. It is remarkable that one only needs to consider the Levi subgroups to build up the representations of G.

Last time we were in the middle of proving a theorem of Jacquet for $G = \mathrm{GL}_n$:

Theorem 12.1 (Jacquet). *A smooth representation π of $G(F_v)$ is quasi-cuspidal if and only if $\mathrm{Res}_M^{G,P}(\pi) = 0$ for all proper parabolics $P \subseteq G$.*

Remark 12.2. This is true generally but we will prove it for $G = \mathrm{GL}_n$ only.

Proof. Recall the notation set up in the previous lecture, as we will freely use it in this proof. In particular, I will always be a subset of the primitive roots of G. We begin with a lemma

Lemma 12.3. *A smooth representation π of $M_I(F)$ is quasi-cuspidal if and only if for all $v \in V$ and $v^\vee \in V^\vee$ there exists $\epsilon \in (0,1]$ such that $\langle v^\vee, \pi(z)v\rangle = 0$ for all $z \in Z_\emptyset I(1)$ with $|\alpha z| \leq \epsilon$ for at least one $\alpha \in I$.*

Proof. By the Cartan decomposition,

$$M_I(F_v) = M_I(\mathcal{O}) Z_\emptyset^I(1) M_I(\mathcal{O})$$

and for $v^\vee \in V^\vee$ we have that $\pi(M_I(\mathcal{O}))v$ and $\pi^\vee(M_I(\mathcal{O}))v^\vee$ are finite by smoothness of the representations. Therefore π is quasi-cuspidal if and only if for each $v \in V$ and $v^\vee \in V^\vee$, the subset $\{z \in Z_\emptyset^I(1) \mid \langle v^\vee, \pi(z)v\rangle \neq 0\}$ is compact modulo $Z_I(F)$.

A closed subset Ω of $Z_\emptyset^I(1)$ is compact modulo $Z_I(F)$ if and only if there exists some $\epsilon \in (0,1]$ such that $|\alpha(\Omega)| \in (\epsilon, 1]$ for all $\alpha \in I$. This completes the proof of the lemma. □

The preceding lemma makes the notion of being compactly supported modulo the center precise. Now we return to the proof of the theorem. By a previous lemma, $\mathrm{Res}_I^J(\pi) = 0$ for all $J \subseteq I$ if and only if $\mathrm{Res}_I^{I-\{\alpha\}}(\pi) = 0$ for all $\alpha \in I$. So take $\alpha \in I$ and assume that this holds for $I - \{\alpha\}$. Thus $V = V(N_{I-\{\alpha\}}^I)$. If $v \in V$ and $v^\vee \in V^\vee$, then there exist compact opens Γ_1, Γ_2 in $N_{I-\{\alpha\}}^I(F)$ such that $v \in \ker \pi_{\Gamma_1}$ and $v^\vee \in (V^\vee)^{\Gamma_2}$.

Fact: there exists $\epsilon > 0$ such that $z\Gamma_1 z^{-1} \subseteq \Gamma_2$ for all $z \in Z_{\emptyset}^I(1)$ with $|\alpha(z)| \leq \epsilon$ (look in Laumon's book [41] for proof). Therefore we have

$$\begin{aligned}\langle v^\vee, \pi(z)v\rangle &= \langle \pi_{\Gamma_2}^\vee v^\vee, \pi(z)v\rangle \\ &= \langle v^\vee, \pi_{\Gamma_2}\pi(z)v\rangle \\ &= \langle v^\vee, \pi(z)\pi_{z^{-1}\Gamma_2 z}v\rangle \\ &= 0\end{aligned}$$

since $\pi_{\Gamma_2}^\vee(v^\vee) = v^\vee$ and $\pi_{\Gamma_1}(v) = 0$ and $z^{-1}\Gamma_2 z \supset \Gamma_1$. The lemma thus shows that π is quasi-cuspidal.

For the other direction, suppose π is quasi-cuspidal and let $J \subseteq I$. We want to show that $V = V(N_J^I(F))$. Let $v \in V$ and let n be an integer > 0 such that $v \in V^{K(n)\cap M_I(F_v)}$. Then

$$\{m \in M(F_v) \mid \pi(e_{K(n)\cap M_I(F_v)})\pi(m_I)v \neq 0\},$$

where $e_{K(n)\cap M_I(F_v)}$ is the idempotent of the compact set $K(n)\cap M_I(F_v)$. But we have the following structural fact:

$$K(n) \cap M_I(F_v) = (K(n) \cap N_J^I(F))(K(n) \cap \widetilde{P}_J^I(F_v))$$

and

$$z^{-1}(K(n) \cap \widetilde{P}_J(F))z \subset K(N) \cap \widetilde{P}_J^I(F_v) \subset K(n) \cap M_I(F_v).$$

This implies that if we take the projector $\pi(e_{K(n)\cap M_I(F_v)})\pi(z)v = \pi(z)\pi_{z^{-1}K(n)\cap N_J^I(F)z^{-1}}v$, then we get that $\pi_{z^{-1}K(n)\cap N_J^I(F_v)z}v = 0$ and $v \in V(N_J^I(F_v))$. This concludes the proof of the theorem. □

Definition 12.4. A smooth representation π of $G(F_v)$ is said to *admit a central character* if for all $z \in Z_G(F_v)$, we have $\pi(z) = \omega_\pi(z)\,\mathrm{Id}$ for $\omega_\pi(z)$ a nonzero complex number. If this is the case, then ω_π defines a smooth character $Z_G(F_v) \to \mathbf{C}^\times$.

Note that some authors call this a quasi-character; a character is sometimes taken to land in the unit circle.

Lemma 12.5 (Schur's lemma, extended to smooth irreducibles by Jacquet)**.** *Let (π, V) be a smooth irreducible representation of $G(F_v)$. Then any endomorphism of V commuting with π is necessarily scalar. In particular, π admits a central character.*

Proof. Let ϕ be an endomorphism of π in $\mathrm{Rep}_s(G)$, the category of smooth representations of G. Assume $\phi \neq \lambda\,\mathrm{Id}$ for all $\lambda \in \mathbf{C}$. Then for all $\lambda \in \mathbf{C}$, the representation $\phi - \lambda\,\mathrm{Id}$ is an automorphism of π in $\mathrm{Rep}_s(G)$, since it is nonzero and π is irreducible. So let $R_\lambda = (\phi - \lambda\,\mathrm{Id})^{-1}$. Then the R_λ are linearly independent over $\mathbf{C}$ as λ varies. Why is this the case: suppose $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ are distinct and let $a_1, \ldots, a_n \in \mathbf{C}^\times$. The linear combination $\sum a_i R_{\lambda_i}$ decomposes as a product

$$\sum_i a_i R_{\lambda_i} = \left(\prod_i R_{\lambda_i}\right) P(\phi),$$

where $P(T) = \sum_i a_i \prod_{j\neq i}(T - \lambda_j)$. Factor $P(T) = a\prod_i (T - m_i)$ for $a \in \mathbf{C}^\times$ and $m_i \in \mathbf{C}$. Then $P(\phi)$ is invertible each $\phi - m_i\,\mathrm{Id}$ is invertible, thus $\sum a_i R_{\lambda_i}$ is invertible, and the R_{λ_i} are thus linearly independent. This proves the claim, which implies that the endomorphisms of π form a $\mathbf{C}$-vectorspace of uncountable dimension.

Now let $v \in V - \{0\}$, so that V is generated by v as a $\mathbf{C}[G]$-module. So it's generated by $\{\pi(g)v\}$ over $\mathbf{C}$ as g varies over G. But v has a large stabilizer by smoothness, so any linearly independent family of elements of V must be countable by properties of compact subsets of G (namely, G/K is countable for any compact open subset K. See 2.6 of [15] for more details). But then the endomorphisms of π must make up a $\mathbf{C}$-vectorspace of at most countable dimension, contradicting the previous claim. We see that we must have $\phi = \lambda \operatorname{Id}$ for some $\lambda \in \mathbf{C}$, which proves the lemma. □

Remark 12.6. One has $\omega_{\pi^\vee} = \omega_\pi^{-1}$.

Let π be an admissible representation of $G(F_v)$. Then for all $f \in C_c^\infty(G(F_v))$, we have defined $\pi(f)$. We claim that this operator is of finite rank. This is because we can find a compact open subset $K \subseteq G(F_v)$ such that $f \in C_c^\infty(K\backslash G(F_v))$. Thus $\pi(f)V \subseteq V^K$, and V^K is finite dimensional by the admissibility of π.

Definition 12.7. The *trace* of $\pi(f)$ is the trace of $\pi(f)|_{\pi(f)V}$.

Remark 12.8. One can compute the trace with respect to *any* finite dimensional subspace W with $\pi(f)V \subset W \subset V$.

We have a distribution $\operatorname{tr}(\pi)\colon C_c^\infty(G(F_v)) \to \mathbf{C}$, sometimes written θ_π, called the *character* of π. Note that this depends on a choice of Haar measure.

We won't prove this, but one should be aware of it, as it is deep and useful.

Theorem 12.9 (Harrish-Chandra)**.** *The distribution θ_π is represented by a locally constant function with support in $G^{reg}(F_v)$.*

This means that for all $\theta_\pi \in C^\infty(G^{reg}(F_v))$, we have

$$\operatorname{tr}\pi(f) = \int_{G(F_v)} \theta_\pi(g) f(g) dg$$

for all $f \in C_c^\infty(G(F_v))$. This is a very useful and nonobvious fact. See Harrish-Chandra's collected works.[17]

Proposition 12.10 (Linear independence of characters)**.** *If $\pi_1, \dots, \pi_n$ is a finite set of admissible irreducible representations, such that $\pi_i \cong \pi_j$ implies $i = j$, then the distributions θ_{π_i} are linearly independent.*

Proof. Fix $K \subseteq G(F_v)$ a compact open such that $V_i^K \neq 0$ for all i. This implies that $\{V_i^K\}$ is a finite family of finite dimensional $\mathbf{C}$-vector spaces with an action of $C_c^\infty(G(F) /\!/ K)$. They are all simple, that is irreducible, for this action. Moreover, they are pairwise nonisomorphic (as Hecke-modules). Let $A =$ the image of $C_c^\infty(G(F_v) /\!/ K)$ in $\prod_i \operatorname{End}_{\mathbf{C}}(V_i^K)$. Then A is a finite dimensional $\mathbf{C}$-algebra and the V_i^K are a finite family of nonisomorphic simple A-modules. Want to show that the traces are linearly independent; now that we've reduced to finite dimensional stuff this follows from Jacobson's density theorem (see [40]). □

We have been owed the following proposition for some time:

Proposition 12.11. *If (π, V) is quasi-cuspidal and irreducible, then π is admissible.*

Proof. Begin with a lemma.

Lemma 12.12. *Let π be smooth. Then the following are equivalent:*

(1) *π is quasi-cuspidal;*
(2) *for all $v^\vee \in V^\vee$ and any compact open $K \subset G(F_v)$, the set*

$$U_{v^\vee,K} = \{g \in G(F_v) \mid \pi(e_k)\pi(h)v \neq 0\}$$

is compact modulo the center.

[17] clean this up and add a more precise citation

Proof of lemma. For any $v \in V$ and any comapct open set $K \subseteq G(F_v)$, and any $v^\vee \in V^\vee$, we have that the support of $\langle \pi v, v^\vee \rangle$ is contained in $U_{v^\vee,K}$. Thus (2) implies (1).

Conversely assume that π is quasi-cuspidal and let $K \subset G(F_v)$ be compact open. For all $v \in V$ let $V_{v,K} = \langle \pi(e_K)\pi(g)v \mid g \in G(F_v),\ v \in V^K \rangle$. We claim that $\dim_{\mathbf{C}} V_{v,K} < \infty$. Indeed, otherwise there exists $(g_n)_{n\in\mathbf{Z}_{>0}} \subseteq G(F_v)$ such that $\pi(e_K)\pi(g_n)v$ are linearly independent. Let $W \subseteq V^K$ be an arbitrary $\mathbf{C}$-vector space such that

$$V^K = W \oplus \text{ span of } \pi(e_k)\pi(h_n)v.$$

As $V = V^K \oplus \ker \pi(e_K)$ by definition of the projector, we can define $\widetilde{v} \in \mathrm{Hom}(V, \mathbf{C})$ such that

$$\langle \widetilde{v}, \pi(e_K)\pi(h_n)v \rangle = n$$

for all n and $\widetilde{v}|_{W\oplus\ker(e_K)} = 0$. This gives $\widetilde{v} \in V^\vee$ is fixed by K. Hence the support of the matrix coefficient $\langle \widetilde{v}, \pi(g_n)v \rangle \cap G(F_v)^2$ is not compact. This implies the claim.

Choose $v_1^\vee, \dots, v_n^\vee$ in $(V^\vee)^K$ separating vectors in $V_{v,K}$. Then

$$U'_{v,K} = U_{v,K} \cap G(F_v) \subset \bigcup_n \text{supp}(\langle \pi()v, v_n^\vee \rangle) \cap G(F_v).$$

Hence $U'_{v,K}$ is compact for all K. Finally, if $g_1, \dots, g_n$ is a system of representatives for $ZG'\backslash G$, we see that

$$U_{v,K} \subset Z\left(\bigcup_i U_{\pi(g_i)v,K} h_i\right)$$

and hence that $U_{v,K}$ is compact modulo K. □

Now we return to the proof of the proposition. Fix $v \in V - \{0\}$ and for all $K \subset G(F_v)$ compact open we have $V^K = \mathrm{im}(\pi(e_K))$ is the linear span of $\pi(e_K)\pi(g)v$ for $g \in G(F_v)$. In other words, $g_1, \dots, g_n$ is a system of representatives of classes in $Z_G(F_v)G(F_v)\backslash G(F_v)$. Have that V^K is the $\mathbf{C}$-linear span of $V_{\pi(g_i)v}$ for $i = 1, \dots, n$. Why is this true? Well thanks to Schur, for any $w \in V$ the space $V_{w,K}$ is the $\mathbf{C}$-linear span of $\pi(g)w$ for $g \in Z_G(F_v)$, which implies that V^K is finite dimensional. □

Let π be an admissible irreducible representation. A *coefficient* of π is a smooth function $f_\pi \in C_c^\infty(G(F_v))$ such that $\mathrm{tr}\,\pi(f_\pi) \neq 0$ and $\mathrm{tr}\,\pi_1(f_\pi) = 0$ for $\pi_1 \ncong \pi$. If $Z_G(F_v)$ is noncompact, we can weaken the last condition to $\pi_1 \ncong \pi \otimes \chi$ for some character $\chi\colon G(F_v) \to \mathbf{C}^\times$. The existence of such functions is a nonarchimedean phenomenon; the Heisenberg uncertainty principle essentially rules out the existence of these things over real groups.

Proposition 12.13. *Assume that $Z_G(F_v)$ is compact. Let π be a supercuspidal representation. Then for all $f \in C_c^\infty(G(F_v))$, there exists a unique $f_\pi \in C_c^\infty(G(F_v))$ such that $\pi(f_\pi) = \pi(f)$ for ?? $\mathrm{End}(\pi)$ and $\pi_1(f_\pi) = 0$ if $\pi_1 \ncong \pi$ for π_1 smooth. Moreover f_π^* is a linear combination of matrix coefficients of $G(F_v)$, we have $f_\pi^*(g) = f_\pi(g^{-1})$[18]. (Thus coefficients exist for supercuspidal representations if $Z_G(F_v)$ is compact)*[19]

Proof. We again begin with a lemma.

Lemma 12.14. *Let $f \in C_c^\infty(G(F_v))$. Then there exists at least one $\pi \in \mathrm{Rep}_S(G)$ such that $\pi(f) \neq 0$ in $\mathrm{End}(\pi)$.*

Note that this implies uniqueness in the proposition above.

[18]Might need to take complex conjugate on right! Check this.

[19]Fix statement of proposition; find reference

Proof of lemma. If $f \in C_c^\infty(G(F_v))$ then
$$f^* * f(1) = \int_{G(F_v)} |f(g)|^2 \, dg$$
if $f \neq 0$ then the same is true of $f^* * f$. Let f_n be the 2^n-th power of $f^* * f = f_0$. By induction $f_n^* = f_n$ implies $f_{n+1} = f_n^* * f_n$, which shows that f_n is not identically zero for all n. Thus f_0 is not nilpotent in $C_c^\infty(G(F_v))$. Fix $K \subset G(F_v)$ such that $f \in C_c^\infty(G(F_v) /\!/ K)$. Then $f^* f_0 \in C_c^\infty(G(F_v) /\!/ K)$ and f_0 is not nilpotent in $C_c^\infty(G(F_v) /\!/ K)$. But the $\mathbf{C}$-algebra $C_c^\infty(G(F_v) /\!/ K)$ is of countable dimension. This implies that there exists at least one maximal ideal $\mathfrak{m} \in C_c^\infty(G(F_v) /\!/ K)$ such that $f_0 \notin \mathfrak{m}$ (this follows from Jacobson Ch1, section 10, Thm 2)[20].

Fixing such an $\mathfrak{m}$ we have that $C_c^\infty(G(F_v) /\!/ K)/\mathfrak{m}$ is an irreducible left $C_c^\infty(G(F_v) /\!/ K)$-module on which f_0 and f act nontrivially. This implies that there exists a smooth representation realizing this module on which $\pi(f_0)$ and $\pi(f)$ are nontrivial.

End of proof of lemma. □

Finish proof next time. □

Remark 12.15. See Cartier [16] in Corvallis possibly for a proof that you can find compact $K \subseteq G(F_v)$ such that $f \in C_c^\infty(G(F_v))$ is bi-invariant under K.

13. Lecture 13: Distinction

13.1. **Supercuspidals admit coefficients.**

Let $G_{/F_v}$ be a connected reductive group.

Proposition 13.1. *Asssume that $Z_G(F_v)$ is compact, and let π be a cuspidal representation of $G(F_v)$. Then for all $f \in C_c^\infty(G(F_v))$ there exists a unique $f_\pi \in C_c^\infty(G(F_v))$ such that*
$$\begin{aligned} \pi(f_\pi) &= \pi(f); \quad \textit{and} \\ \pi_1(f_\pi) &= 0 \qquad \textit{if } \pi_1 \neq \pi. \end{aligned}$$

Note that, as an immediate consequence, we see that coefficients exist for supercuspidals. We need some preparation before giving the proof. Given an admissible representation π of $G(F_v)$, we have an admissible representation σ of $G(F_v) \times G(F_v)$ on $\mathrm{End}(\pi)^\infty$ given by
$$\sigma(g_1, g_2)\phi = \pi(g_2) \circ \phi \circ \pi(g_1).$$
This representation is indeed admissible, since for any compact subgroups $K_1, K_2 \subset G(F_v)$, we can find a compact $K \subset G(F_v)$ with $K \times K \subset K_1 \times K_2$, and therefore the subspace of fixed vectors $\mathrm{End}(\pi)^\infty \supset \mathrm{End}(\pi)^{K_1 \times K_2} \cong \mathrm{End}(V_\pi^{K_1 \times K_2}) \cong V_\pi^{K_1} \otimes (V_\pi^\vee)^{K_2} \subset V_\pi^K \otimes (V_\pi^\vee)^K$ is of finite dimension. By a similar reasoning, the natural map
$$\alpha : \pi^\vee \otimes \pi \longrightarrow (\mathrm{End}(\pi)^\infty, \sigma) \qquad \alpha(\tilde{v} \otimes v)(v_1) = \langle \tilde{v}, v_1 \rangle v,$$
is an isomorphism, since we can reduce to the finite dimensional case. Consider also the map
$$\beta : (\mathrm{End}(\pi)^\infty, \sigma) \longrightarrow (C_c^\infty(G(F_v)), \rho) \qquad \beta(\phi)(g) = \mathrm{tr}(\pi(g) \circ \phi),$$
where ρ acts via $\rho(g_1, g_2)(f)(h) = f(g_1^{-1} h g_2)$.

Proof of Prop. 13.1. Let r denote the restriction $\sigma|_{C_c^\infty(G(F_v))}$; it is a smooth representation of the product $G(F_v) \times G(F_v)$. Since $Z_G(F_v)$ is compact by assumption, and π is supercuspidal, we have $\beta(\mathrm{End}(\pi)^\infty) \subset C_c^\infty(G(F_v))$. Notice that β is not identically zero: indeed, there exists $\tilde{v}$ such that

[20] Which Jacobson!?

$\langle \tilde{v}, v\rangle \neq 0$, i.e. $\langle \tilde{v}, \pi(1)v\rangle \neq 0$, and since $\alpha \circ \beta$ is not identically zero, β is not either. Since $\pi \circ \pi^\vee$ is an irreducible representation of $G(F_v) \times G(F_v)$, β is an embedding. Consider

$$\beta' : f \mapsto \pi(f) : (C_c^\infty(G(F_v)), r) \to (\mathrm{End}(\pi)^\infty, \sigma).$$

Then $\beta' \circ \beta$ is an endomorphism of the irreducible representation $\mathrm{End}(\pi)^\infty$ of $G(F_v) \times G(F_v)$. Hence $\beta' \circ \beta$ is scalar by Schur's lemma, say $\beta' \circ \beta = \lambda \mathrm{Id}$ ($\lambda \in \mathbf{C}$). We will show that $\lambda \neq 0$; the content of the proposition will then follow taking f_π to be $\lambda^{-1}(\beta \circ \beta(f^*))^*$, where $f^*(g) = f(g^{-1})$. Let (π_1, V_1) be a smooth irreducible representation of $G(F_v)$, and let $v_1 \in V_1$ be a non-zero vector. Let $\gamma_1 : f \mapsto \pi_1(f^*)v_1 : (C_c^\infty(G(F_v)), r|_{1\times G(F_v)}) \to (V_1, \pi_1)$. Note that $\mathrm{End}(\pi)^\infty|_{1\times G(F_v)}$ is, as a representation of $G(F_v)$, isomorphic to a number of copies of $\pi^\vee \otimes \pi$, and that the same is true of $\gamma_1(\beta(\mathrm{End}(\pi)^\infty, r|_{1\times G(F_v)})^{G(F_v)}$. Thus $\gamma_1(\beta(\mathrm{End}(\pi)^\infty)) = 0$ unless $\pi_1 = \pi$ (since whenever the former is nonzero we obtain an intertwining operator between π_1 and π). Moreover, by construction, we have

$$\pi(\beta \circ \beta'(f)^*) = \beta' \circ \beta \circ \beta'(f) = \lambda \beta'(f) = \lambda \pi(f^*)$$

for all $f \in C_c^\infty(G(F_v))$, and we see that $\lambda \neq 0$: in fact, we can find $f \in C_c^\infty(G(F_v))$ such that $\beta'(f) = \pi(f^*) \neq 0$, and since β is an embedding, also $\beta \circ \beta'(f) \neq 0$, hence $\lambda \neq 0$, and the proof is complete. ☐

13.2. **Trace Formulae and Relative Trace Formulae.** Let G be a reductive F-group, and $H \subset G$ a reductive subgroup, which we will not assume to be necessarily connected. Let

$$\iota : V_\pi \hookrightarrow L_0^2(G(F)A_G \backslash G(\mathbf{A}_F))$$

be an embedding. For the ease of notation, let L^2 and L_0^2 denote the spaces $L^2(G(F)A_G\backslash G(\mathbf{A}_F))$ and $L_0^2(G(F)A_G\backslash G(\mathbf{A}_F))$ respectively. Let $\chi : H(\mathbf{A}_F) \to \mathbf{C}$ be a quasi-character trivial on $(A_G \cap H(\mathbf{A}_F)).H(F)$.

Proposition 13.2 (Ash–Ginzburg–Rallis). *Supppose that $|\chi(h)| = 1$ for all $h \in H$ (i.e. that χ is a character in a common – though not universal – terminology). Then for all rapidly decreasing functions $\phi \in L^2$, the period integral*

$$P_\chi(\phi) = \int_{(A_G\cap H(\mathbf{A}_F)).H(F)\backslash H(\mathbf{A}_F)} \phi(g)\chi(g)\mathrm{d}g$$

is absolutely convergent.

Before we commence the proof, we need to introduce an important concept.

Definition 13.3. A cuspidal automorphic representation (π, ι) of $A_G\backslash G(\mathbf{A}_F)$ is said to be (H, χ)-*distinguished* if $P_\chi(\phi) \neq 0$ for some $\phi \in \iota(V_\pi)$. When (H, χ) is understood, or irrelevant, we simply say that it is *distinguished*.

Example 13.4. (1) Consider the diagonal embedding $\Delta : H \to H \times H$. We ask ourselves which representations π' of $\Delta H \subset H \times H$ are distinguished. Any such representation π' can be decomposed as $\pi_1 \times \pi_2$ with π_1, π_2 representations of $H(\mathbf{A}_F)$. As a map

$$(\phi_1, \phi_2) \mapsto P(\phi_1 \otimes \phi_2) : V_{\pi_1} \times V_{\pi_2} \to \mathbf{C},$$

the period integral is invariant under $\Delta H(\mathbf{A}_F)$. Thus $\pi' = \pi_1 \times \pi_2$ is distinguished if and only if $\pi_1 \cong \pi_2^\vee$. That is, the representations π' of the form $\pi \times \pi^\vee$ for π a representation of $H(\mathbf{A}_F)$ are the only distinguished representations of ΔH.

(2) Take $G = \mathrm{GL}_{2/\mathbf{Q}}$ and $H = \mathrm{Res}_{K/\mathbf{Q}} G$ for a quadratic extension $K/\mathbf{Q}$. This data corresponds to an embedding $K \hookrightarrow \mathrm{GL}_2(\mathbf{Q})$. Then the notion of distinction is related to Heegner points. We note that $A_G \cap H(\mathbf{A}_F).H(F)\backslash H(\mathbf{A})/K_H$ is a *finite* number of points for every $K_H \subset H(\mathbf{A}_F)$ compact subgroup.

(3) For G and H as in the previous example, with $K/\mathbf{Q}$ *real* quadratic, this relates to modular curves and Hilbert modular surfaces.

(The definition of distinction is due to Harder, Langlands and Rapoport, with later contributions due Jacquet.) Let $T_G \subset B_G \subset G$, with T_G a maximal torus, and B_G a maximal F-Borel (but note that may not be a Borel over $\bar{F}$), and let $K_{G,\infty} \subset G(\mathbf{R} \otimes_{\mathbf{Q}} F)$ be a maximal compact subgroup. We make similar definitions for H and omit G from the previous notation. Assume $T_H \subset T$, $B_H \subset B$ and $K_{H,\infty} \subset K_\infty$.

Proof of Prop.13.2. We assume $F = \mathbf{Q}$. Let Δ be the set of simple root attached to (B,T). Take $A^G = T(\mathbf{R})^+/(T(\mathbf{R}) \cap A_G)$, a connected real Lie group, and for each $r \in \mathbf{R}_{>0}$, define the subset

$$A^G_r := \{t \in A^G \ : \ \alpha(t) > r \ \text{ for all } \alpha \in \Delta\} \subset A^G.$$

We say that ϕ is *rapidly decreasing* if, for some r, $|\phi(t)| \leq C\alpha(t)^p$ holds for some $p \in \mathbf{Z}$ and $C > 0$, and all $t \in A^G_r$, $\alpha \in \Delta$ and $x \in \Omega$. We note that the constant C is allowed to depend on Ω and r, but can be taken to be independent of α. (Exercise: relate this notion of rapidly decreasing to a previous one.) Fix Δ for G, and let ℓ denote the $\mathbf{Q}$-rank od H. Choose a basis of cocharacters $t_1, \ldots, t_\ell \in X_*(H)$ and let $\alpha_1, \ldots, \alpha_\ell \in \Delta$ be such that

$$\alpha_i(t(\tau_j)) = \begin{cases} 1 & \text{if } i \neq j \\ \tau^{v_i} & \text{if } i = j \quad (v_i \in \mathbf{Q}). \end{cases}$$

We need to assume the truth of the following statement, which follows from *reduction theory*:

Claim 13.5. *$H(\mathbf{A}_F) = H(F)A^H_r N^o_H M_H$ for some $0 < r < 1$, with N^o_H a relatively compact subgroup of $N_H(\mathbf{A})$ (with N_H the unipotent radical of B_H), $A^H_r = \{t \in A^H \ : \ \beta(t) \geq r$ for all $\beta \in A_H\}$ as before, and M_H a compact set in $H(\mathbf{A}_F)$.*

Thus, since χ is unitary, $P_\chi(\phi)$ converges absolutely provided that $\int_{A^H_r N^o_H M_H} |\phi(anm)| \mathrm{d}a\mathrm{d}n\mathrm{d}m$ converges. Now, the Weyl group $W(G,T)$ acts on A^G and partitions it into so-called Weyl chambers (which are just fundamental domains for this action). Let S be the closure of a Weyl chamber. Hence the integral we are interested in is dominated by

$$\sum_{w \in W(G,T)} \int_{A^H \cap wS} |\phi(anm)| \mathrm{d}a\mathrm{d}n\mathrm{d}m.$$

Since ϕ is automorphic and w is represented by an element of $G(\mathbf{Q})$, we have $\phi(anm) = \phi(wanm) = \phi(waw^{-1}wnm)$. However, $\phi(waw^{-1}wnm)$ is rapidly decreasing on $wSN^o_H M_H$. Thus using $\alpha_1, \ldots, \alpha_\ell$ as coordinates in A^H, we conclude the proof of the proposition. ☐

13.3. **How to study distinction?** This will be our motivating question for a while in the following. For $f \in C^\infty_c(A_G \backslash G(\mathbf{A}_F))$, consider the integral operator

$$\begin{aligned} R(f) : L^2 &\longrightarrow L^2 \\ \phi &\mapsto \int_{A_G \backslash G(\mathbf{A}_F)} f(x)\phi(gx)\mathrm{d}x. \end{aligned}$$

Just manipulating formally for the moment, we see that

$$\begin{aligned} R(f)\phi(x) &= \int_{A_G\backslash G(\mathbf{A}_F)} f(y)R(y)\phi(x)\mathrm{d}y \\ &= \int_{A_G\backslash G(\mathbf{A}_F)} f(x)\phi(xy)\mathrm{d}y \\ &= \int_{A_G\backslash G(\mathbf{A}_F)} f(x^{-1}y)\phi(y)\mathrm{d}y \\ &= \int_{A_G\backslash G(\mathbf{A}_F)} \sum_{\gamma\in G(F)} f(x^{-1}\gamma y)\phi(y)\mathrm{d}y. \end{aligned}$$

In other words, $R(f)$ is an integral operator with kernel

$$K_f(x,y) := \sum_{\gamma\in G(F)} f(x^{-1}\gamma y).$$

This is the *geometric interpretation* of the kernel. Note that $K_f(x,y)$ is smooth on $G(\mathbf{A}_F)\times G(\mathbf{A}_F)$: indeed, if $(x,y)\in\Omega_1\times\Omega_2\subset G(\mathbf{A}_F)\times G(\mathbf{A}_F)$, then the sum $\sum_{\gamma\in G(F)} f(x^{-1}\gamma y)$ is finite, over the set of $\gamma\in G(F)$ such that $\gamma\in\Omega_1\mathrm{Supp}(f)\Omega_2$.

In contract, there is the *spectral realization* of the kernel. Recall that an operator $A : L^2\to L^2$ is *Hilbert-Schmidt* if the image $A(L^2)$ has a countable basis $(\phi_i)_{i=1}^\infty$ consisting of eigenvectors for A, say $A(\phi_i)=\lambda_i\phi_i$, such that $\sum_{i=1}^\infty|\lambda_i|^2$ is finite. On the other hand, A is said to be *of trace class* if $\sum_{i=0}^\infty|\lambda_i|$ is finite, with λ_i as before. In the former case, we let $|A|=|A|_{HS}$ denote $\sum_{i=1}^\infty|\lambda_i|^2$, and in the latter, $\mathrm{tr}A$ denote $\sum_{i=1}^\infty|\lambda_i|$. Clearly, an operator is Hilbert-Schmidt if it is trace class, but a HS operator need not be of trace class.

We have the following *fundamental fact*, due to Donnely: $R_0(f) := R(f)|_{L_0^2}$ is of trace class. Hence the multiplicity of cuspidal representations is finite. In fact more is true: it is a result due to Müller that $R_{\mathrm{discr}}(f)$, the restriction of $R(f)$ to the discrete spectrum L^0_{discr}, is of trace class. Thus we can write

$$\mathrm{tr}(R_0(f)) = \sum_{\substack{\text{equiv.classes of}\\ \text{cusp.aut.reps.}\\ \pi\text{ of }A_G\backslash G(\mathbf{A}_F)}} m(\pi)\mathrm{tr}(\pi(f));$$

we call $m(\pi)$ the (finite!) *multiplicity* of π. Thus $R_0(f)$ has kernel $\sum_\pi K_{\pi(f)}(x,y)$, where $K_{\pi(f)}(x,y) = K^{\mathrm{cusp}}_{\pi(f)}(x,y) = \sum_{\phi\in B_\pi}\pi(f)\phi(x)\overline{\phi(y)}$, for B_π an orthonormal basis of $L_0^2(\pi)$, the π-isotypical subspace of L_0^2.

Remark 13.6.

- Note that $\pi(f)B_\pi$ need not be finite in general; it is so only if f is K_∞-finite.
- As defined, $K_{\pi(f)}(x,y)$ is not even a honest function: it may not make sense to evaluate it at any point. But note that integrating $K_{\pi(f)}(x,y)$ along $\Delta : G\to G\times G$ we obtain

$$\sum_{\substack{\text{equiv.classes of}\\ \text{cusp.aut.reps.}\\ \pi\text{ of }A_G\backslash G(\mathbf{A}_F)}} \int_{A_GG(F)\backslash G(\mathbf{A}_F)} K_{\pi(f)}(x,y)\mathrm{d}x = \sum_{\substack{\text{equiv.classes of}\\ \text{cusp.aut.reps.}\\ \pi\text{ of }A_G\backslash G(\mathbf{A}_F)}} m(\pi)\mathrm{tr}(\pi(f)),$$

and this makes sense, because $(\phi_1,\phi_2)\mapsto\int_{A_GG(F)\backslash G(\mathbf{A}_F)}\phi_1(g)\overline{\phi_2(g)}\mathrm{d}g$ is the pairing defining the metric on L^2.

14. Lecture 14: Relative Trace Formula

We have defined for $f \in C_c^\infty(\mathbf{A}_G \backslash G(\mathbf{A}_F))$ the kernal functions:

$$K_f(x,y) : \mathbf{A}_G \backslash G(\mathbf{A}_F) \times \mathbf{A}_G \backslash G(\mathbf{A}_F) \to \mathbf{C}$$

And

$$K_f^{cusp}(x,y) = \sum_{pi \text{ cusp}/\sim} \sum_{phi \in B_\pi} \pi(f)\phi(x)\overline{\phi}(y)$$

We have that we can integrate and get:

$$\int_{\mathbf{A}_G \backslash G(\mathbf{A}_F)} K_f^{cusp}(x,x)dx = \sum_\pi mult(\pi)tr(\pi f)$$

Exercise 1. If f is K_∞-finite and π is cuspidal then the image of $\pi(f)$ consists of K_∞-finite vectors.

The above exercise implise that we can choose a finite orthonormal basis $\{\phi_i\}$ for the image of $\pi(f)$ for each f. In this case the formula we had (*)[21] makes sense at the level of smooth functions.

We now wish to make the following definition:

Definition 14.1. Let $H \subset G \times G$ and $\chi : H(\mathbf{A}_f) \to \mathbf{C}^*$ be trivial on $H(F)$ and $\mathbf{A}_G \cap H(\mathbf{A}_F)$. We may then look at:

$$\int_{(H(\mathbf{A}_F) \cap \mathbf{A}_G) \backslash H(\mathbf{A}_F)} K_{\pi(f)}(h_l, h_r) dh_l h_r = \sum_{\phi \in B_\pi} P_\chi(\pi(f)\phi, \overline{\phi})$$

Where we have:

$$P_\chi(\phi_1, \phi_2) = \int_{(H(\mathbf{A}_F) \cap \mathbf{A}_G) \backslash H(\mathbf{A}_F)} \phi_1(h_l)\overline{\phi_2(h_r)}\chi(h_l h_r) dh_l h_r$$

The above expression is known as a *relative trace* with respect to the (almost) hermitian form P_χ.

(This idea is due to Jacquet).

Example 14.2. We consider the special case where G has no proper parabolic subgroups over F (except for the center) which is the case if and only if $G(F)\mathbf{A}_G \backslash G(\mathbf{A}_F)$ is compact. In this case

[21] ref to whatever * is???

we have that $L_0^2 = L^2$. And for the case where H is the diagonal embedding of G we get:

$$\begin{aligned} tr(R_0(f)) = tr(R(f)) &= \int_{G(F)\mathbf{A}_G\backslash G(\mathbf{A}_F)} K_f(x,x)dx \\ &= \int_{G(F)\mathbf{A}_G\backslash G(\mathbf{A}_F)} \sum_{\gamma\in G(F)} f(x^{-1}\gamma x)dx \\ &= \int_{G(F)\mathbf{A}_G\backslash G(\mathbf{A}_F)} \sum_{\gamma/\sim} \sum_{\delta\in C_G(F)\backslash G(F)} f(x^{-1}\delta^{-1}\gamma\delta x)dx \\ &= \sum_{\gamma/\sim} \int_{C_\gamma(F)\backslash G(\mathbf{A}_F)} f(x^{-1}\gamma x)dx \\ &= \sum_{\gamma/\sim} \int_{C_\gamma(F)\backslash C_\gamma(\mathbf{A}_F)} \int_{C_\gamma(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} f(x^{-1}\delta^{-1}\gamma\delta x)d\delta dx \\ &= \sum_{\gamma/\sim} Vol(C_\gamma(F)\backslash C_\gamma(\mathbf{A}_F)) \int_{C_\gamma(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} F(x^{-1}\gamma x)d\dot{x} \end{aligned}$$

We thus define the *orbital integral*:

$$O_\gamma(F) = \int_{C_\gamma(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} f(x^{-1}\gamma x)d\dot{x}$$

And the *Tamigawa number*:

$$\tau(G_\gamma) = Vol(C_\gamma(F)\backslash C_\gamma(\mathbf{A}_F))$$

Both of these must be defined relative to the same measure (most often the Tamagawa measure) to make sense of this. We thus have that:

$$tr(R(F)) = \sum_\gamma \tau(C_\gamma)O_\gamma(f)$$

(One should think of this as a generalized class number).

Remark 14.3. One should note that $\tau(C_\gamma)$ is not naively factorizable but $O_\gamma(f)$ is. That is:

$$O_\gamma(\otimes_\nu f_\nu) = \prod_\nu O_{\gamma_\nu}(f_\nu)$$

where $O_{\gamma_\nu}(f_\nu) = \int_{C_\gamma(F_\nu)\backslash G(F_\nu)} f_\nu(x^{-1}\gamma x)d\dot{x}$.

The goal now is to give a single relative trace formula which can be made to work for non-compact quotients.

We give now wish to give a theorem of Hahn [25] which follows from the work of Jacquet, Rogowski, Delign, Kazhdan. We will define the terms used in the theorem in what follows.

Theorem 14.4. *Let $f = \otimes_\nu f_\nu \in C_c^\infty(G(\mathbf{A}_F))$ be such that there exist places ν_1, ν_2, ν_3 such that:*

(1) *f_{ν_1} is supported on relatively elliptic elements.*
(2) *f_{ν_2} is supported on (strongly) regular elements.*
(3) *f_{ν_3} is F-supercuspidal.*

Then:

$$\sum_{\{\gamma\}} \tau(C_\gamma)RO_\gamma^\chi(f) = \sum_{\pi\ cusp/\sim} RT_\chi(\pi(f'))$$

[25]

For the case when f is K_∞ finite we define:

$$RT_\chi(\pi(f')) = \sum_{\phi \in B_\pi} P_\chi(\pi(f')\phi, \phi)$$

(here f' is a certain integral of f over the center).

Definition 14.5. A representation f_ν is said to be *F-supercuspidal* if $\int_{U(F_\nu)} f(g)dg = 0$ for all parabolics $P = MU$ of G defined over F.

Exercise 2. Assume (for simplicity) that $Z_G(F_\nu)$ is compact then the matrix coefficients of supercuspidal representations are F-supercuspidal. (use $V \cong V/V(N)$).

For $H \subset G \times G$ we have a natural action of H on G via:

$$(h_l, h_r) \circ (g) \mapsto h_l g h_r^{-1}$$

for $\gamma \in G(R)$ we let H_γ be the stabilizer of γ. It is a linear algebraic group over F [43]. Moreover denote by $O(\gamma)$ the orbit of γ under this action.

Definition 14.6. An element $\gamma \in G(R)$ is said to be:

- *relatively semi-simple* if $O(\gamma)$ is closed (this implies H_γ is reductive [4].
- *relatively elliptic* if H_γ is anisotropic mod Z_G.
- *relatively regular* if $O(\gamma)$ has maximal dimension (with respect to $\gamma' \in G(F)$).
- *strongly relatively regular* if γ is regular and H_γ is connected.

Definition 14.7. We define the *set of relative classes* to be $\Gamma_r(R) = H(R)\backslash G(R)$. And the *set of geometric relative classes* to be $\Gamma_r^{geom} = Im(G(R) \to (H\backslash G)(R))$.

Example 14.8. For the case where H is the diagonal embedding of G in $G \times G$ alot of this coincides with the usual notions [54].

In this case we have that:

- $H_\gamma(R) = C_\gamma(R)$
- $O(\gamma)$ is the conjugacy class of γ
- γ is regular eliptic if and only if $\mathbf{Q}[C_\gamma]$ is a field of degree n over $\mathbf{Q}$ (the torus which stabilizes γ comes from a field).
- γ is strongly regular and semi-simple if and only if it is regular and semi-simple. That is that the centralizers of semi-simple elements are connected in GL_n (It is moreover known that if G^{der} is simply connected then the centralizers of semi-simple elements are connected [22])
- $\Gamma_r(R)$ is the set of conjugacy classes over R
- $\Gamma_r^{geom}(R)$ is the set of $\overline{F}$ conjugacy classes of R-points.

  The difference in the above is where Galois cohomology comes into play. [23]

14.1. Relative Orbital Integrals.

Definition 14.9. Let γ_ν be a relative semi-simple element. We say that γ_ν is *relevant* if X_ν is trivial on $H_\gamma^0(F_\nu)$.

We say that $\gamma \in G(F)$ is relevant if it is relavant for all ν.

[22] cite: Steinberg-Kotwitze

[23] for a reference see: Labless Asterique on Galois Cohomology

Definition 14.10. For $f_\nu \in C_0^\infty(F_\nu)$ and γ_ν relevant we define the *relative orbital integral*:

$$RO_\gamma^\chi(f_\nu) = \int_{H_\gamma^0\backslash H(F_\nu)} \chi_\nu(h_l h_r^{-1}) f_\nu(h_l^{-1}\gamma_\nu h_R)\frac{dh_r h_l}{df_\gamma}$$

We remark that $dh_l h_r$ is Haar measure on $H(F_\nu)$ and df_γ is Haar measure on H_γ so the resulting measure is a Radon measure (inner regular and locally finite).

Proposition 14.11. *Suppose χ is unitary then the relative orbital integral is absolutely convergeant.*

Remark 14.12. We would like to drop the word unitary, this is the work of Friedburg-Jacquet[24].

Proof. [25] Consider the map $A : H_\gamma^0(F_\nu)\backslash(F_\nu) \to O(\gamma)(F_\nu)$ defined by $(h_l, h_r) \mapsto h_l^{-1}\gamma h_r$. Then $A_* dh_l h_r$ gives a Radon measure on $O(\gamma)$ so it suffices to show that $A(H_\gamma^0(F_\nu)\backslash H(F_\nu)) \cap supp(f_\nu)$ is compact. To show this it suffices to show that $Im(A) \subset G(F_\nu)$ is closed as we have:

$$H_\gamma^0\backslash H \overset{finite}{\to} H_\gamma\backslash H \overset{\sim}{\to} O(\gamma) \hookrightarrow G$$

Where the final arrow is a topological embedding. Thus we must show that the map

$$H_\gamma^0(F_\nu)\backslash H(F_\nu) \to (H_\gamma\backslash H)(F_\nu)$$

is closed. In particular we need to show $H(F\nu)$ has closed image in $(H_\gamma\backslash H)(F_\nu)$

Exercise 3. Show that the image is indeed closed. [25]

□

Now $f \in C_c^\infty(G(\mathbf{A}_F))$ and γ is relevant. We have:

$$RO_\gamma^\chi = \int_{H_\gamma^0(\mathbf{A}_F)\backslash H(\mathbf{A}_F)} \chi(h_l^{-1}h_r) f(h_l^{-1}\gamma h_r)\frac{dh_l h_r}{df_\gamma}$$

To make sense of this we shall assume that we have $dh_l h_r = \otimes dh_l dh_R$, $df_\gamma = \otimes df_\gamma$ with $dh_l hr(K_\nu \cap H(F_\nu)) = df_\gamma(K_\nu \cap H(F_\nu)) = 1$ for allmost all ν.

15. Lecture 15: More on the relative trace formula

Let $H \subseteq G \times G$ be a reductive subgroup, where G is a reductive group, all defined over a number field F. Assume that all are conneceted. We'd like to study periods of automorphic forms on $G \times G$ over H. Given this data, there is an action of H on G. At the level of points it acts in the following way:

$$(h_l, h_r) \cdot g = h_l g h_r^{-1}.$$

Last time we wrote down several properties pertaining to this action (relatively semi-simple, etc).

Here is a special case which is not so well-known: take H to be the diagonal subgroup. Then $\mathcal{O}(\gamma)$, the orbit of γ, is just the conjugacy class of γ.

Facts: Suppose that G is quasi-split and the derived group G^{der} is simply connected. Then:

(1) if H is the diagonal group, the natural map $G(k) \to (H\backslash G)(k)$ is surjective, where k/F is a field extension. This result is by Kottwitz and Steinberg; for details see Kottwitz's paper "Rational conjugacy classes in algebraic groups" [38].
(2) the centralizer C_γ of any γ is connected, cf.[55, 8.1] May be due to Steinberg).

[24] Reference?

[25] Does this follow from fact that the imagem equals the kernel of a map into H^1 and is thus closed?

What are the analogues in the general case? This is an open question.

Note: Last time we wanted to show that if γ is relatively semi-simple, that is, its orbit is closed, then the relative orbital integral

$$\mathrm{RO}_\gamma(f) = \int_{H^0_\gamma(F_v)\backslash H(F_v)} \int f(h_l \gamma h_r^{-1}) dh_l dh_r / d\gamma$$

The measure $dh_l dh_r/dt_\gamma$ is a Radon measure on $H^0_\gamma(F_v)\backslash H(F_v)$. So to show the relative orbital integral is well-defind, it would be enough to construct a pull-back of

$$C_c^\infty(G(F_v)) \to C_c^\infty(\mathcal{O}(\gamma)(F_v)) \to C_c^\infty(H^0_\gamma(F_v)\backslash H(F_v))$$

to a diagram

$$H^0_\gamma(F_v)\backslash H(F_v) \to \mathcal{O}(\gamma)(F_v) \to G(F_v)$$

The rightmost map is fine as it is a closed embedding in the v-adic topology; we saw this at the start of the course [26]. What about

$$H^0_\gamma(F_v)\backslash H(F_v) \to \mathcal{O}(\gamma)(F_v)?$$

It is certainly continuous, but we want to show that it is *proper*, that is, the image of a closed set is a closed set. Then the pullback is defined and our integrals are well-defined.

15.1. **v-adic quotients.** Let M be an affine scheme over F_v. Assume we have an action $G \times M \to M$ where G is redcutive (above our G would actually be H). It is a fact, see Geometric Invariant Theory [44], that the categorical quotient $G\backslash M$ exists and is represented by $\mathrm{Spec}(F_v[M]^G)$. The natural map $M(F_v) \to H\backslash M(F_v)$ may not have dense image; for example the left side could be empty but not the right!

Definition 15.1. Let $[G(F_v)\backslash M(F_v)]$ be the set of closed $G(F_v)$-orbits in $M(F_v)$. This is the v-adic quotient of $M(F_v)$ by $G(F_v)$.

There exists a natural map $p\colon M(F_v) \to [H(F_v)\backslash M(F_v)]$ that sends x to the unique closed orbit containing x in the Zariski closure of $H(F_v)x$. The quotient $[G(F_v)\backslash M(F_v)]$ is Hausdorff, and its topology is the quotient topology induced by p.

See "Spherical characters on p-adic symmetric spaces" by Rader-Rallis [47], Proposition 2.5.

We certainly have a surjection

$$G(F_v)\backslash M(F_v) \to [G(F_v)\backslash M(F_v)]$$

which is continuous. In fact, the two topologies have the same open sets, but this projection collapses points. In particular, the map is proper. We'll use this nice map to factor the previous one that was giving us trouble.

We have a natural map from the v-adic quotient to the categorical

$$[G(F_v)\backslash M(F_v)] \to (G\backslash M)(F_v).$$

It sends a $G(F_v)$-orbit to the G-orbit it defines. One has the following

Theorem 15.2 (Rader-Rallis)**.** *This map is proper with finite fibers.*

For details see the same reference [47] as above.

[26] Reference?

Now we're fine because we can factor

$$\begin{array}{ccc} H^0_\gamma(F_v)\backslash H(F_v) & \longrightarrow & \mathcal{O}(\gamma)(F_v) \\ & \searrow \quad \nearrow & \\ & [H^0_\gamma(F_v)\backslash H(F_v)] & \end{array}$$

where the diagonal arrows are both proper. So the map that was perhaps troublesome is proper and we're done. So all of the local relative orbital integrals are well-defined. Now we'd like to show that the tensor product is also well-defined

Proposition 15.3. *Suppose that $\gamma \in G(F)$ is relevant and strongly relatively regular semi-simple (closed orbit and stabliser is connected). Then for almost every v,*

$$\mathrm{RO}^{\chi}_{\gamma}(\mathbf{1}_{K_v}) = \int_{H_\gamma(F_v)\cap K_v\times K_v\backslash H(F_v)\cap K_v\times K_v} dh_l dh_r/dt_\gamma = 1,$$

where $K_v = G(\mathcal{O}_{F_v})$ is hyperspecial and the Haar measures gives volume 1 to the interesction with $K_v \times K_v$.

Remark 15.4. Getz does not know how to prove the analogue when H_γ is not connected. For conjugacy classes one uses the theory of z-extensions, but don't know what the analogue is here.

Proof. We'll need the following facts: (1) Lang's theorem: if G is a connected linear algebraic group over a finite field $\mathbf{F}$ then $H^1(\mathrm{Gal}(\overline{F}F/\mathbf{F}), G)$ is a singleton set.

(2) $X\backslash O_{F_v}$ is smooth affine of finite type, then $X(\mathcal{O}_{F_v}) \to X(\mathbf{F}_v)$ is surjective[27].

Let $\gamma \in G(F)$ be a relatively regular element and let $\mathcal{O}(\gamma)$ be its orbit. We have maps

$$H^0_\gamma\backslash H \to H^0_\gamma\backslash H \to O(\gamma)$$

over F. The first is finite etale and the second is an isomorphism. Choose an embedding $G \to \mathrm{GL}_n$ over F, for some n. Let $\mathcal{G}$ be the schematic closure of G inside GL_n over $\mathcal{O}_F$. Similarly define $\mathcal{H}$, $\mathcal{H}_\gamma$ and $\mathcal{O}(\gamma)$.

Now assume $\mathcal{H}_\gamma$ is connected. For N sufficiently large, if v is coprime to N, then

$$\mathcal{H}_\gamma(\mathcal{O}_{F_v}) = K_v \times K_v \cap \mathcal{H}_\gamma(F_v)$$

(see Tits's paper [56] in Corvallis)and it is a hyperspecial subgroup. This implies in particular that $\mathcal{H}^0_{\gamma,\mathbf{F}}$ is *connected* because this reduction has the same type as the model over $\mathcal{O}_{F_v}$.

Let $\gamma' \in O(\gamma)(F_v)$ and consider the set of all $\overline{h}_l, \overline{h}_r \in \mathcal{H}(\overline{\mathcal{O}}_F)$ such that $\overline{h}_l\gamma\overline{h}_r = \gamma'$. This is a nontrivial $\mathcal{H}_\gamma$-torsor [28], call it $\mathcal{T}$. Then Lang's theorem implies that $\mathcal{T}(\mathbf{F}) \neq \emptyset$. This implies, by the other fact $\mathcal{T}(\mathcal{O}_{F_v})$ is not empty. Thus, there exist

$$(h_l, h_r) \in \mathcal{H}(\mathcal{O}_{F_v})$$

such that $h_l\gamma h_r\gamma'$, and this implies the proposition. ∎

Thanks to this proposition, we get well-defined adelic integrals.

[27] cite neron models

[28] add proof of simple transitivity of action

15.2. **Justification of previous trace formula derivation.** We begin with some notation.

$$A_{H,G} = A_H cap (A_G \times A_g)$$
$$A = \{(z,z) \in A_{H,G}\},$$
$$A = A_{H,G} \cap \Delta G(\mathbf{A}_F).$$

Choose Haar measures on A and $A_{H,G}$; one should be careful here, but for simplicity we will ignore these details.

For $f \in C_c^\infty(G(\mathbf{A}_F))$ define

$$f'(x) = \int_{A_G \backslash A_{H,G}} f(z_r^{-1} z_l x) dz_r z_l / da$$

Define

$$K^0_{f'}(x,y) = \sum_{[\gamma] \in K(F)} f'(x^{-1}\gamma y)$$ [29]

Theorem 15.5. *Suppose that H is connected. Then there exist places v_1 and v_2 of F such that f_{v_1} is supported on relatively elliptic elements and f_{v_2} is supported on strongly relatively regular elements. Then*

$$\sum_{[\gamma \in \Gamma_\gamma(F)]} \tau(H_\gamma)\, \mathrm{RO}^\chi_\gamma(f) = \int_{H(F)A_{H,G} \backslash H(\mathbf{A}_F)} \chi(x^{-1}, y) K_{f'}(h_l, h_r) dh_r dh_l.$$

Here $\tau(H_\gamma) = \mathrm{vol}(H_\gamma(F) \backslash H_\gamma(\mathbf{A}_F))$.

Note that the proof will show that the sum on the left is *finite* and the integral converges absolutely.

Proposition 15.6. *Suppose that H is connected and let $C \subseteq G(\mathbf{A}_F)$ be a compact subset, such that C_v is hyperspecial for almost all v. Then there exist only finitely many $H(F)\gamma \subseteq \Gamma_\gamma(F)$ with γ strongly relatively regularly semi-simple and $H(\mathbf{A}_F) \cap C \neq \emptyset$.*

We will first assume this proposition and prove the previous theorem:

Proof of theorem. Suppose that γ is strongly regular elliptic semi-simple. Then

$$\left|\mathrm{RO}^\chi_\gamma(f)\right| < \infty$$

by earlier work and $|\tau(H_\gamma)| < \infty$ by ellipticity. Let C be the closure of the support of f, so $C \subseteq G(\mathbf{A}_F)$. Then

$$\sum_{H(F)\gamma} \left|\tau(H_\gamma)\, \mathrm{RO}^\chi_\gamma(f)\right| < \infty$$

by the proposition. This implies that

$$\sum_\gamma \tau(H_\gamma)\, \mathrm{RO}^\chi_\gamma(f) = \sum_\gamma \tau(H_\gamma) \int_{H_\gamma(\mathbf{A}_F) \backslash H(\mathbf{A}_F)} \chi(h_l h_r^{-1}) f(h_l^{-1}\gamma h_r) dh_l h_r / dt_\gamma$$
$$= \sum_\gamma \tau(H_\gamma) \int_{(A \backslash H_\gamma(\mathbf{A}_F)) \backslash (A_{H,G} \backslash H(\mathbf{A}_F))} \chi(h_l^{-1}\gamma h_r) dh_l hr / dt_\gamma.$$

Notice that

$$\int_{H_\gamma(\mathbf{A}_F) \backslash H(\mathbf{A}_F)} \chi(h_l h_r^{-1}) f(h_l^{-1}\gamma h_r) dh_l h_r = 0$$

if γ is not relevant, because

$$\int_{H_\gamma(F)\backslash H_\gamma(\mathbf{A}_F)} \chi(g_1, g_2^{-1}) dg_1 dg_2 = 0$$

in this case. Thus

$$\begin{aligned}
&\sum_{\text{relevant } \gamma} \int_{A\backslash H_\gamma(F)\backslash A_{G,H}\backslash H(\mathbf{A}_F)} \chi(h_l, h_r) f'(h_l \gamma h_r) dh_l h_r \\
&= \int_{A\backslash H(F) A_{H,G} H(\mathbf{A}_F)} \chi(h_l, h_r^{-1}) \sum_{\gamma \in G(F)} f'(h_l^{-1} \gamma h_r) dh_l h_r \\
&= \int_{A\backslash H_\gamma(F)\backslash A_{G,H}\backslash H(\mathbf{A}_F)} \chi(h_l, h_r^{-1}) H_f^\rho(h_l, h_r) dh_l h_r.
\end{aligned}$$

□

Now we must prove the proposition:

Proof. Let $B\colon G \to H\backslash G$ be the projection and $C \subseteq G(\mathbf{A}_F)$ be a compact such that C_v is hyperspecial for almost all v. Then $B(C) \cap H\backslash G(F)$ is a finite set, because $(H\backslash G)(F) \subseteq (H\backslash G)(\mathbf{A}_F)$ is discrete. If γ is relatively semisimple then $O(\gamma) \subseteq G$ is Zariski closed. Thus if γ, γ' is relatively semisimple and $B(\gamma) = B(\gamma'$, then there exists $\overline{h} \in H(\overline{F})$ such that $\overline{h}\gamma = \gamma'$. This implies that γ is equivalent with γ' in $\Gamma_r^{geo}(F)$. Therefore, there exist only finitely many elements of $\Gamma_r^{geo}(F)$ that intersect C.

Now we need to show that there exist only finitely many strongly relatively regular semisimple γ in a given geometric class that intersect C. Let γ be strongly relatively regular semisimple from above, there exists a finite set of places V such that if $v \notin V$, then $\gamma_v \in C_v$ and if γ_v' is in the geometric class of γ_v and its class intersects C_v, then $H(F_v)\gamma_v = H(F_v)\gamma_v'$. This is the content of the result that the relative orbital integrals are almost always 1.

On the other hand, Galois cohomology implies that there are only finitely many semisimple elements of $\Gamma_r(F_v)$ in a given geometric class, that is, elements of $\Gamma_r(\overline{F})$.

In sum, there exists finitely many $H(\mathbf{A}_F)\gamma \in \Gamma_r(\mathbf{A}_F)$ such that $H(\mathbf{A}_F)\gamma' \cap C \neq \emptyset$ and $\gamma_v \in H(F_v)\gamma_v'$ for all v.

Now, the set of elements of $\Gamma_r(F)$ contained in a given element of $\Gamma_r(\mathbf{A}_F)$ (which is storngly relatively regularly semisimple). It injects into a group

$$\mathfrak{E}(H_\gamma, H, \mathbf{A}_F/F)$$

which is finite by Lemma 1.8.5 of Labesse's Asterisque paper "Cohomologie, stabilisation, changement de base" [39]. □

16. Lecture 16: The Simple Trace Formula

16.1. **Towards the simple trace formula.** We were half-way in establishing the simple trace formula. Recall that we have a picture $H \leqslant G \times G$ of connected reductive groups over a number field F, and that for $f \in C_c^\infty(G(\mathbf{A}_F))$ satisfying certain properties and $\chi : H(\mathbf{A}_F) \to \mathbf{C}^\times$ a character trivial on $H(F)A_{H,G}$,

$$\sum_{\gamma \in \Gamma_r(F)} \mathrm{RO}_\gamma^\chi(f) = \int_{H(F)A_G\backslash H(\mathbf{A}_F)} \chi(h_\ell, h_r^{-1}) K_f(h_\ell, h_r) \mathrm{d}h_\ell h_r.$$

This is for the geometric side of the trace formula. Our next goal is to obtain a spectral expansion of the kernel.

Consider $f^1 \in C_c^\infty(A_G\backslash G(\mathbf{A}_F))$ given by $f^1(g) = \int_{A_{H,G}} f(z_\ell z_r^{-1}g)\mathrm{d}z_\ell z_r$, and write $R_0(f^1) = \sum_\pi m(\pi)\pi(f^1)$, where the sum is over a system of representatives π of equivalence classes of cuspidal automorphic representations of $A_G\backslash G(\mathbf{A}_F)$ (two such representations being considered equivalent when they differ by a twist of a quasi-character of A_G). The space $L_0^2(G(F)A_G\backslash G(\mathbf{A}_F))$ (to be denoted L_0^2 from now on) can be decomposed according to the action $R_0(f)$ as $L_0^2 = \sum_\pi V_\pi$, where $V_\pi \subset L_0^2$ is the subspace consisting of m_π copies of π. Let B_π be an orthonormal basis of V_π with respect to the pairing $\langle \phi_1, \phi_2\rangle = \int_{G(F)A_G\backslash G(\mathbf{A}_F)} \psi_1(g)\overline{\phi_2(g)}\mathrm{d}g$. Then $m_\pi\pi(f^1)$ has kernel $K_{\pi(f^1)}(x,y) = \sum_{\phi\in B_\pi}(\pi(f^1)\psi)(x)\phi(y)$.

Claim 16.1. *There exists a unique square integrable function that is smooth in x and y separately, and represents $K_{\pi(f^1)}$ (cf.[2] Lemma 4.5, 4.8).*

Assume that $H \leqslant G\times G$ can be decomposed as $H = H_\ell \times H_r$. Define the *relative trace*

$$\mathrm{RT}_H^\chi(\pi(f^1)) := \int_{H(F)A_G\backslash H(\mathbf{A}_F)} \chi(h_\ell, h_r^{-1})K_{\pi(f^1)}(h_\ell,h_r)\mathrm{d}h_\ell h_r.$$

Note that this definition makes sense. In fact, as claimed above, $K_{\pi(f^1)}(x,y)$ is smooth in each variable separately, and it is a fact due to Harish-Chandra, that if a function in L_0^2 is smooth, then it is rapidly decreasing (cf.[27]§4). Therefore, by a Fubini-type argument, using Prop. 13.2 for each variable, we see that $\mathrm{RT}_H^\chi(\pi(f^1))$ is indeed well-defined and finite.

Remark 16.2. Suppose that $\mathrm{RT}_H^\chi(\pi(f^1))$ does not vanish. Then π is both (H_ℓ,χ_ℓ)- and (H_r,χ_r)-distinguished. (But note that this gives no info on the embedding $\iota : V_\pi \hookrightarrow L_0^2$ for which this happens.)

Let's refine $\mathrm{RT}_H^\chi(\pi(f^1))$. First note that $(f^1)^\infty$ is K^∞-finite for some $K^\infty \subset G(\mathbf{A}_F^\infty)$, but that this is not necessarily so for $(f^1)_\infty$. So we assume further that $(f^1)_\infty$ is K_∞-finite. Then $\dim_{\mathbf{C}}(\mathrm{im}\ \pi(f^1))$ is finite, since K_∞-vectors in L_0^2 are admissible by a result of Harish-Chandra mentioned in a previous lecture. In this case, $\mathrm{RT}_H^\chi(\pi(f^1)) = \sum_{\pi\in B_\pi} P_{\chi_\ell}(\phi)\overline{P_{\chi_r}(\phi)}$ (where the orthonormal basis B_π is constructed starting with a basis of K_∞-finite vectors, and completed to a basis for V_π), where

$$P_{\chi_?}(\phi) = \int_{(A_{H_?}\cap A_G)H_?(F)\backslash H_?(\mathbf{A}_F)} \chi_?(g)\phi(g)\mathrm{d}g, \quad (? \in \{\ell, r\}).$$

A great deal of the work today will be devoted to the proof of the following:

Proposition 16.3. *Let $f \in C_c^\infty(G(\mathbf{A}_F))$, and assume that $R(f^1)$ has image in L_0^2. Then*

$$\int_{H(F)A_{H,G}\backslash H(\mathbf{A}_F)} \chi(h_\ell, h_r^{-1})K_f(h_\ell,h_r)\mathrm{d}h_\ell h_r = \sum_\pi \mathrm{RT}_H^\chi(\pi(f^1)).$$

Moreover, the integral on the left and the sum on the right are absolutely convergent.

Proof. Recall that $K_f(x,y)$ is smooth in each variable separately, and that viewed this way it is in $L_0^2\times L_0^2$; so applying Harish-Chandra's result alluded to before, we see that the kernel is rapidly decreasing in each variable, and that the same is true of $K_{\pi(f^1)}(x,y)$. Some formal considerations give:

$$\begin{aligned}\int_{H(F)A_{H,G}\backslash H(\mathbf{A}_F)}\chi(h_\ell,h_r^{-1})K_f(h_\ell,h_r)\mathrm{d}h_\ell h_r &= \int_{H(F)A_{H,G}\backslash H(\mathbf{A}_F)}\chi(h_\ell,h_r^{-1})\sum_\pi K_{\pi(f^1)}(h_\ell,h_r)\mathrm{d}h_\ell h_r \\ &= \sum_\pi\int_{H(F)A_{H,G}\backslash H(\mathbf{A}_F)}\chi(h_\ell,h_r^{-1})K_{\pi(f^1)}(h_\ell,h_r)\mathrm{d}h_\ell h_r \\ &= \sum_\pi \mathrm{RT}_H^\chi(\pi(f^1)).\end{aligned} \tag{16.1.0.2}$$

Of course, the equality (16.1.0.2) requires some justification. By a result that can be found in [2]§4, we have point-wise convergence, $K_{f^1}(x,y)=\sum_\pi K_{\pi(f^1)}(x,y)$; so by Lebesgue's dominated convergence theorem, is suffice to show that $\sum_\pi|K_{\pi(f^1)}(x,y)|$ is integrable over $H(F)A_{H,G}\backslash H(\mathbf{A}_F)$. Choose $m>0$ sufficiently large. There exists K_∞-finite functions $g_i\in C_c^m(A_G\backslash G(\mathbf{A}_F))$ $(i=1,2)$ and K_∞-finite Z in the universal enveloping algebra $\mathrm{U}(\mathrm{Lie}(A_G\backslash G(\mathbf{A}_F^\infty)\otimes\mathbf{C}))$ such that is $r_0>\deg(Z)$ and $h\in C_c^\infty(A_G\backslash G(\mathbf{A}_F))$, then we can write

$$h=\sum_{i=1}^2 h_i+g_i,$$

where $h_1:=h*Z$ and $h_2:=h$ (cf. [2] Cor. 4.2.). Notice that the h_1 is K_∞-finite if h is.

Write $g^*(x)$ for $\overline{g(x^{-1})}$, and assume for the moment that $h\in C_c^\infty(A_G\backslash G(\mathbf{A}_F))$ is K_∞-finite for some $r>r_0$. Then

$$\begin{aligned}\left|K_{\pi(h)}(x,y)\right| &= \left|\sum_{\phi\in B_\pi}\pi(h)\phi(x)\overline{\phi(y)}\right| \\ &= \left|\sum_{i=1}^2\sum_{\phi\in B_\pi}\pi(h_i)\phi(x)\overline{\pi(g_i^*)\phi(y)}\right| \\ &\leq \sum_{i=1}^2\left(\sum_{\phi\in B_\pi}|\pi(h_i)\phi(x)|\right)\cdot\left(\sum_{\phi\in B_\pi}|\pi(g_i^*)\phi(y)|\right) \qquad \text{(Cauchy-Schwarz)} \\ &= \sum_{i=1}^2 K_{\pi(h_i*h_i^*)}(x,x)\cdot K_{\pi(g_i^**g_i)}(y,y) \qquad (\text{since}\geq 0 \text{ and using } K_\infty\text{-finiteness}).\end{aligned}$$

Therefore we see that, assuming h is K_∞-finite,

$$\left|\sum_\pi K_{\pi(h)}(x,y)\right|\leq\sum_{i=1}^2\left(\sum_\pi K_{\pi(h_i*h_i*)}(x,x)\right)\cdot\left(\sum_\pi K_{\pi(g_i^**g_i)}(y,y)\right). \tag{16.1.0.3}$$

Claim 16.4. *The inequality (16.1.0.3) holds for arbitrary h.*

A first idea to try to show this could be using some continuity argument. But there is a subtlety due to the relevant topology in $C_c^r(A_G\backslash G(\mathbf{A}_F))$ (defined by seminorms from certain differential operators): the K_∞-finite functions do *not* constitute a dense subspace. The trick is then to view an element in $C_c^r(A_G\backslash G(\mathbf{A}_F))$ as in the larger space $C_c^{r-\ell}(A_G\backslash G(\mathbf{A}_F))$, for suitably chosen ℓ, where then any h as in the claim can be approximated (cf. Hahn).

Our next goal is to show that each of the factor in the RHS of (16.1.0.3) is rapidly decreasing. Choose $L>0$ and enlarge m if necessary so that g becomes smooth. Then it follows from [1]§2 that

$$\sum_{\pi} \left|\Lambda_2^T K_{\pi(h_i * h_i^*)}(x,x)\right| \leq C(h_i)\|x\|^{-L},$$

and that

$$\sum_{\pi} \left|\Lambda_2^T K_{\pi(g_i * g_i^*)}(y,y)\right| \leq C(g_i)\|y\|^{-L},$$

where Λ_2^T are Arthur's *truncation operators*. Now since $R(f^1)$ has image in L_0^2 by assumption, $K_{\pi(h_i*h_i^*)}(x,x)$ and $K_{\pi(g_i^* * g_i)}(y,y)$ are not altered by Λ_2^T ([1]§1), so they are rapidly decreasing by the above estimates, and we can apply Prop. 13.2. The proposition follows. □

Remark 16.5. Points to work out or think about:

- Relationship between the different notions of *rapidly decreasing* (cf. Borel, Borel-Jacquet).
- In Prop. 13.2, is the smoothness assumption really necessary?

This formula has following two notable specializations:

16.2. The simple trace formula.

The last step to prove the simple trace formula is the following criterion for things to live in the cuspidal subspace:

Lemma 16.6. *Suppose $f = \otimes_v f_v \in C_c^\infty(A_G\backslash G(\mathbf{A}_F))$. If f_v is F-supercuspidal for some v, then* $\operatorname{im} R(f^1)$ *is in L_0^2.*

Proof. For $\psi \in L_0^2$, and $P = MN$ an F-rational parabolic, we have

$$\begin{aligned}\int_{N(F)\backslash N(\mathbf{A}_F)} R(f^1)\psi(n^{-1}x)\mathrm{d}n &= \int_{N(F)\backslash N(\mathbf{A}_F)} \int_{N(F)A_G\backslash G(\mathbf{A}_F)} \sum_{y\in N(F)} f^1(x^{-1}nyg)\psi(g)\mathrm{d}g\mathrm{d}n \\ &= \int_{N(F)A_G\backslash G(\mathbf{A}_F)} \int_{N(\mathbf{A}_F)} f(x^{-1}ng)\mathrm{d}n\psi(g)\mathrm{d}g \\ &= 0,\end{aligned}$$

since $\int_{N(\mathbf{A}_F)} f(x^{-1}ng)\mathrm{d}n = \prod_v \int_{N(F_v)} f_v(x_v^{-1}n_v g_v)\mathrm{d}n = 0$ (a.e.), by the assumption on f at some place v. □

We have thus concluded to proof (assuming $H \leqslant G \times G$ connected and of the form $H_\ell \times H_r$) of the following result – the relative simple trace formula – originally due to Hahn:

Theorem 16.7. *Let $f = \otimes_v f_v \in C_c^\infty(G(\mathbf{A}_F))$ be such that there exists place v_1 and v_2, and some place v_3 such that the following hold:*

1. *f_{v_1} is supported on relative elliptic elements;*
2. *f_{v_2} is supported on relative strongly regular elements;*
3. *f_{v_3} is F-supercuspidal.*

Then

$$\sum_{\gamma\in\Gamma_r(F)} \tau(H_\gamma)\mathrm{RO}_\gamma^\chi(f) = \sum_\pi \mathrm{RT}_H^\chi(\pi(f^1)),$$

and both sides converge absolutely.

The usual trace formula. Take $H = G \times G$, and consider the diagonal embedding $(g_1, g_2) \mapsto (g_1, g_2; g_1, g_2) : H \longrightarrow G \times G$. Then (exercise!):

$$\mathrm{RT}_H(\pi(f_1 \times f_2) = m(\pi)\mathrm{tr}(\pi(f_1 * f_2^\vee)),$$

where $f_2^\vee(g) := f_2(g^{-1})$, and

$$\mathrm{RO}_{(\gamma_1,\gamma_2)}(f_1 \times f_2) = \mathrm{O}_{\gamma_2\gamma_1^{-1}}(f_1 * f_2^\vee),$$

where $\mathrm{O}_\gamma(f) = \int_{C_\gamma(\mathbf{A}_F)\backslash C(\mathbf{A}_F)} f(g^{-1}\gamma g)\mathrm{d}g$.

Note that given $f \in C_c^\infty(G(\mathbf{A}_F))$, there exist f_1, f_2 such that $\sum_i f_{i,1} \times f_{i,2} = f$. This is clearly true over F_v, for v a finite place, and at ∞ this is a theorem due to Dixmier and Malliavin (cf. [22]).

The twisted trace formula. Let σ be an automorphism of G, and consider the embedding of $(g_1, g_2) \mapsto (g_1, g_2^\sigma; g_1, g_2)$ of $H = G \times G$ into $G \times G \times G \times G$. Then $\mathrm{RT}_H(\pi_1\pi_2(f_1 \times f_2))$ is related to $\mathrm{tr}\ \pi_1(f_1)\pi_2(f_1 \times \sigma)$ if $\pi_1 \cong \pi_2^\vee$ and $\pi_1 \cong \pi_1^\sigma$, and also

$$\mathrm{RO}_{(\gamma_1,\gamma_2)}(f_1 \times f_2) = \mathrm{TO}_{\gamma_2\gamma_1^{-1}}(f_1 * (f_2^\vee)^\sigma),$$

where $\mathrm{TO}_\gamma(f) = \int_{C_\gamma^\sigma(\mathbf{A}_F)\backslash G(\mathbf{A}_F)} f(g^{-\sigma}\gamma g)\mathrm{d}g$, with $C_\gamma^\sigma = \{g \in C(R) \ : \ g\gamma g^{-\sigma} = \gamma\}$.

Remark 16.8. The theory of *endoscopy* deals with the comparison of trace formulæ and relative twisted trace formulæ for a well chosen pair of groups and automorphism (the twists correspond to *functorial lifts* of representations to another group that is dictated by endoscopy).

Corollary 16.9 (of the simple trace formula). *Let G be a (connected) reductive group with $Z_G(F_{v_i})$ compact for $v_1, \dots, v_n$, and let $\rho_{v_1}, \dots, \rho_{v_n}$ be a collection of supercuspidal representations. Then there exists a cuspidal automorphic representation π of $G(\mathbf{A}_F)$ such that $\rho_{v_i} \cong \pi_{v_i}$ for all i.*

Proof. Let $f \in C_c^\infty(G(\mathbf{A}_F))$, and assume that for each i, f_{v_i} is a coefficient of ρ_{v_i}.

Claim 16.10. *f_{v_i} is F-supercuspidal.*

In fact, for $P = MN$ an F-parabolic, we have $\int_{N(F_v)} f(ng)\mathrm{d}n = 0$, since otherwise the assignment $\psi \mapsto (g \mapsto \int_{N(F_{v_i})} \phi(ng)\mathrm{d}n)$ would give a non-zero map $V_{\rho_{v_i}} \longrightarrow V_{\rho_{v_i}}/V_{\rho_{v_i}}(N)$, and therefore $V_{\rho_{v_i}}/V_{\rho_{v_i}}(N)$ would be non-zero, and ρ_{v_i} not supercuspidal by a theorem of Jacquet, contrary to the hypothesis.

So we need to show that the assumptions (1) and (2) in Theorem 16.7 can be satisfied and that the geometric side is non-zero. But we can always do that, since we have a submersion $g \mapsto g^{-1}tg : T^{\mathrm{reg}}(F_v) \times G(F_v) \longrightarrow G(F_v)$, as follows from *Weil's integration*. ☐

17. Lecture 17: ?

[30]

Remark 17.1. There was a corrective comment to previous notes regarding definition of elliptic [31]

Last time we showed that supercuspidals can be realized as local factors in cuspidal automorphic representations (that is they can be globalized). The same is true for the discrete series (that is that matrix coefficients are in $L^2(G(F_\nu))$), but we won't prove this.

[30] title

[31] Should correct in place and not here

The moral of the relative trace formula is that spectral information is controlled by geometric information. For an illustration of this, consider the question "Do there exist cuspidal automorphic representations?" One method to show their existance is to explicitely construct them using Poincare series. For $\mathrm{SL}_2(\mathbf{Z})$ this amounts to averaging a function invariant under $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and averaging over the quotient. This method however won't work if G_F is anisotropic (mod center) as in this case there are no cusps. However, even in this context the trace formula does work, and the point is that we only need to show that there exist elliptic elements.

If we compare to the above the case of $\Gamma \subset \mathrm{PGL}_2(\mathbf{R})$ where Γ is not arithmetic then cusp forms are hard to come by ([45]). Indeed, generically $L^2_0(\Gamma \backslash \mathrm{PGL}_2(\mathbf{R}))$ is small.

A comment is that in order to exhibit cusp forms via the relative trace formula one must exhibit eliptic elements. These always exists [32].

Another question might ask is, given fixed (H, χ) with $H \subset G$ a reductive subgroup and $\chi : H(\mathbf{A}_F) \to \mathbf{C}$ a quasicharacter trivial on $(\mathbf{A}_G \cap H(\mathbf{A}_F))H(F)$ do there exist cuspidal automorphic representations π which are (H, χ) distinguished?

We have the following vanishing theorem:

Theorem 17.2 (AGR[33]). *If (G, H) is one of the following pairs of groups over $\mathbf{Q}$ then no H-distinguished cuspidal automorphic representations of $G(\mathbf{A}_\mathbf{Q})$ exist.*

(1) $\mathrm{GL}_{n+k}, \mathrm{SL}_n \times \mathrm{SL}_k$ *for* $n \neq k$.
(2) GL_{2n} *or* GL_{2n+1} *with* Sp_{2n}.
(3) $\mathrm{SO}(n, n)$ *with* SL_n *for* n *odd.*
(4) $\mathrm{Sp}_{2(n+k)}$ *with* $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2k}$.
(5) Sp_{2n} *with* $\mathrm{Sp}_{2(n-l)}$ *with* $4l < n$.
(6) $O(Q)$ *with* $O(Q_l)$ *where* $Q = Q_l \oplus Q_l^\perp$ *with* $2dim(Q_L^\perp)$ *less than the witt index of* Q.

Note that there may still be distinquished automorphic forms, just not cuspidal ones. Moreover, distinction may not be the end of the story, we could refine the notion and consider "Arithmetic Distinction" [34].

We now proceed with the question of how to actually construct distinguished representations.

We have the following corrolary to the relative trace formula:

Corollary 17.3. *Suppose that there exists an element $\gamma \in G(F)$ which is both strongly relatively eliptic regular semi-simple and eliptic regular semi-simple. Then, there exists a cuspidal representation π of $G(\mathbf{A}_F)$ which is distinguished by H.*

A question that arises from this is: "Is the converse true?" The answer might follow from a more general relative trace formula.

Lemma 17.4. *Let ν be a place of F, the set of elements of $G(F_\nu)$ which are eliptic semi-simple is open. Moreover, the set of elements which are stronly relatively eliptic is open.*

Proof of Corrallary. Choose a positive real valued function of $C_c^\infty(G(\mathbf{A}_F))$ and places ν_1, ν_2 such that γ_{ν_1} is relatively eliptic semi-simple and γ_{ν_2} is eliptic. Choose f_{ν_1} to have support in the strongly relatively regular eliptic semi-simple elements and f_{ν_2} to have support in the elliptic regular semi-simple elements. Apply the relative trace formula. ☐

Remark 17.5. It is unclear where the assumption of regularity is used in the proof.

[32] find a reference
[33] cite
[34] ref W. Zhang

Non-relative part of lemma. Let $\{T_\alpha\} \subset G_F$ be a set of representatives of the conjugacty classes of eliptic maximal tori in G_{F_ν}. It is a finite list, and moreover the image of the map $(\cup_\alpha T_\alpha^{reg}) \times G(F_\nu) \to G(F_\nu)$ given by $(t, g) \mapsto g^{-1}tg$ equals the set of regular eliptic elements of $G(F_\nu)$. Moreover, this map is a subversion [28] [46]. This result can be viewed as part of the Weyl Integration formula over $\mathbf{R}$ proved in [35][36] □

For the relative part of the lemma we shall need more machinery. We point out that the above essentially uses the Chevalley restriction theorm which says that if $T \subset G$ is a maximal torus then $T/W(T,G) \simeq G/\text{conj}$.

Now, recall that if G acts on $X \times Y$ then we have the notation $X \times_G Y = X \times Y/G$. Let X be an affine G-variety over F and choose $x \in X(F)$ such the G-orbit of X is closed (that is so that G_x is reductive [42, Matsuchima's Theorem].) Let $S \subset X$ be a G_x stable subvariety (for example T_α^{reg} in the previous case) with $x \in S(F)$. Let G_x act on $G \times S$ on the right via $(g, s) \circ g_0 = (gg_0, g_0^{-1}s)$. Consider the natural map $\theta : G \times S \to X$, then θ is constant on G_x orbits and induces a map $\varphi : G \times_{G_x} S \to X$ and we moreover have that $(G \times_{G_x} S)/G \to S/G_x$.

Definition 17.6. We say that S as above is an *etale slice* of the action of G on X if φ and φ/G are etale and the cartesian diagram:

$$\begin{array}{ccc} G \times_{G_x} S & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ (G \times_{G_x} S)/G \simeq S/G & \xrightarrow{\varphi/G} & X/G \end{array}$$

is commutative.

Theorem 17.7. *If G_x is closed then Etale slices exist.* [13]

The relative part. Now suppose that G and X in the above are H and G from the lemma. Suppose that $x \in G(F_\nu)$ is relatively regular semi-simple elliptic (so that the H orbit is closed and of maximal dimension). Let $P : G \to G/H$ be the quotient map (under the inclusion $H \subset G \times G$). There exists an open $U \subset G/H$ such that $P(\gamma) \in U(F)$ and if $y \in P^{-1}(U)$ then H_y is conjugate to a subgroup of H_γ. (this follows from the etale slice theorem). But since γ is regular we have that $H_y \subset H_\gamma$ implies $H_y = H_\gamma$. Thus U is Zarisky open which implies that $U(F_\nu)$ is open. □

We remark that this gives a geometric criterion for the existance of distinguished representations. Some usefull projects to extend this would be: Apply this to produce families of distinguished representations or be more ambitious and do "De George-Wallak".

Exercise 4. Prove that there are no elements of GL_3 wich are both relatively elliptic with respect to $\mathrm{GL}_2 \times \mathrm{GL}_1$ and elliptic. Compare this to the case of $\mathrm{GL}_1 \times \mathrm{GL}_1 \subset \mathrm{GL}_2$.

The problem of understanding distinction for a general H would currently be too ambitious, we shall instead focus on nice families which seem more tractable. For example $H_l = H_r = G^\sigma$ where $\sigma : G \to G$ is an automorphism.

In the case where σ has order two then G/H_l is what is known as a symmetric space.

A usefull fact in this area is that we can classify the automorphisms of (nice) G.

Proposition 17.8. *There exists an exact sequence:*

$$1 \to (G/Z)(\overline{F}) \to Aut(G_{\overline{F}})^{35}$$

[35] This is obviously missing a third term... B. Conrad says: If we map automorphisms to automorphisms of based root datum, then it is always surjective and split. If one maps to automorphisms of the Dynkin diagram, then one must assume that G is semisimple in order for the map to be defined.

If G is semi-simple and either simply connected or adjoint then the last map is a surjection and the sequence splits. [37]

Example 17.9.
- Conjugations, for example by $\begin{pmatrix} id_m & 0 \\ 0 & -id_n \end{pmatrix}$ gives $G^\sigma = G_m \times G_n$.
- $g \mapsto g^{-t}$ gives the orthogonal group (this is the outer automorphism for the Dynkin diagram of time A_n).
- Composition of conjugation by $\begin{pmatrix} 0 & id_n \\ id_n & 0 \end{pmatrix}$ with g^{-t} gives $\mathrm{Sp}_2\, n$.
- Let M/F be a quadratic extension and $\bar{\cdot}$ be the nontrivial galois element then have:

$$U_{n_1,n_2}(R) = \{g \in \mathrm{GL}_n(M \otimes R) | \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{g}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = g\}$$

18. Lecture 18: Even more on distinction

18.1. Cases when one can characterize distinction.

Theorem 18.1 (Jacquet)**.** *Let M/F be a quadratic extension, let $G = R_{M/F}\,\mathrm{GL}_n$ and let H_l be a quasi-split unitary group. Then a cuspidal automorphic representation π of $G(\mathbf{A}_F)$ is distinguished by H if and only if $\widetilde{\pi}_a \cong \pi$, where the tilde denotes the action of $\mathbf{G}_a(M/F)$.*

Note that the condition in the theorem above is equivalent to π being a lift from $\mathrm{GL}_n(\mathbf{A}_F)$, by work of Arthur and Clozel.More concretely, π is a base change of ρ on $\mathrm{GL}_n(\mathbf{A}_F)$ in this setting if and only if $L(s,\pi) = L(s,\rho)L(s,\rho\otimes\chi)$ where the group generated by χ is exactly $\mathrm{Gal}(M/F)^\vee$. See "Kloosterman identities over a quadratic extension" part I and II in [31] and [32].

For another example, take $H_l = \mathrm{GL}_n \times \mathrm{GL}_n \subseteq \mathrm{GL}_{2n}$ to be the the fixed points of $\begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix}$. We have a character

$$\mu_s \colon H(\mathbf{A}_F) \to \mathbf{C}^\times$$

given by the formula

$$\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \mapsto |\det q_1/\det q_2|^{s-1/2}\,\chi(\det q_1/\det q_2)\eta(\det q_2),$$

where χ and η are characters of $\mathbf{A}_F$. In this setting one has the following

Theorem 18.2 (Friedberg-Jacquet)**.** *A cuspidal automorphic representation π on $G(\mathbf{A}_F)$ is (H,μ_s)-distinguished if and only if $L(s_0,\Lambda^2\pi\otimes\eta)$ has a pole at $s=1$ and $L(s,\pi\otimes\chi)\neq 0$.*

See Frieberg-Jacquet's "Linear Periods" [23] for details.

Remark 18.3. Note that in this field we always take the critical strip to be in the region with real parts in $(0,1)$, unlike the usual normalizations for L-functions of modular forms, say.

Note in the theorem above that if $\eta = 1$ then the theorem is the same as saying that π is a lift from Sp_{2n} or $O_?$ (don't remember the number; look it up!). Ash and Ginzburg use this to construct p-adic L-functions under a technical hypothesis; see their Inventiones paper for GL_{2n} [3], for example.

Here is another case when we can characterize distinction.

Theorem 18.4 (Jacquet-Rallis)**.** *There are no cuspidal automorphic representations on GL_{2n} which are distinguished by Sp_{2n}.*

See their paper "Symplectic Periods" [33] for more details.

Now we consider a different situation. Take $H_l = \mathrm{GL}_{n/F}$ which embeds into $G = \mathrm{Res}_{M/F}\,\mathrm{GL}_{n/F}$. This can be treated via the Rankin-Selberg method (Flicker). Taking $H_l = O_n \hookrightarrow \mathrm{GL}_n$, one can consult work of Jacquet on Waldspurger's theorem [30] in the case $n = 2$.

In complete generality, one would like an answer to the following:

Question: Given $H_l \subseteq G$, can we predict which cuspidal representations π will be distinguished?

The most sophisticated look into this is due to Sakellaridis and Venkatesh, but their paper is currently in preparation. They have proved results of the following form: if $H_l \backslash G$ is *spherical* and everything splits, then we can say which spherical π_v are distinguished for v local.

Definition 18.5. An algebraic subgroup $H_l \subseteq G$ is said to be *spherical* if $\mathrm{Hom}_{H_l}(V, V_{triv})$ is at most one dimensional for all algebraic finite irreducible $\rho\colon G \to V$.

(Recall that we proved at some point that $G_F(V)$ along with a hyperspecial subgroup make up a Gelfand pair. This should suggest a connection between this definition of global spherical resp and our earlier local one.)

The condition that everything splits above is very annoying and restrictive. A lot of work remains to remove this condition. These results are taken from Sak in Compositio.

18.2. **Predicting functorial lifts for distinguished representations.** In the usual trace formula, the geometric side is indexed by conjugacy classes. Let $T \subseteq G$ be a maximal torus. We have the Chevalley restriction theorem, so that

$$T/W(T, G) \cong G/\text{conjugacy}.$$

Thus $T/W(T, G)$ "controls the conjugacy class", so we should look for something like it in the general relative setting.

Let's now put ourselves in the following situation: suppose $\sigma\colon G \to G$ is an involution. Set $H = (G^\sigma)^0 \times (G^\sigma)^0 \hookrightarrow G \times G$. Now we'll examine $G/H \cong G^\sigma \backslash G/G^\sigma$. This parameterizes the geometric side of the relative trace formula for $H \subseteq G$.

Have the *moment map*,

$$B_\sigma\colon G \to G$$

taking $g \mapsto gg^{-\sigma}$. Let Q denote the scheme theoretic image of B_σ. Thus $Q(R) = \{g \in G(R) \mid g = g^{-\sigma}\}$. We have a map

$$H_l \backslash G/G^\sigma \xrightarrow{B_\sigma} H_l \backslash G,$$

where the action on the right is via conjugation. This is defined concretely as $B_\sigma(hgg\sigma) = hgg^{-\sigma}h^{-\sigma}$.

Definition 18.6. A torus $T \subseteq G$ is said to be *σ-split* if $t^{-1} = t^\sigma$ for $t \in T(R)$.

Remark 18.7. Richardson's Inventiones paper "Orbits and invariants attached to involutions of reductive groups" [49] is a good reference, and Vast also has papers on this. Beware that Richardson says "σ-anisotropic" instead of our σ-split.

Here are some facts: (1) if T is σ-split, then $T \subseteq Q$. Why? Because have an isogeny $T \to T$ given by $t \mapsto tt^{-\sigma} = t^2$. This is a surjective map of schemes (hence not necessarily surjective on points!). Since Q is the scheme theoretic image of the map B_σ, we're okay. (2) σ-split tori exist and a maximal σ-split torus is contained in a maximal σ-stable torus of G. (Helmink, "Tori invariant under an involutory.."; see his website[36]).

[36] I checked but could not find this reference

If T is a σ-stable torus, let $T_\sigma \subseteq T$ be the maximal σ-split subtorus. Then $T = T^\sigma T_\sigma$ where T^σ is the subtorus fixed by σ. Note that $T_\sigma \cap T^\sigma$ is a finite group scheme consisting of elements of order 2. We have a Weyl group attahced to this torus

$$W(T_\sigma, H) = N_H(T_\sigma)/C_H(T_\sigma),$$

sometimes called the "*little Weyl group*".

Richardson: Let $T_\sigma \subseteq G$ be a maximal σ-split torus. The inclusion $T_\sigma \hookrightarrow Q$ induces an isomorphism

$$T_\sigma/W(T_\sigma, H) \xrightarrow{\sim} H\backslash Q.$$

For reference see the Inventiones paper [49] cited above.

Remark 18.8. Take $G = H_l \times H_l$ and $\sigma\colon G \to G$ to be the map $(h_1, h_2) \mapsto (h_2, h_1)$. Then we recover the original Chevalley restriction theorem, so that this is an honest generalization.

Given $q \in Q(F)$, it admits a Jordan decomposition $q = q_s q_n$ with $q_s, q_n \in Q(\overline{F})$ with q_s and q_n commuting, q_s is semisimple and q_n is unipotent. The orbit Hq_s is the unique closed orbit in Hq and Hq is closed if and only if $q = q_s$.

Let $x_0 = \chi \in X^*(T_\sigma)$ be a coroot such that its restriction to $(T \cap Z(G))^\sigma$ is trivial. Let

$$\Psi(G, \sigma) = (X^*(T_\sigma), \Phi_{T_\sigma}, X_0, \Phi_{T_0}^\vee)$$

In the same paper as above, Richardson proves that this is a root datum and thus determines a group $\widetilde{G}$ over $\overline{F}$.

Conjecture 18.9 (Jacquet). *A cuspidal π on $G(\mathbf{A}_F)$ is distinguished by H if and only if π is a functorial lift from a specific form of $\widetilde{G}$ over F. (We've omitted describing the Galois action to nail down the particular form of the group $\widetilde{G}$)*

Remark 18.10. This is the local obstruction to being distinguished. Sometimes there exists a global obstruction (e.g. nonvanishing of L-function in Frieberg-Jacquet). The global obstruction vanishing should be enough to give the implication: functorial lift implies H distinguished.

18.3. **Applications of existence of automorphic representations.** We first apply our earlier existence results to the Friedberg-Jacquet case:

Corollary 18.11. *There exist $\mathrm{GL}_n \times \mathrm{GL}_n$ distinguished cuspidal representations of $\mathrm{GL}_{2n}(\mathbf{A}_F)$.*

Proof. Let v be a finite place of F and take $\beta \in \mathrm{GL}_n(F_v)$ a regular elliptic semisimple element. Let $K \subseteq M_n(F_v)$ be its centralizer (collection of things such that $g\beta = \beta g$ since not a group). Then K is a field. By changing β if necessary, can assume that (1) β does not have ± 1 as eigenvalues and (2) $\beta^2 - 1$ is not a square in K.

Now consider

$$\gamma = \begin{pmatrix} \beta & I \\ \beta^2 - I & \beta \end{pmatrix} \in \mathrm{GL}_{2n}(F_v).$$

In their paper "Uniqueness of Linear Periods" [34] , Jacquet-Rallis Lemma 4.3 gives: γ is of the form $gg^{-\sigma}$ where $g \in \mathrm{GL}_{2n}(F_v)$ and σ is conjugation by

$$\begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}$$

The eigenvalues of γ are $\iota_i(\beta) \pm \sqrt{\iota_i(\beta 0^2 - 1}$ for the various embeddings $\iota_i\colon K \to \overline{F}$. By (2), γ is semisimple regular, which implies that it is relatively semisimple. The centralizer satisfies

$$C_{\gamma, \mathrm{GL}_{2n}}(R) = \left\{ \begin{pmatrix} g_1 & g_2 \\ g_2(\beta^2 - 1) & g_1 \end{pmatrix} \mid g_i \in K \otimes_F R \text{ and } N_{K/F}(g_1^2 - (\beta^2 - 1_n)g_2^2) \in F^\times \right\}.$$

Thus $C_{\gamma,\mathrm{GL}_{2n}}$ is elliptic and g is relatively elliptic because we have an isomorphism

$$H_g(R) \to C_{g_1\,\mathrm{GL}_n\times\mathrm{GL}_n}(R)$$

mapping $(h_l, h_r) \mapsto h_l$ (exercise). Thus $C_{g,\mathrm{GL}_n\times\mathrm{GL}_n} \cong C_{g,\mathrm{GL}_n} \cap \mathrm{GL}_n \times \mathrm{GL}_n$ is elliptic, as claimed. To finish, we need to argue that there exists a g as in the decomposition of γ such that g is semisimple elliptic.

Consider $C_{\gamma,\mathrm{GL}_{2n}} \cap Q$. Note that $C_{\gamma,\mathrm{GL}_{2n}}$ is σ-stable. Its intersection with Q contains a σ-split torus T_σ which contains γ, by the theorem of Richardson. It is enough to show that the map $C_{\gamma,\mathrm{GL}_{2n}} \to T_\sigma$ taking $t \mapsto tt^{-\sigma}$ has image intersecting $T_\sigma^{rel\ reg}(F)$. Have that

$$\{\beta \in T_\sigma^{reg}(F_v) \mid \beta^2 = \beta\beta^{-\sigma} \in T_\sigma^{\mathrm{reg}}(F_v)\} = B_\sigma^{-1}(T_\sigma^{reg}(F_v)) \cap T_\sigma^{reg}(F_v);$$

the right set in the intersection above is the set of principal points in the sense of Richardson and Vast. The set on the left of the intersection is open in $T(F_v)$ and on the one on the right is dense in $T(F_v)$, so this intersection is nonempty. We may thus choose a global g such that g_v is in this intersection. □

19. Lecture 19: Cohomology of Symmetric Spaces

19.1. **Symmetric Spaces.** The topic in the rest of these lectures will be the automorphic description of the cohomology of arithmetic locally symmetric spaces.

Let $G_{/F}$ be a connected reductive group over a number field F, $K \subset G(\mathbf{A}_F^\infty)$ a compact open, and $K_\infty \subset G(F_\infty)$ a maximal compact subgroup. Recall that $G(F)A_G\backslash G(\mathbf{A}_F)/K_\infty K$ is a finite union of locally symmetric spaces. We will use the following notation:

$$X := A_G\backslash G(F_\infty)/K_\infty,$$

and $\mathrm{Sh}^K \equiv \mathrm{Sh}(G,X)^K$ will denote $G(F)\backslash X \times G(\mathbf{A}_F^\infty)/K$. We say that Sh^K is a *Shimura manifold*, and we note that it admits the following description: take a set of representatives $(g_i)_{i\in I}$ for $G(F)\backslash G(\mathbf{A}_F^\infty)/K$; then $\bigsqcup_{i\in I} \Gamma_i\backslash X \xrightarrow{\sim} \mathrm{Sh}^K$ via $\Gamma_i x \mapsto G(F)(x, g_i)K$, where $\Gamma_i = g_i^{-1}Kg_i \cap G(F)$.

In the following paragraph we will define what if means for K to be neat.

Claim 19.1. *If K is* neat*, the Sh^K is a smooth manifold.*

A subgroup $\Gamma \subset G(F)$ is a *congruence subgroup* if it is of the form $G(F) \cap K$ for some $K \subset G(\mathbf{A}_F^\infty)$ compact open; it is *arithmetic* if it is commensurable with some congruence subgroup (i.e. its intersection with some such subgroup is of finite index in each of the two).

An element $g \in G(F)$ is *neat* if the subgroup of $\bar{F}^\times$ generated by its eigenvalues is torsion-free. An arithmetic subgroup $\Gamma \subset G(F)$ is *neat* if every $\gamma \in G(F)$ is neat. Clearly, if Γ is neat, then all its subgroups and homomorphic images (over F) are neat. It is harder to see (exercise!) that every sufficiently small congruence subgroup of $G(F)$ is neat (hint: choose an embedding $G \hookrightarrow \mathrm{GL}_n$; then check for $\mathrm{GL}_n(\mathbf{Z})$).

Let $g = (g_v)'$ be an element in $\mathrm{GL}_n(\mathbf{A}_F^\infty)$, and $\Gamma_v \subset \bar{F_v}^\times$ the subgroup generated by the eigenvalues of g. Fix an embedding $\bar{F} \hookrightarrow \bar{F}_v$, and consider $(\bar{F}^\times \cap \Gamma_v)_{\mathrm{tors}}$ (note that since every subgroup of $\bar{F}^\times$ consisting of root of unity is normalized by $G_{/F}$, this is independent on the choice of embedding.

Definition 19.2. We say that g as above is *neat* if $\bigcap_v (\Gamma_v \cap \bar{F}^\times)_{\mathrm{tors}} = \{1\}$, that $g \in G(\mathbf{A}_F^\infty)$ is *neat* if its image under a faithful representaton into GL_n is neat, and that K is *neat* if all its elements are neat.

All sufficiently small K are neat, and neatness is preserved under taking subgroups and homomorphic images (cf. [59]§5).

We next define the action of Hecke operators in this context (recall that these were introduced before as operators on $C_c^\infty(G(\mathbf{A}_F)))$. Let $K, K' \subset G(\mathbf{A}_F^\infty)$ be compact open subgroups, and $g \in G(\mathbf{A}_F^\infty)$ be such that $K' \subset gKg^{-1}$. The we have a map

$$T_g : \mathrm{Sh}^{K'} \longrightarrow \mathrm{Sh}^K$$

given by $G(F)(x, hK') \mapsto G(F)(x, hgK)$. On the other hand, if $\Gamma, \Gamma' \subset G(F)$ are arithmetic subgroups, and $\gamma \in G(F)$ is such that $\Gamma' \subset \gamma\Gamma\gamma^{-1}$, we also have

$$T_\gamma : \Gamma'\backslash X \longrightarrow \Gamma\backslash X$$

given by $\Gamma' x \mapsto \Gamma\gamma^{-1}x$; these are a finite étale maps.

19.2. **Local systems.** Consider the following classical example: by the Shimura isomorphism, the cohomology of a modular curve with constant coefficients, say $H^1(\Gamma_0(N)\backslash\mathfrak{H}, \mathbf{C})$, *only* accounts for weight 2 modular forms on $\Gamma_0(N)$. One way to capture cohomologically all other (higher) weights, is introducing non-constant coefficients, the local systems.

Let $\Gamma \subset G(F)$ be a neat arithmetic subgroup. (Note that in the following we will usually assume the neatness of Γ implicitly.) Let V be a left Γ-module equipped with the discrete topology, and form the quotient $\mathcal{F}^\Gamma(V) = \Gamma\backslash(V \times X)$ by the diagonal action of Γ on the product. We say that the diagram given by the natural projection

$$\mathcal{F}^\Gamma(V) \longrightarrow \Gamma\backslash X$$

is a *local system*. For example, if V is a representation of Γ over $\mathbf{C}$, then this is the total space of a locally constant sheaf of $\mathbf{C}$-vector spaces. The rule that to each open $U \subset \Gamma\backslash X$ assigns the set of sections $\mathcal{F}^\Gamma(V)|_U := \{s : U \to \mathcal{F}^\Gamma(V)\}$ is a sheaf on $\Gamma\backslash X$, and one can therefore consider its sheaf cohomology $H^\bullet(\Gamma\backslash X, \mathcal{F}^\Gamma(V))$. Note that there is a natural identification $H^0(\Gamma\backslash X, \mathcal{F}^\Gamma(V)) = V^\Gamma$, and in fact, that this extends to an isomorphism in the derived category $\mathrm{D}(\mathrm{Mod}_\Gamma)$:

$$\mathbf{R}\Gamma(\Gamma\backslash X, -)\mathcal{F}^\Gamma = \mathrm{R}\Gamma(-)$$
$$H^\bullet(\Gamma\backslash X, \mathcal{F}^\Gamma(V)) = H^\bullet(\Gamma, V),$$

the point being that Γ acts *freely* on X.

Now for $g \in G(F)$ and Γ, Γ' with $\Gamma' \subset g\Gamma g^{-1}$ as before, consider the map $\theta = (\gamma' \mapsto g^{-1}\gamma' g) : \Gamma' \longrightarrow \Gamma$. This induces pullbacks $\theta^* : \mathrm{Mod}_\Gamma \longrightarrow \mathrm{Mod}_{\Gamma'}$, and similarly in the derived category.

Claim 19.3. $\mathcal{F}^\Gamma\theta^* \cong T_g^*\mathcal{F}^\Gamma$.

Consider the map

$$\Gamma'\backslash(V \times X) \longrightarrow \Gamma\backslash(V \times X) \times_{\Gamma\backslash X} \Gamma'\backslash X$$

given by $(m, x) \mapsto (m, g^{-1}x.x)$. If $\Gamma = \Gamma'$, then $V \cong \theta^*V$ via $m \mapsto g^{-1}m$, and also

$$\mathbf{R}\Gamma(\Gamma, V) = \mathbf{R}(\Gamma\backslash X, \mathcal{F}^\Gamma(V)) \to \mathbf{R}\Gamma(\Gamma\backslash X, T_g^*\mathcal{F}^\Gamma(V)) \cong \mathbf{R}\Gamma(\Gamma\backslash X, \mathcal{F}^\Gamma(V)) \cong \mathbf{R}\Gamma(\Gamma, V).$$

We now turn to the adelic setting. Let v be a place of F, and V a representation of the local group G_{F_v}. For $K \subset G_{F_v}$ compact open, define

$$\mathcal{F}^K(V) := G(F)\backslash V \times X \times G(\mathbf{A}_F^\infty)/K,$$

where for $\gamma \in G(F)$, $\gamma(v, x, gK) = (\gamma.v, \gamma.x, \gamma gK)$. We thus have a natural map $\mathcal{F}^K(V) \longrightarrow \mathrm{Sh}^K$. For $g \in G(\mathbf{A}_F^\infty)$, and K, K' with $K' \subset gKg^{-1}$, we have an isomorphism in the derived category $\mathrm{D}(\mathrm{Rep}_{G(F_v)})$,

$$\mathcal{F}^K(V) \xrightarrow{\sim} T_g^*\mathcal{F}^K(V) = G(F)\backslash V \times X \times G(\mathbf{A}_F)/K \times_{\mathrm{Sh}^K} \mathrm{Sh}^{K'},$$

via $(v, (x, hK)) \mapsto (v, (xhgK, xhK))$.

The relationship between the two given descriptions is the following. Fix $g \in G(\mathbf{A}_F^\infty)$, and let $\Gamma = gKg^{-1} \cap G(F)$. We then have an immersion $\iota : \Gamma\backslash X \hookrightarrow \mathrm{Sh}^K$, $\Gamma x \mapsto (xgK)$, and on the corresponding local systems, if $V \in \mathrm{Rep}_{G(F_v)}$, then $\iota^*\mathcal{F}^K(V) \cong \mathcal{F}^\Gamma(V)$, where V is regarded as a Γ-module via $\Gamma \hookrightarrow G(F) \hookrightarrow G(F_v)$.

For $g \in G(\mathbf{A}_F^\infty)$, consider the following diagram:

$$\begin{array}{ccccc} & \mathcal{F}^{gKg^{-1}\cap K}(V) & \xrightarrow{T_g} & \mathcal{F}^{K\cap g^{-1}Kg}(V) & \\ \swarrow^{\pi_1} & & & & \searrow^{\pi_2} \\ \mathcal{F}^K(V) & & & & \mathcal{F}^K(V) \end{array}$$

and let $T(g) := \pi_2 \circ T_g \circ \pi_1^*$. (Note that the fact that $T_g^*\mathcal{F}^K \cong \mathcal{F}^{K'}$, allows us to define the map in the middle.) This induces maps

$$T(g) : H^\bullet(\mathrm{Sh}^K, \mathcal{F}^K(V)) \longrightarrow H^\bullet(\mathrm{Sh}^K, \mathcal{F}^K(V)).$$

Example 19.4. Let $G = \mathrm{GL}_{2,\mathbf{Q}}$, and $K = K_1(c) = \{\left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right) \pmod c\}$, and $h = (\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)) \in M_2(\hat{\mathbf{Z}}) \cap \mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^\infty)$. Let $g = h^{-1}\det(h)$. Then $T(g)$ is just the usual Hecke operator $T(p)$ if $(p,c) = 1$. Indeed, by the Shimura isomorphism, $S_k(\Gamma_1(c)) \xrightarrow{\omega\oplus\bar{\omega}} H^\bullet(\mathrm{Sh}^K, \mathcal{F}^K(V))$, with

$$\omega : (f : \mathfrak{H} \to \mathbf{C}) \mapsto f(z)(-X + zY)^{k-2}\mathrm{d}z,$$

where $(-X+zY)^{k-2}$ is seen in the space $V = \mathrm{Symm}^{k-2}(V_{\mathrm{st}})$ of symmetric polynomials of degree $k-2$, V_{st} being the standard representations of GL_2. (Cf. [24].)

A basic question in the subject if *how to describe the cohomology groups* $H^\bullet(\mathrm{Sh}^K, \mathcal{F}^K(V))$. Via de Rham cohomology, it gives the connection of these groups to automorphic representations via $(\mathfrak{g}, K)$-cohomology. On the other hand, when Sh^K is algebraic over a number field, the description via étale cohomology gives the link to Galois representations. In fact, the interaction between these two descriptions is *the only* known tool for proving modularity theorems.

19.3. **Shimura data.** Recall the definition of *Serre's torus* $\mathbb{S} := \mathrm{Res}_{\mathbf{C}/\mathbf{R}}(\mathbb{G}_{\mathrm{m}})$. Note that $\mathbb{S}(\mathbf{R}) = \mathbf{C}^\times$. The following definition is essentially due to Deligne, *d'après* Shimura (cf.[20]).

Definition 19.5. A pair (G, X) is a *Shimura datum* if

- $G_{/\mathbf{Q}}$ is a connected reductive group; and
- X is a $G(\mathbf{R})$-conjugacy class of homomorphisms $s : \mathbb{S} \to G_\mathbf{R}$

such that the following three conditions are satisfied:

(SV1) For $h \in X$, the characters $z/\bar{z}$, 1, and $\bar{z}/z$ occur in the representation of $\mathbb{S}$ on $\mathrm{Lie}(G^{\mathrm{ad}})_\mathbf{C}$ defined by h (where $G^{\mathrm{ad}} := G/Z_G$ is the adjoint group);

(SV2) $\mathrm{ad}(h(\sqrt{-1}))$ is a *Cartan involution* of G^{ad};

(SV3) G^{ad} has no $\mathbf{Q}$-factors on which the projection of h is trivial.

Note that condition (SV1) and (SV2) imply that X is a (hermitian) symmetric space for G. In particular, $X = A_G\backslash G(\mathbf{R})/K_\infty$. Also, in (SV2), an involution $\theta : G_\mathbf{R} \longrightarrow G_\mathbf{R}$ is a Cartan involution if $G^{(\theta)}(\mathbf{R}) := \{g \in G(\mathbf{C}) \;:\; g = \theta(\bar{g})\}$ is compact. (For $\theta = \mathrm{ad}(h(\sqrt{-1}))$, we have $G^{(\theta)}(\mathbf{R}) = K_\infty$.)

We have the following classification: the unitary types correspond to A_n; there is one type corresponding to B_n and C_n, respectively; 3 types corresponding to C_n; 2 types corresponding to E_6; 1 type corresponding to E_7; and no type corresponding to either E_8, F_4 or G_2.

A natural question is whether $\mathrm{Sh}(G,X)^K$ can be realized as the complex points of some variety. For each $x \in X$, we have a co-character $u_x(z) := h_{x_\mathbf{C}}(z, 1)$, where $x_\mathbf{C}$ denotes be base change of x to $\mathbf{C}$. This defines an element in $X(T)/W(T,G)(\mathbf{C})$ (where as usual, T denotes a maximal torus). Let $E \subset \mathbf{C}$ be the field of definition of u_x.

Claim 19.6. *$E = E(G,X)$ is a number field, independent of the choice of $x \in X$. It is called the* reflex field *of (G,X).*

Theorem 19.7. *There exists a smooth quasi-projective variety $M(G,X)^K$ defined over the reflex field E of (G,X) such that*

$$\mathrm{Sh}(G,X)^K = M(G,X)^K(\mathbf{C}).$$

Furthermore, there is a canonical such model, characterized by the Galois action on certain special points. *Furthermore, all T_g are defined over E.*

Definition 19.8. The *Shimura variety*

$$M(G,X) := \varprojlim_K M(G,X)^K$$

is the unique projective limit of quasi-projective canonical models of $\mathrm{Sh}^K = \mathrm{Sh}(G,X)^K$ such that $\varprojlim_K \mathrm{Sh}^K$ is isomorphic to $M(G,X)^K$ together with $G(\mathbf{A}_F^\infty)$-actions.

Definition 19.9. A morphism of Shimura data, $(G,X) \longrightarrow (G',X')$, is a morphism of algebraic groups $G \to G'$ (over F) sending X to X'.

Definition 19.10. A morphism of Shimura varieties, $\mathrm{Sh}(G,X)^K \longrightarrow \mathrm{Sh}(G',X')^K$, is an inverse system of regular maps compatible with the $G(\mathbf{A}_F^\infty)$-action.

Theorem 19.11. *A morphism of Shimura data induce a morphism of Shimura varieties over the compositum $E(G,X)E(G',X')$. Moreover, it is a closed immersion if $G \to G'$ is injective.*

If T is a maximal torus defined over a CM extension, a *reciprocity law* $(T,X) \to G(X)$ specifies the Galois action on the corresponding points. These prescriptions suffice to uniquely determine the action.

We end with some examples:

(1) $G = \mathrm{GL}_{2,\mathbf{Q}}$; $h(a+b\sqrt{-1}) = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$.
(2) $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$; for example, for $F/\mathbf{Q}$ totally real, $h = \prod_\sigma \left(\begin{smallmatrix} a^\sigma & b^\sigma \\ -b^\sigma & a^\sigma \end{smallmatrix}\right)$;
(3) $G = \mathrm{GSp}_{2n}$, where $\mathrm{GSp}_{2n}(\mathbf{R}) = \{g \in \mathrm{GL}_{2n}(\mathbf{R}) \;:\; g^t J g = c(g)J(c(g) \in \mathbf{R}^\times)\}$ for $J = \left(\begin{smallmatrix} & & 1 \\ & \diagup & \\ 1 & & \end{smallmatrix}\right)$; $h(a+b\sqrt{-1}) = \left(\begin{smallmatrix} aI & -bJ \\ bJ & aI \end{smallmatrix}\right)$.

20. Lecture 20: Cohomology of Locally Symmetric Spaces

Recall that we have G_F a reductive group. $K_\infty \subset G(F \otimes_{\mathbf{Q}} \mathbf{R})$ such that K_∞^+ is a maximal connected compact subgroup. We have $K \subset G(\mathbf{A}_F)$. From this we constructed the Shimura manifold:

$$\mathrm{Sh}^K = \mathrm{Sh}(G,X)^K = G(F)\backslash X \times G(\mathbf{A}_F^\infty)/K$$

Where $X = \mathbf{A}_G \backslash G(F \otimes_{\mathbf{Q}} \mathbf{R})/K_\infty$.

Remark 20.1. For $G = \mathrm{GL}_2$ the usual Shimura datus is given by the class of $h(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Under this interpretation the stabilizer of h is $\mathrm{SO}_2(\mathbf{R})$ and not $O(\mathbf{R})$. As a consequence of this we have that $X = \mathbf{C} \setminus \mathbf{R}$ and not $\mathfrak{H}$. The point here is that the K_∞ that appears in the definition of the shimura datum is not necissarily maximal.

For $V \in \operatorname{Rep} G_F$ we then have that:

$$\mathcal{F}(V)^K = G(F)\backslash(V \times X) \times G(\mathbf{A}_F^\infty)/K \to G(F)\backslash X \times G(\mathbf{A}_F^\infty)/K = \mathrm{Sh}^K$$

For K as above and $\gamma \in G(F)$ we constructed a correspondance $\pi_2 \circ T_\gamma \circ \pi_1^* = T(\gamma)$ which acts on $H^\bullet(\mathrm{Sh}^K, \mathcal{F}(V)^K)$.

Exercise 5. Verify that the map $G(F) \to \mathrm{Aut}(H^0(\mathrm{Sh}^K, \mathcal{F}(V)^K))$ given by $\gamma \mapsto T(\gamma)$ induces an action of $C_c^\infty(G(\mathbf{A}_F^\infty) \,/\!/\, K)$ on $H^0(\mathrm{Sh}^K, \mathcal{F}(V)^K)$

The pressent goal is to understand the structure of $H^0(\mathrm{Sh}^K, \mathcal{F}(V)^K)$ as a Hecke module.
Recall that there exists an embedding:

$$H^\bullet(\Gamma\backslash X, \mathcal{F}(V)^\Gamma) \hookrightarrow H^\bullet(\mathrm{Sh}^K, \mathcal{F}(V)^K)$$

given by $\Gamma_X \mapsto (x, g)$ where $\Gamma = gKg^{-1} \cap G(F)$.
Some references for what follows are: [7][11][37].
We will start by looking at $\Gamma\backslash X$.
Consider Γ a group and V a Γ-module. Let $L^q = Map_{sets}(\Gamma^{q+1}, V)$ be the set of maps from Γ^{q+1} to V. We define an action of Γ on L^q as follows:

$$(\gamma f)(x_0, \ldots, x_q) = \gamma f(\gamma^{-1}x_0, \ldots, \gamma^{-1}x_q)$$

Let $C^q = C^q(\Gamma, V) = (L^q)^\Gamma$ and define a differential $d : C^\bullet \to C^\bullet$ on the complex by:

$$(d_q f)(x_0, \ldots, x_q) = \sum_i (-1)^i f(x_0, \ldots, \hat{x}_i, \ldots, x_q)$$

We then define $H^q(\Gamma, V) = \ker d_q / \Im d_{q-1}$, this is the cohomology of Γ with coefficients in V.

Remark 20.2. Observe that L^q is an injective resolution of V and thus these cohomology froms are the derived functor cohomology groups for $V \to V^\Gamma$.
Note that these coboundary maps are not the ones one often uses to concretely define group cohomology.

We have some basic facts:

Claim 20.3. *Assume that V is a finite dimensional vector space in characteristic 0. Then:*

- *if $|\Gamma| < \infty$ then $H^q(\Gamma, V) = 0$ for $q \neq 0$.*
- *for $\Gamma' \lhd \Gamma$ we have $H^q(\Gamma, V) = H^q(\Gamma', V)^{\Gamma/\Gamma'}$*

That the characterist is 0 is needed here.

Let X be a space on which Γ acts freely and properly and such that $\pi_i(X) = 1$ for $i \geq 0$ then we have the space $\Gamma\backslash X$ is an *Eilenberg-Maclean space* for Γ and:

$$H^0(\Gamma, V) = H^\bullet(\Gamma\backslash X, \mathcal{F}(V)^\Gamma)$$

For $\mathcal{F}(V)^\Gamma = \Gamma\backslash(V \times X)$.
For now we shall assume (without loss of generality) that $F = \mathbf{Q}$ and V is a $\mathbf{C}$-vector space. Suppose $\Gamma \subset G(\mathbf{R})$ is a discrete subgroup which acts freely on $X = \mathbf{A}_G\backslash G(\mathbf{R})/K_\infty$. We have an isomorphism:

$$H^\bullet(\Gamma, V) = H^\bullet(A^\bullet(X, V)^\Gamma)$$

Where $A^\bullet(X, V)$ is the complex of differential forms on X with coefficients in V. Note that $A^\bullet(X, V) = A^\bullet(X, \mathbf{C}) \otimes_{\mathbf{C}} V$.

[37] These references are not in the refs.bib file and should be added – cam

Remark 20.4. We point out that X need not always be connected, if K_∞ is a maximal compact then $\pi_i(X) = 0$ for all i and $\Gamma\backslash X$ is an Eilenberg-Maclean space.

Now, let $\mathfrak{g} \supset \mathfrak{k}$ be the Lie algebras (over $\mathbf{C}$) for G and K respectively. Let $\mathcal{V}$ be a $\mathfrak{g}$-module (with no assumption on dimension). Let $C(\mathfrak{g}, \mathcal{V}) = \mathrm{Hom}(\Lambda^\circ \mathfrak{g}, \mathcal{V}) = \Lambda^\circ \mathfrak{g}^* \otimes_{\mathbf{C}} \mathcal{V}$. We define a differential on the complex by:

$$(df)(x_0, \dots, x_g) = \sum (-1)^i f(x_0, \dots, \hat{x}_i, \dots, x_g) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_g)$$

We then define in the usual way:

$$H^\bullet(\mathfrak{g}, \mathcal{V}) = H^\bullet(C(\mathfrak{f}, \mathcal{V}), d)$$

Now consider:

$$C^\bullet(\mathfrak{g}, \mathfrak{k}, \mathcal{V}) = \{f \in C(g, V) | \iota_x f = \theta_x f = 0 \;\forall x \in \mathfrak{k}\}$$

Where ι_x and θ_x are the interior product and derivative respectively. So $(\iota_x f)(x_1, \dots, x_g) = f(x, x_1, \dots, x_g)$ and $(\theta_x f)(x_1, \dots, x_g) = \sum_i f(x_1, \dots, [x, x_i], \dots, x_g) + x f(x_1, \dots, x_g)$. (That is we have $C^\bullet(\mathfrak{g}, \mathfrak{k}, \mathcal{V}) = \mathrm{Hom}_{\mathfrak{k}}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{V})$.) Notice that one has $\theta_x = d \circ \iota_x + \iota_x \circ d$ for $x \in \mathfrak{g}$ and thus this new complex is d stable. Consequently we can take the cohomology of this complex. This is what is called the $(\mathfrak{g}, \mathfrak{k})$-cohomology of $\mathcal{V}$.

Because of the fact that $\mathrm{Lie}\, K_\infty = \mathrm{Lie}\, K_\infty^+$ this cohomology can't be quite what we want.

Assume now that $\mathcal{V}$ is a smooth $G(\mathbf{R})$ or $(\mathfrak{g}, K_\infty)$-module. Now set $\mathfrak{g} = \mathrm{Lie}(G(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathfrak{k} = \mathrm{Lie}(K_\infty) \otimes_{\mathbf{R}} \mathbf{C}$ then $K_i nfty$ acts in a natural manner on $C(\mathfrak{g}, \mathfrak{k}, \mathcal{V})$ via the action of K_∞ on $\mathcal{V}$ and the adjoint action on $\mathfrak{g}$. For this action define $C(\mathfrak{g}, K_\infty, \mathcal{V}) = C(\mathfrak{g}, \mathfrak{k}, \mathcal{V})^{K_\infty}$.

Definition 20.5. The $(\mathfrak{g}, K_\infty)$-cohomology of the $(\mathfrak{g}, K_\infty)$-module $\mathcal{V}$ is the cohomology of this complex. Thus we have:

$$H^\bullet(\mathfrak{g}, K_\infty; \mathcal{V}) = H^\bullet(C(\mathfrak{g}, \mathfrak{k}; \mathcal{V})^{K_\infty}) = H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{V})^{K_\infty/K_\infty^+}$$

We would now like to relate the $(\mathfrak{g}, K_\infty)$-cohomology and the cohomology of $\Gamma\backslash X$.

Fix a basis ω^i ($i = 1, \dots, m = \dim_{\mathbf{R}}(G)$) of left invariant 1-forms on $\mathbf{A}_G\backslash G(\mathbf{R})$. For $I = \{i_1, \dots, i_q\} \subset \{1, \dots, m\}$ with $i_l < i_n$ for $l < n$ set $\omega^I = \omega^{i_1} \wedge \dots \wedge \omega^{i_q}$ then we have that the ω^i frame the cotangeant bundle $T^\bullet(\mathbf{A}_G\backslash G(\mathbf{R}))$ and hence of $T^\bullet(\Gamma\mathbf{A}_G\backslash G(\mathbf{R}))$ and moreover, ω^I frame $\Lambda^\circ(T^\bullet(\Gamma\mathbf{A}_G\backslash G(\mathbf{R})))$ which is to say that any smooth q-form on $\Gamma\mathbf{A}_G\backslash G(\mathbf{R})$ can be written as:

$$\eta = \sum_I f_I \omega^I$$

with $f_I \in C^\infty(\Gamma\mathbf{A}_G\backslash G(\mathbf{R}))$. Likewise for $\eta \in \Lambda^q(\Gamma(\mathbf{R}, V))$ we have that $\eta = \sum_I f_I \omega^I$ with $f_I \in C_c^\infty(\Gamma\backslash G(\mathbf{R})) \otimes V$.

Let $\mathcal{V} = C^\infty(\Gamma\mathbf{A}_G\backslash G(\mathbf{R})) \otimes V$, we thus have obtained an identification:

$$A(\Gamma\mathbf{A}_G\backslash G(\mathbf{R}), V) = C(\mathfrak{g}, \mathcal{V})$$

which commutes with the natural differentials on either side.

We also have the map $\pi : G(\mathbf{R}) \to X$ which descends to a map $\pi : \Gamma\mathbf{A}_G\backslash G(\mathbf{R}) \to \Gamma\backslash X$ and so induces maps:

$$A^\circ(X, V)^\Gamma \xrightarrow{\pi^*} A^\circ(\Gamma\mathbf{A}_G\backslash G(\mathbf{R}), V) \simeq C^\circ(\mathfrak{g}, \mathcal{V})$$

With the image of the map being $C^\circ(\mathfrak{g}, K_\infty, \mathcal{V})$. This gives us an isomorphism:

$$H^\bullet(\Gamma\backslash X, \mathcal{F}(V)^\Gamma) \simeq H^\bullet(\mathfrak{g}, K_\infty, \mathcal{V})$$

So let $f_i \in C^\infty(G(\mathbf{Q})\mathbf{A}_G\backslash G(\mathbf{R})\times G(\mathbf{A}^\infty)/K)\otimes V)$, we can form the corresponding $\eta \in A^\bullet(G(\mathbf{Q})\mathbf{A}_G\backslash G(\mathbf{R})\times G(\mathbf{A}^\infty)/K, \tilde{\mathcal{F}(V)}^K)$ (where $\tilde{\mathcal{F}(V)}$ is the fullback under the projection $\Gamma\mathbf{A}_G\backslash G(\mathbf{R}) \to \Gamma\backslash X$) by $\eta = \sum_I f_I\omega^I$. By doing this we obtain an isomorphsim to:

$$A^\circ(G(\mathbf{Q})\mathbf{A}_G\backslash G(\mathbf{R}) \times G(\mathbf{A}^\infty)/K, V) \simeq C^\bullet(\mathfrak{g}, C^\infty(G(\mathbf{Q})\mathbf{A}_G\backslash G(\mathbf{R}) \times G(\mathbf{A}^\infty)/K) \otimes V)$$

Passing to the K_∞ invariants we obtain:

$$A^\bullet(\mathrm{Sh}^K, \mathcal{F}(V)^K) = C^\bullet(\mathfrak{g}, K_\infty, C^\infty(G(\mathbf{Q})\mathbf{A}_G\backslash G(\mathbf{A})/K) \otimes V)$$

This gives an explicit link between cohomology and automorphic representations. We remark that this map is Hecke equivaraiant in the sense that $T(\gamma) \mapsto 1_{K\gamma K}$.

This whole construction motivates the following definition:

Definition 20.6. Let $\phi \in C^\infty(G(F)\mathbf{A}_G\backslash G(\mathbf{A}_F))^K$, we say that ϕ is *cohomological* or a *cohomological vector* if there exists a representation V of G_F, $v \in V$ and ω^I on $G(F\otimes\mathbf{R})$ such that $\phi\omega^I \otimes v \in C^\bullet(\mathfrak{g}, \mathfrak{k}_\infty, C^\infty(G(F)\mathbf{A}_G\backslash G(\mathbf{A}_F))\otimes V)$ defines a non-zero class in $H^\bullet(\mathfrak{g}, K_\infty, C^\infty(G(F)\mathbf{A}_G(\backslash G(\mathbf{A}_F)))$

We have that $C^\infty(G(F)\mathbf{A}_G\backslash(\mathbf{A}_F)) \hookrightarrow L^2$ and so if π is a cuspidal (or any) automorphic representation realized as action on $C^\infty(G(F)\mathbf{A}_G\backslash G(\mathbf{A}_F))$ we would naturally consider $H^\bullet(\mathfrak{g}, K_\infty, \pi\otimes V)$ (we note that this cohomology group only depends on π_∞).

Let $\{pi\}$ be a set of representatives for equivalence classes of cuspidal automorphic representations of $\mathbf{A}_G\backslash G(\mathbf{A}_F)$ so that we have:

$$L_0^2 = \oplus m(\pi)\pi$$

and

$$H^\bullet(\mathrm{Sh}^K, C^K) \supset H^\bullet_{cusp}(\mathrm{Sh}^K, \mathcal{F}(V)^K) = H^\bullet(\mathfrak{g}, K_\infty, C^\infty(G(F)\mathbf{A}_F\backslash G(\mathbf{A}) \cap L_0^2) \otimes V)^K$$

where we have equality if G is anisotropic (we in general don't have equality and the complement is what is known as *Eisenstein Cohomology*. We remark that intersection cohomology and L^2-cohomology are important here.

So we have that:

$$H^\bullet_{cusp}(\mathrm{Sh}^K, \mathcal{F}(V)^K) = \oplus_\pi m(\pi)H^\bullet(\mathfrak{g}, K_\infty, \pi_\infty \otimes V) \otimes (\pi^\infty)^K$$

as a module under $C_c^\infty(G(\mathbf{A}_F^\infty) \mathbin{/\!/} K)$.

Definition 20.7. An automorphic representation π of $\mathbf{A}_G\backslash G(\mathbf{A}_F)$ is *cohomological* if $H^\bullet(\mathfrak{g}, K_\infty, \pi_\infty\otimes V) \neq 0$.

ϕ in the space of π is *cohomological* if there exists an embedding $\pi \to L_0^2$ such that the image of ϕ is cohomological.

A fundamental unanswered question in the theory of distinction is the following:

Given π, cuspidal, cohomological on $\mathbf{A}_G\backslash G(\mathbf{A}_)$ that is distinguished by (H_l, X_l) does there exist a cohomological ϕ in the space of π such that $P_{X_l}(\phi) \neq 0$.

So why do we care?

If such a ϕ exists and we have $\mathrm{ord}(X_l) < \infty$ then π-distinguished implies the existance of a non-zero cycle class attached to (H_l, X_l) inside $H^\bullet(\mathrm{Sh}^K, \mathcal{F}(V)^K)$.

Exercise 6. Suppose $H_l \subset G$ and $H_{l\infty} \hookrightarrow \Delta H_{l\infty} \subset H_{l\infty} \times H_{l\infty} = G_{F_\infty}$, in this case we have that a cohomological π which is distinguished contains a cohomological vector.

(Hint: Use Shuurs Lemma)

21. Lecture 21: More on the cohomology of Shimura varieties

21.1. **More on $(\mathfrak{g}, K_\infty)$-cohomology.** The cohomology functors $H^\bullet(\mathfrak{g}, V)$ are the right derived functors of $V \mapsto V^{\mathfrak{g}}$. The $(\mathfrak{g}, K_\infty)$ cohomology can be put in a more functorial perspective using Ext groups.

Nice properties of $(\mathfrak{g}, K_\infty)$-cohomology. Suppose that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as Lie algebras and $K_\infty = K_1 \oplus K_2$, $V = V_1 \otimes V_2$. Then one has the Kunneth forumla:

Theorem 21.1 (Kunneth formula). *One has natural isomorphisms*

$$H^\bullet(\mathfrak{g}, K_\infty, V) = \bigoplus_{p+q=k} H^p(\mathfrak{g}_1, K_1, V_1) \oplus H^q(\mathfrak{g}_2, K_2, V_2).$$

One also has access to Poincare duality for this cohomology theory:

Theorem 21.2 (Poincare duality). *If $m = \dim X$ (where as usual $X = A_G \backslash G / K_\infty$), then*

$$H^q(\mathfrak{a}\backslash\mathfrak{g}, K_\infty, \pi \otimes V) \cong H^{m-q}(\mathfrak{a}\backslash\mathfrak{g}, K_\infty, \pi_\infty^\vee \otimes V^\vee)^\vee.$$

The point of all this is the following: to compute $(\mathfrak{g}, K_\infty)$ cohomology, it suffices to understand the case where $\mathfrak{g}$ is simple over $\mathbf{R}$ and, morally at least, when $q \leq m/2$ (where again $m = \dim X$).

Now we return to our setting: given a connected reductive F-group, G, and such that for each $K \subseteq G(\mathbf{A}_F^\infty)$ compact open we have $K_\infty \subseteq G(F \otimes_{\mathbf{Q}} \mathbf{R})$ a compact subgroup such that K_∞^+ is a maximal connected compact subgroup of $G(F \otimes_{\mathbf{Q}} \mathbf{R})$, then we have a Shimura manifold $\mathrm{Sh}(G, X)^K$, where $X = A_G \backslash G(F \otimes_{\mathbf{Q}} \mathbf{R}) / K_\infty$, and for each representation V of G_{F_v} we have a local system $\mathcal{F}(V)^K \to \mathrm{Sh}(G, X)^K$. The point of the previous discussion is the following: as $C_c^\infty(G(\mathbf{A}_F^\infty /\!/ K))$-modules,

$$H^\bullet_{cusp}(\mathrm{Sh}^K, \mathcal{F}(V)^K) = \bigoplus_{\pi} H^\bullet(\mathfrak{g}, K, \pi_\infty \otimes V)^{m(\pi)} \otimes (\pi^\infty)^K$$

This sum is over the irreducible cuspidal representations of $A_G \backslash G(\mathbf{A}_F)$. Note that the cuspidal cohomology is a subcomplex of the full sheaf cohomology groups:

$$H^\bullet_{cusps}(\mathrm{Sh}^K, \mathcal{F}(V)^K) \subseteq H^\bullet(\mathrm{Sh}^K, \mathcal{F}(V)^K).$$

The complement is described in terms of cuspidal cohomology of parabolic subgroups. It is another illustration of Harrish-Chandra's philosophy of cusp-forms; it relies on Langland's theory of Eisenstein series. See [6].

Now we turn to the investigation of the cohomology groups $H^\bullet(\mathfrak{g}, K_\infty, \pi_\infty \otimes V)$! It suffices to consider the situation where G is defined over $\mathbf{Q}$, and $K_\infty \subseteq G(\mathbf{R})$ is as above (that is, its connected component is a maximal compact subgroup). Again $A_G \subseteq G$ is the connected component of a maximal $\mathbf{Q}$-split torus. As usual $X = A_G \backslash G(\mathbf{R}) / K_\infty$ is a Riemannian symmetric space of negative curvature. Here K_∞^+ is equal to the connected component of the fixed points of a Cartan involution θ. Let $\mathfrak{g} = \mathrm{Lie}(G_{\mathbf{R}}) \otimes_{\mathbf{R}} \mathbf{C}$, which can be decomposed as $\mathfrak{g} = \mathfrak{a} \oplus k \oplus \beta$ where $\mathfrak{a} = \mathrm{Lie}(A_G)$, $k = \mathrm{Lie}(K_\infty)$ and β is the -1 eigenspace of θ. We have $[k, \beta] \subseteq \beta$ and $[\beta, \beta] \subseteq k$. If $G_{\mathbf{R}}$ is without anisotropic factors modulo its center, then the second inclusion is an equality (To see that it's not always the case, consider for example a division algebra $B_{\mathbf{R}} \cong \mathbf{H}^n \oplus M_2(\mathbf{R})^{m-4n}$).

Lie algebra complex. Let π_∞ be an admissible irreducible $(\mathfrak{g}, K_\infty)$-module and let V be an irreducible representation of $G_{\mathbf{R}}$. Then

$$C^\bullet = C^\bullet(\mathfrak{a}\backslash\mathfrak{g}, K, \pi_\infty \otimes V) = \Lambda\beta^\bullet \otimes (\pi_\infty \otimes V).$$

We don't even know that this cohomology is nonzero! To study it we will introduce the *Killing form*, which has nothing to do with murder. It defines a scalar product on $\Lambda\beta^\bullet$ and we have a

K_∞-invariant inner product on $\pi_\infty \otimes V$. For the definition recall the earlier discussion of $(\mathfrak{g}, K_\infty)$-modules. In this way we obtain a positive definite inner product on $C^\bullet$, say $(,)$, which is invariant under K_∞.

Let $\partial\colon C^q \to C^{q-1}$ be the adjoint of d with respect to this inner product. The *Laplacian* is $\Delta = d\partial + \partial d$. We say that a form $\eta \in C^\bullet$ is *harmonic* if $\Delta(\eta) = 0$. This is equivalent to η being *closed* and *coclosed* (that is, $d\eta = 0$ and $\partial\eta = 0$).

Let $\mathfrak{C}$ be a Casimir element of $\mathfrak{a}\backslash\mathfrak{g}$ in the universal enveloping algebra. Recall that this is defined as follows: take a basis $\{X_i\}$ for $\mathfrak{a}\backslash\mathfrak{g}$ and let $\{X_i^\vee\}$ be the corresponding dual basis with respect to the Killing form. Then the Casimir is the second order differential operator in the universal enveloping algebra $U(\mathfrak{g})$ defined by $\sum X_i \otimes X_i^\vee$. One can show that $\mathfrak{C}$ is independent of the choice of basis, up to a scalar, and that $\mathfrak{C} \subseteq Z(\mathfrak{g})$ where $Z(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$. Since it's in the center it acts by a scalar, say r, on π_∞ and as a scalar s on V. Even though $\mathfrak{C}$ is defined up to scaling, and r and s depend on this scaling, we'll only care about whether s and r are equal or not. This is independent of the scaling, so we'll be okay.

Lemma 21.3 (Kuga). *We have $\Delta\eta = (r-s)\eta$ for $\eta \in C^q$.*

We won't prove this. See Borel[38]. This lemma has some nice consequences:

Corollary 21.4. *If $r \neq s$ then $H^\bullet(\mathfrak{g}, K_\infty, \pi_\infty \otimes V) = 0$. If $r = s$ then every cochain is harmonic and $H^\bullet(\mathfrak{g}, K_\infty, \pi_\infty \otimes V) = C^\bullet(\mathfrak{g}, K_\infty, \pi_\infty \otimes V)$, which recall by definition is just $\mathrm{Hom}_{K_\infty}(\Lambda\beta, \pi_\infty^\infty \otimes V)$. Finally*

$$H^\bullet(\mathfrak{g}, K_\infty, V) = \left\{ \begin{array}{ll} 0 & \text{if } V \text{ is nontrivial,} \\ (\Lambda\beta^\bullet)^{K_\infty} & \text{if } V \text{ is trivial.} \end{array} \right\}.$$

Proof of corollary. For the first claim, if η is a cocycle, that is $d\eta = 0$, then $\Delta\eta = d\partial\eta$ and

$$\eta = (r-s)^{-1}\Delta\eta = (r-s)^{-1} d\partial\eta,$$

so that η is also a coboundary. It is hence zero in cohomology. The proof of the second claim is similarly easy.

For the third claim, note that Casimir operates as zero $(= r)$ since we are considering π_∞ as the trivial representation. Since V is irreducible this implies that it must in fact be trivial. □

In the third part above we have a (trivial) example of a nonzero cohomology group. What does it mean, though? To study this take $V = \mathbf{C}$ to be trivial and suppose that G is anisotropic over $\mathbf{Q}$ but not over $\mathbf{R}$ (so no cusps, like a Shimura curve). Then we have maps

$$j\colon H^q(\mathfrak{a}\backslash\mathfrak{g}, K_\infty, \mathbf{C}) \to H^q(\mathrm{Sh}^K)$$

which are always injective, because a nonzero harmonic form is not a coboundary. A reference for this material is the book by Borel and Wallach [8].

Theorem 21.5 (Matsushima). *The map j above is bijective for $j \leq m(\mathfrak{a}\backslash\mathfrak{g})$. Moreover if $q \leq m(\mathfrak{a}\backslash\mathfrak{g})$ then every harmonic form in $H^q(\mathrm{Sh}^K)$ is invariant under $G(\mathbf{R})$.*

Remark 21.6. The assumption that G is anisotropic is not really necessary, but the statement above will need to be modified. The integer $m(\mathfrak{a}\backslash\mathfrak{g})$ is defined in terms of a bilinear form which depends on the Killing form. It is probably related to the curvature tensor.

Here is an example: when $G = \mathrm{SL}_2(\mathbf{R})$ and $K_\infty = \mathrm{SO}_2(\mathbf{R})$ then $X = \mathfrak{H}$ and these are precisely the constant functions and $(dz \wedge d\bar{z})/y^2$, up to scaling. For more examples see the book of Getz-Goresky [24].

[38] Did not find the appropriate reference

Moral: the cohomology appearing in Matsushima is easy to understand (the invariant forms); they do not depend on K (or Γ) and by Matsushima they explain everything in low enough degrees.

Another way of looking at Kuga's lemma. Recall that the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by a character on any irreducible $\mathfrak{g}$-module, say by χ_{π_∞} on π_∞ and by χ_V on V.

Lemma 21.7 (Wigner). *If $H^\bullet(\mathfrak{g}, K_\infty, \pi_\infty \otimes V) \neq 0$ then $\chi_{\pi_\infty}^\vee \neq \chi_V$. If $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet \beta, \pi_\infty \otimes V) \neq 0$ then this implies that $r = s$ if and only if $\chi_{\pi_\infty^\vee} = \chi_V$.*

Terminology: We think χ_{π_∞} is called the *Harrish-Chandra* parameter of π_∞. It is quite a coarse invariant to associate to a representation; for instance, it does not distinguish nonisomorphic representations.

Now that we've studied the "simple" part of the cohomology, how is the remaining cohomology described?

Example 21.8. Suppose that π_∞ is a $(\mathfrak{sl}_2, \mathrm{SO}_2(\mathbf{R}))$-module. In this case, if π_∞ is nontrivial then $H^\bullet(\mathfrak{sl}_2, \mathrm{SO}_2(\mathbf{R}), \pi_\infty \otimes V) \neq 0$ if and only if π_∞ is discrete series. For this see Harder [26].

Aside: to show that $H^\bullet(\mathrm{Sh}^K)$ is nonzero, by above it suffices to exhibit a cohomological automorphic representation of $A_G \backslash G(\mathbf{A}_F)$. How does one exhibit such cohomological representations? One way is to apply the trace formula with a nice choice of test functions to kill off the noncohomological parts. This strategy is succesful thanks to the theory of Lefschetz functions. See Borel-Labesse-Schwarmer for this [6]. These functions are also important in attaching Galois representations to cohomological automorphic representations. For this see the Paris book project [29].

What should we expect of the cohomology? If X is a smooth projective n-dimensional variety over a field k, and if $Y \subseteq X$ denotes a hyperplane section such that $X - Y$ is smooth (this will hold generically), then one has

Theorem 21.9 (Lefschetz). *The induced map on cohomology*

$$H^\bullet_{et}(X, \mathbf{Q}_l) \to H^\bullet_{et}(Y, \mathbf{Q}_l)$$

is an isomorphism for $k \leq n-1$ and injective for $k = n-1$.

The moral here is that new cohomology occurs in the middle dimension. It's difficult to do something similar for the Shimura manifolds that we've been considering, but there is a speculatory analogue: instead of a hyperplane section Y, consider $\mathrm{Sh}(G', X')^{K'}$ where $G' \hookrightarrow G$ and $X' \hookrightarrow X$ and $K' = G'(F) \cap K$. Then have a map $\mathrm{Sh}^{K'} \to \mathrm{Sh}^K$ and one can ask the question: *in what range of degrees does there exist G', X' such that the induced map*

$$H^\bullet(\mathrm{Sh}^K) \to H^\bullet(\mathrm{Sh}^{K'})$$

is bijective? (Note that this is naive, and one should allow translates by $G(\mathbf{A}_F)$ as well). There are some results in this direction due to Bergeron and Clozel. This leads one to consider Arthur's conjectures.

As with the etale cohomology above, the middle cohomologies in our setting contain the "new" cohomology (though the corresponding automorphic representations may still be lifts from smaller groups). In the nonalgebraic case, the middle is replaced by a range of degrees.

Example 21.10. Take $\mathfrak{g} = \mathfrak{sl}_2 \otimes_{\mathbf{R}} \mathbf{C}$ and $K_\infty = \mathrm{SU}_2(\mathbf{R})$. Then $H^i(\mathfrak{g}, K_\infty, \pi \otimes V) \neq 0$ for $i = 1, 2$ if π is a suitable cohomological representation. See Harder's paper again [26].

Vogan-Zuckerman theory. Let $\mathfrak{g}$ be complex semisimple and let K_∞ be connected, for simplicity (say they are basechanges of real things $\mathfrak{g}_0$ and k_0). Since K_∞ is compact, the adjoint $\mathrm{Ad}(x)$ acting on $\mathfrak{g}$ is diagonalizable for all $x \in k_0$, with real eigenvalues, and complex conjugation switches positive and negative eigenvalues. Let q denote the sum of the non-negative eigenspaces, let u be the sum of the positive eigenspaces, and let l be the sum of the zero eigenspaces. Then $q \subseteq \mathfrak{g}$ is a parabolic subalgebra and $q = l + u$. Then q is called a θ-*stable* parabolic subalgebra, though not all θ-stable parabolic subalgebras arise in this way. We'll only care about ones which do, though.

Suppose that V is a representation of $\mathfrak{g}$ such that the highest weight with respect to a θ-stable Borel subalgebra is fixed by θ. Then Vogan-Zuckerman define a $(\mathfrak{g}, K_\infty)$-module $A_q(V)$, where our notation follows Borel-Wallach and not Vogan-Zuckerman.

Theorem 21.11 (Vogan-Zuckerman). *We have $H^\bullet(\mathfrak{g}, K_\infty, A_q(V) \otimes V) \neq 0$. If π_∞ is irreducible, then $\pi_\infty \cong A_q(V)$ for some q and V.*

For a proof see their original paper [57].

Why is this cohomology important, even if one only cares about the Langlands program? Consider the following result of Sug Woo Shin, following Eichler-Shimura, Deligne, Carayol, Kottwitz, Clozel, Harris-Taylor, Yoshida, Morel, etc, etc.

Theorem 21.12 (Shin). *Let F be a CM-field. If π is a cohomological representation of $\mathrm{GL}_n(\mathbf{A}_F)$ with $\overline{\pi} \cong \pi^\vee$, then there exists a continuous semisimple representation*

$$R_l(\pi)\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$$

such that π and $R_l(\pi)$ are associated under local Langlands for all $\mathfrak{p} \not| l$.

This result can be found in Shin's forthcoming Annals paper, which is available on his website [52].

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