

INTERSECTION NUMBERS OF HECKE CYCLES ON HILBERT MODULAR VARIETIES

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ABSTRACT. Let \mathcal{O} be the ring of integers of a totally real number field E and set $\mathbf{G} := \text{Res}_{E/\mathbb{Q}}(\text{GL}_2)$. Fix an ideal $\mathfrak{c} \subset \mathcal{O}$. For each ideal $\mathfrak{m} \subset \mathcal{O}$ let $T(\mathfrak{m})$ denote the \mathfrak{m} th Hecke operator associated to the standard compact open subgroup $U_0(\mathfrak{c})$ of $\mathbf{G}(\mathbb{A}^f)$. Setting

$$X_0(\mathfrak{c}) := \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_\infty U_0(\mathfrak{c}),$$

where K_∞ is a certain subgroup of $\mathbf{G}(\mathbb{R})$, we use $T(\mathfrak{m})$ to define a Hecke cycle

$$Z(\mathfrak{m}) \in IH_2[E:\mathbb{Q}](X_0(\mathfrak{c}) \times X_0(\mathfrak{c})).$$

Here IH_\bullet denotes intersection homology. We use Zucker's conjecture (proven by Looijenga and independently by Saper and Stern) to obtain a formula relating the intersection number $Z(\mathfrak{m}) \cdot Z(\mathfrak{n})$ to the trace of $*T(\mathfrak{m}) \circ T(\mathfrak{n})$ considered as an endomorphism of the space of Hilbert cusp forms on $U_0(\mathfrak{c})$.

1. INTRODUCTION

We begin by recalling the main theorem of Hirzebruch and Zagier's famous paper [HZ]. In their paper they examine the intersection numbers of certain ‘‘Hirzebruch-Zagier cycles’’ Z_m . These cycles are sums of the closures of (affine) modular curves and certain compact Shimura curves inside a toroidal compactification of the Hilbert modular surface $\text{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p})}) \backslash \mathfrak{H}^2$. Here \mathfrak{H} is the usual complex upper half plane, $p \equiv 1 \pmod{4}$ is a prime and $\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$ is the ring of integers of $\mathbb{Q}(\sqrt{p})$. In particular, if $Z_m \cdot Z_n$ denotes the ‘‘number of intersections’’ of Z_m and Z_n , then the bulk of [HZ] is devoted to proving that for each $m \in \mathbb{Z}_{>0}$ the generating series

$$(1.1) \quad \sum_{n=0}^{\infty} (Z_m \cdot Z_n) q^n$$

is a weight 2 modular form for $\Gamma_0(p)$ with character $(\frac{\cdot}{p})$. Here $Z_m \cdot Z_0$ is essentially the volume of Z_m and $q := e^{2\pi iz}$ for $z \in \mathfrak{H}$.

In this paper we provide a kind of generalization of Hirzebruch-Zagier. Since their seminal work, a number of cohomology theories useful for studying topological intersection theory on Hermitian locally symmetric spaces such as Hilbert modular varieties have developed, including intersection homology, L^2 -cohomology, and Harder's theory of Eisenstein cohomology. These theories have intrinsic interest, but our treatment of them is utilitarian; they provide a natural framework for studying the interplay between intersection theory on modular varieties and the coefficients of modular forms. Our goal is to revisit Hirzebruch-Zagier armed with these theories with the idea of proving comparison theorems for pairings that arise naturally in the context of intersection homology on the one hand, and Hecke algebras on the other.

Concretely, we consider the graphs of Hecke correspondences on the product of two Hilbert modular varieties associated to totally real fields of arbitrary dimension. In order to make this precise, we must develop some notation: Let E be a totally real number field with degree $[E:\mathbb{Q}] = n$, ring of integers \mathcal{O} , and narrow class number h^+ ; thus h^+ is the order of the ray class group modulo β_∞ , the product of the infinite places. To every ideal $\mathfrak{c} \subset \mathcal{O}$ we associate in (2.3) the Hilbert modular variety

$$Y_0(\mathfrak{c}) := \bigcup_{j=1}^{h^+} \Gamma_j \backslash \mathfrak{H}^n$$

2000 *Mathematics Subject Classification.* 11F23.

Key words and phrases. Intersection homology, Hilbert modular forms, Lefschetz numbers.

This research was supported by the ARO through the NDSEG Fellowship program.

via a well-known construction (see [Ge, §I.7]). This variety has h^+ components, each of complex dimension n (the Γ_j are discrete subgroups of $(\mathrm{GL}_2(\mathbb{R})^+)^n$). For convenience, denote its Baily-Borel compactification by $X_0(\mathfrak{c})$. We then define in §5, for every ideal $\mathfrak{m} \subset \mathcal{O}$, a Hecke cycle

$$Z(\mathfrak{m}) \in IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})).$$

Here IH_\bullet denotes intersection homology (with middle perversity, see §4). The class $Z(\mathfrak{m})$ is determined by the Hecke operator $T(\mathfrak{m})$ for $X_0(\mathfrak{c})$ (see (3.6) for the definition of $T(\mathfrak{m})$).

In (2.2) we recall the definition of the standard compact open subgroups $U_0(\mathfrak{c}) \leq (\mathrm{Res}_{E/\mathbb{Q}}\mathrm{GL}_2)(\mathbb{A}^f)$, where $\mathfrak{c} \subset \mathcal{O}$ is an ideal as above and \mathbb{A}^f denotes the finite adèles of \mathbb{Q} . Let $\mathbb{T}_{\mathfrak{c}}$ denote the Hecke algebra associated to $U_0(\mathfrak{c})$ (see §3); thus $\mathbb{T}_{\mathfrak{c}}$ acts on the space $M(U_0(\mathfrak{c}))$ of weight $(2, \dots, 2)$ Hilbert modular forms on $U_0(\mathfrak{c})$ (see (2.8)). Moreover, this action preserves the subspace $S(U_0(\mathfrak{c}))$ of Hilbert cusp forms (see (2.9)). There are two natural bilinear pairings associated to $\mathbb{T}_{\mathfrak{c}}$. The first is just

$$(1.2) \quad \begin{aligned} \mathbb{T}_{\mathfrak{c}} \times \mathbb{T}_{\mathfrak{c}} &\longrightarrow \mathbb{C} \\ (T_1, T_2) &\longmapsto \mathrm{Tr}(*T_1 \circ T_2), \end{aligned}$$

where we view T_1 and T_2 as endomorphisms of $S(U_0(\mathfrak{c}))$ and the adjoint is taken with respect to the Petersson inner product (see (3.15)). Secondly, attached to each element $\sum c_{\mathfrak{m}}T(\mathfrak{m})$ of the subspace of $\mathbb{T}_{\mathfrak{c}}$ spanned by the $T(\mathfrak{m})$, we have the cycle $\sum c_{\mathfrak{m}}Z(\mathfrak{m})$, and thus can consider the pairing on these cycles induced by the generalized Poincaré pairing

$$IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})) \times IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})) \longrightarrow H_0(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})) \longrightarrow \mathbb{C},$$

(see §5).

Theorem 1.1 below, the main result of this work, is essentially a comparison of these two pairings. To state it, we set notation for the finite sum

$$(1.3) \quad \sigma'_c(\mathfrak{a}) := \sum_{\substack{\mathfrak{n} \supseteq \mathfrak{a} \\ \mathfrak{n} + \mathfrak{c} = \mathcal{O}}} N(\mathfrak{n}^{-1}\mathfrak{a}),$$

where $N(\mathfrak{n}) := \mathrm{Norm}_{E/\mathbb{Q}}(\mathfrak{n})$. The reader may recognize $\sigma'_c(\mathfrak{a})$ as a generalized divisor function that appears in the Fourier coefficients of a certain Eisenstein series (see (8.1)). One can also relate $\sigma'_c(\mathfrak{m})$ in a simple manner to the volume of $Z(\mathfrak{m})$.

We are now ready to state Theorem 1.1. In our setting, in which we consider the product of two copies of a Hilbert modular variety $X_0(\mathfrak{c})$ associated to a totally real number field E of arbitrary degree n and narrow class number h^+ , this theorem gives a formula for $Z(\mathfrak{m}) \cdot Z(\mathfrak{n})$ in terms of traces of Hecke operators and coefficients of Eisenstein series:

Theorem 1.1. *Let $\mathfrak{m}, \mathfrak{n} \subset \mathcal{O}$ be ideals. If $\mathfrak{m}, \mathfrak{n}$ are ideals that are equivalent in the ray class group modulo β_∞ , then we have the formula*

$$Z(\mathfrak{m}) \cdot Z(\mathfrak{n}) = 2^n \left((-1)^n \mathrm{Tr}(*T(\mathfrak{m}) \circ T(\mathfrak{n})) + h^+ \sigma'_c(\mathfrak{m}) \sigma'_c(\mathfrak{n}) \right).$$

Otherwise, $Z(\mathfrak{m}) \cdot Z(\mathfrak{n}) = 0$.

Here $*T(\mathfrak{m})$ is the dual of $T(\mathfrak{m})$ with respect to the Petersson inner product (see (3.16)).

Remark. We remark that the method of proof of Theorem 1.1, which is different from the approach of Hirzebruch and Zagier, boils down to a formal argument using a Lefschetz coincidence formula in intersection homology and Zucker's conjecture (now a theorem). Both of these hold for arbitrary Hermitian locally symmetric varieties and cohomology groups with coefficients in a representation, which indicates that some analogue of the formula in Theorem 1.1 should hold in great generality.

From the shape of the formulae in Theorem 1.1, one would expect that some analogue of the generating series (1.1) would be modular. This is indeed the case. More precisely, we have the following:

Theorem 1.2. *If $S(U_0(\mathfrak{c})) = S^{new}(U_0(\mathfrak{c}))$ and either $[E : \mathbb{Q}] > 1$ or $\mathfrak{c} \neq \mathcal{O}$, then for each (integral) ideal $\mathfrak{m} \subset \mathcal{O}$ the formal Fourier expansion*

$$\Phi_{\mathfrak{m}, \mathfrak{c}} \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) := \delta(\mathfrak{c})(Z(\mathfrak{m}) \cdot Z(0))(y)|y| + \sum_{0 \ll \xi \in E} Z(\mathfrak{m}) \cdot Z(\xi y)|y| e(\mathrm{tr}(\xi i y_\infty)) \chi_E(\xi x)$$

defines an element of $M(U_0(\mathfrak{c}))$. Here we set $Z(\mathfrak{a}) = 0$ if the ideal associated to \mathfrak{a} is not integral, and $0 \ll \xi$ means that ξ is totally positive.

Here

$$\delta(\mathfrak{c}) := \begin{cases} 1 & \text{if } \mathfrak{c} = \mathcal{O}_E \\ 0 & \text{otherwise} \end{cases}$$

and $S^{\text{new}}(U_0(\mathfrak{c}))$ is the new subspace (also known as the primitive subspace, see [Sh2, p. 652]). For the definition of $e(\text{tr}(\xi iy_\infty))\chi_E(\xi x)$, see (2.11). The constant term $(Z(\mathfrak{m}) \cdot Z(0))(y)$ is essentially the volume of $Z(\mathfrak{m})$ in one component and zero elsewhere (see (8.3)).

Remark. One should compare Theorem 1.2 with the formula for the generating series (1.1) obtained by Zagier in [Za], which we now recall. Let $(,)$ denote the Petersson inner product (we won't specify the normalization), and denote by \widehat{f} the Doi-Naganuma lift of an elliptic modular form f for $\Gamma_0(p)$ with nebentypus $(\frac{p}{\cdot})$ (see [Za]). From Zagier's results, one can reformulate the main identity of [HZ] as

$$(1.4) \quad \sum_{n=0}^{\infty} (Z_m \cdot Z_n) q^n = \pi_+ \left(t(m) E_{2,p}(z) - \frac{4}{p^2} \sum_{n=1}^{\infty} \left(\sum' \frac{(\widehat{f}, \widehat{f})}{(f, f)^2} a_f(m) a_f(n) \right) q^n \right).$$

Here the prime indicates the summation over the basis of normalized weight two elliptic newforms (i.e. eigenforms for all the Hecke operators)

$$f(z) := \sum_{n=1}^{\infty} a_f(n) q^n$$

for $\Gamma_0(p)$ with character $(\frac{p}{\cdot})$, π_+ is the canonical projection to the plus space, $t(m)$ is a rational number depending only on m , and $E_{2,p}$ is a weight two Eisenstein series for $\Gamma_0(p)$ with character $(\frac{p}{\cdot})$ (see [Za, (98-99)]). In the course of the proof of Theorem 1.2 in §8, we shall see that $\Phi_{\mathfrak{m}, \mathfrak{c}}$ can be similarly decomposed as the sum of an Eisenstein series and a sum of weighted newforms.

We can rephrase Theorem 1.2 in the language of modular forms as follows: let $M(\Gamma_j)$ be the space of classical Hilbert modular forms of weight $(2, \dots, 2)$ for Γ_j (see (2.6)). Moreover, let s_1, \dots, s_{h+} be idèles of E with $(s_j)_\nu = 1$ for each infinite place ν such that the fractional ideals associated to s_1, \dots, s_{h+} form a set of representatives for the classes in the ray class group modulo β_∞ (see §2). Using this notation we can phrase Corollary 1.2 in terms of classical Hilbert modular forms simply by inverting the formulae (2.10-2.11) for converting classical Fourier expansions to adelic Fourier expansions:

Corollary 1.3. *Suppose that \mathfrak{m} is in the ray class of s_j^{-1} modulo β_∞ . If $U_0(\mathfrak{c})$ is new (in the sense of §3), and either $[E : \mathbb{Q}] > 1$ or $\mathfrak{c} \neq \mathcal{O}$, then*

$$\delta(\mathfrak{c}) N(s_j) (Z(\mathfrak{m}) \cdot Z(0))(s_j^{-1}) + \sum_{\substack{\xi \in s_j \\ 0 \ll \xi}} Z(\mathfrak{m}) \cdot Z(\xi s_j^{-1}) N(s_j) e(\text{tr}(\xi z)) \in M(\Gamma_j(\mathfrak{c})).$$

Remark. The case $n = 1$, $E = \mathbb{Q}$, $\mathfrak{c} = \mathbb{Z}$ is not covered by Theorem 1.2 or Corollary 1.3. However, using Theorem 1.1 in this case and an easy analogue of the argument in §8, one can prove that

$$-\frac{1}{24} E_2(z) := -\frac{1}{24} + \sum_{n \geq 1} \left(\sum_{d|n} d \right) e(nz) = N(s_1) (Z(\mathfrak{m}) \cdot Z(0))(s_1^{-1}) + \sum_{\substack{\xi \in s_1 \\ 0 \ll \xi}} Z(\mathfrak{m}) \cdot Z(\xi s_1^{-1}) N(s_1) e(\text{tr}(\xi z)).$$

Here we choose $s_1 = \mathbb{Z}$. The function $E_2(z)$ is not a modular form (the space $M(\text{SL}_2(\mathbb{Z})) = M_2(\text{SL}_2(\mathbb{Z}))$ is zero dimensional). However, this ‘‘quasi-modular’’ function is often useful in the classical theory of elliptic modular forms. Furthermore, it is in some sense the first nontrivial example of a p -adic modular form (see [Se]).

Under the assumptions of Theorem 1.2, one can use the automorphic forms $\Phi_{\mathfrak{m}, \mathfrak{c}}$ to define a \mathbb{C} -linear map Ψ by

$$(1.5) \quad \Psi : \begin{array}{ccc} HC(\mathfrak{c}) & \longrightarrow & M(U_0(\mathfrak{c})) \longrightarrow S(U_0(\mathfrak{c})) \\ \sum \alpha_{\mathfrak{m}} Z(\mathfrak{m}) & \longmapsto & \sum \alpha_{\mathfrak{m}} \Phi_{\mathfrak{m}, \mathfrak{c}} \end{array}$$

where $HC(\mathfrak{c})$ is the subspace of $IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c}))$ spanned by the Hecke cycles $Z(\mathfrak{m})$ as \mathfrak{m} varies over the ideals of \mathcal{O} and the second map is the canonical projection. In §8 we prove that this map has full image:

Theorem 1.4. *If $S^{\text{new}}(U_0(\mathfrak{c})) = S(U_0(\mathfrak{c}))$ and either $\mathfrak{c} \neq \mathcal{O}$ or $\mathfrak{c} = \mathcal{O}$ and $n \neq 1$, then the map*

$$\Psi : HC(\mathfrak{c}) \longrightarrow S(U_0(\mathfrak{c}))$$

of (1.5) is surjective.

Here $S^{\text{new}}(U_0(\mathfrak{c}))$ denotes the new subspace (see [Sh2]).

Now that we have stated the main theorems of this paper, we pause to point the reader to some related results in the literature:

Remark.

- (1) Kudla and Millson prove theorems similar to Theorem 1.2 in their study of special cycles on orthogonal and unitary groups (see [KM]).
- (2) Arithmetic analogues of (1.1) and Theorem 1.2 have appeared in ([BBK], [BKK]) and the program of Kudla, Rapoport, and Yang (see [Ku], [KRY]).
- (3) Lefschetz fixed point numbers of Hecke correspondences on certain Hermitian locally symmetric varieties in weighted cohomology and L^2 -cohomology are computed in [GKM] and [St], respectively.

As one might guess from the above, the nature of this work involves collecting a variety of theories into one place, and therefore requires a substantial amount of preparatory material. The author would like to stress that the first four sections of the paper are essentially expository and were written for the purpose of recalling relevant results and notation. With this in mind, we now outline the contents of this paper. In the next section, we fix notation for Hilbert modular varieties and modular forms. Section 3 recalls the relevant Hecke operators on holomorphic modular forms. In §4 we fix notation for Hecke correspondences and state the Zucker conjecture in our situation. This theorem allows us to identify L^2 -cohomology and intersection homology as Hecke-modules.

We finally define the cycles $Z(\mathfrak{m})$ in §5 and prove a preliminary Lefschetz-coincidence formula relating their intersection numbers to an alternating sum of traces of Hecke operators on L^2 -cohomology. For the purpose of evaluating this alternating sum, we recall an explicit description of the L^2 -cohomology groups of Hilbert modular varieties due to Harder and Hida in §6. We combine this explicit description with the Lefschetz coincidence formula mentioned above to prove Theorem 1.1 in §7. Finally, §8 investigates the relationship of Theorem 1.1 to the coefficients of certain automorphic forms; this is where we prove Theorem 1.2 and Theorem 1.4.

2. HILBERT MODULAR VARIETIES AND HILBERT MODULAR FORMS

For concreteness, we begin the body of the paper by recalling notation and defining conventions concerning Hilbert modular varieties and forms, our main objects of study. This notation will be used throughout the following sections. As in the introduction, let E/\mathbb{Q} be a totally real number field with ring of integers \mathcal{O} , $[E : \mathbb{Q}] = n$ and $|\text{CL}^+(E)| = h^+$ (here $\text{CL}^+(E)$ denotes the narrow class group of E). Moreover, let

$$\mathbf{G} := \text{Res}_{E/\mathbb{Q}}(\text{GL}_2).$$

Thus \mathbf{G} is a reductive algebraic group whose \mathbb{Q} -rational points are in one-to-one correspondence with the E -rational points of GL_2 . We note that $\mathbf{G}(\mathbb{R}) = (\text{GL}_2(\mathbb{R}))^n$ has 2^n connected components; we let $\mathbf{G}(\mathbb{R})^0$ be the component containing the identity. We have a homomorphism

$$\begin{aligned} \mathbb{C}^\times &\longrightarrow \mathbf{G}(\mathbb{R}) \\ x + iy &\longmapsto \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \dots, \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \right). \end{aligned}$$

The centralizer of the image of this homomorphism is

$$K_\infty := \left\{ \left(\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}, \dots, \begin{pmatrix} x_n & -y_n \\ y_n & x_n \end{pmatrix} \right) : x_k^2 + y_k^2 \neq 0 \right\} \cong (\text{SO}_2(\mathbb{R})^n \cdot \mathbb{R}^\times)^n.$$

We put a complex structure on $\mathbf{G}(\mathbb{R})/K_\infty$ via the identification

$$(2.1) \quad \begin{aligned} \mathbf{G}(\mathbb{R})/K_\infty &\longleftrightarrow (\mathbb{C} - \mathbb{R})^n \\ (h_1, \dots, h_n) &\longmapsto (h_1 i, \dots, h_n i), \end{aligned}$$

where, if h_j is the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{R})$ in the quotient, we define $h_j z = \frac{az+b}{cz+d}$ to be the usual fractional linear transformation.

Let $\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{A}_{\mathbb{Q}})$ be the adelic points of \mathbf{G} and $\mathbf{G}(\mathbb{A}^f)$ the subgroup associated to the finite adèles. We now define, for every integral ideal \mathfrak{c} of E , a compact open subgroup $U_0(\mathfrak{c})$ of $\mathbf{G}(\mathbb{A}^f)$. Let \mathfrak{d} be the different of E/\mathbb{Q} . As in [Sh2], define for every prime ideal $\mathfrak{p} \subset \mathcal{O}$,

$$\begin{aligned} V_{\mathfrak{p}} &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(E_{\mathfrak{p}}) : a\mathcal{O}_{\mathfrak{p}} + \mathfrak{c}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}, b \in \mathfrak{d}_{\mathfrak{p}}^{-1}, c \in \mathfrak{c}_{\mathfrak{p}}\mathfrak{d}_{\mathfrak{p}}, d \in \mathcal{O}_{\mathfrak{p}} \right\} \\ W_{\mathfrak{p}} &:= \{x \in V_{\mathfrak{p}} : \det x \in \mathcal{O}_{\mathfrak{p}}^{\times}\}. \end{aligned}$$

Here $\mathfrak{c}_{\mathfrak{p}}$ (resp. $\mathfrak{d}_{\mathfrak{p}}$) denotes the completion of \mathfrak{c} (resp. \mathfrak{d}) at the place associated to \mathfrak{p} . We then define

$$(2.2) \quad \begin{aligned} U_0(\mathfrak{c}) &:= \prod_{\mathfrak{p}} W_{\mathfrak{p}} \\ W &:= \mathbf{G}(\mathbb{R})^0 \times \prod_{\mathfrak{p}} W_{\mathfrak{p}} = \mathbf{G}(\mathbb{R})^0 \times U_0(\mathfrak{c}) \\ V &:= \mathbf{G}(\mathbb{A}) \cap \left(\mathbf{G}(\mathbb{R})^0 \times \prod_{\mathfrak{p}} V_{\mathfrak{p}} \right). \end{aligned}$$

We denote by

$$Y_0(\mathfrak{c}) := \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_{\infty} U_0(\mathfrak{c}).$$

the Hilbert modular variety associated to the compact open subgroup $U_0(\mathfrak{c})$. In order to write down an analytic realization of $Y_0(\mathfrak{c})$, we pause to set some notation. Let \mathbb{A}_E denote the adèles of E . Now, for each $s \in \mathbb{A}_E^{\times}$ (the idèles of E), let $i(s)$ denote the associated fractional ideal of E . Let β_{∞} denote the product of the infinite places of E ; thus h^+ is the order of the ray class group modulo β_{∞} . Choose h^+ elements $s_1, \dots, s_{h^+} \in \mathbb{A}_E^{\times}$ with $(s_j)_v = 1$ for each infinite place v such that $i(s_1), \dots, i(s_{h^+})$ form a complete set of representatives for the ideal classes modulo β_{∞} . Furthermore, define x_j by

$$x_j := \begin{pmatrix} 1 & 0 \\ 0 & s_j \end{pmatrix}.$$

By strong approximation for GL_2 , we have a decomposition

$$(2.3) \quad \begin{aligned} \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K_{\infty} U_0(\mathfrak{c}) &= \bigcup_{j=1}^{h^+} x_j \mathbf{G}(\mathbb{R})^0 U_0(\mathfrak{c}) x_j^{-1} \cap \mathbf{G}(\mathbb{Q}) \backslash (\mathbf{G}(\mathbb{R})^0 / K_{\infty}) \\ &= \bigcup_{j=1}^{h^+} \Gamma_j \backslash \mathfrak{H}^n \end{aligned}$$

for some congruence subgroups Γ_j of $\mathbf{G}(\mathbb{A})^+ \cap \mathbf{G}(\mathbb{Q})$ (here $\mathbf{G}(\mathbb{A})^+$ is the subgroup of $\mathbf{G}(\mathbb{A})$ consisting of elements whose projection to the archimedean places is an element of $\mathbf{G}(\mathbb{R})^0$). Specifically,

$$(2.4) \quad \Gamma_j := \Gamma_j(\mathfrak{c}) := x_j W x_j^{-1} \cap \mathbf{G}(\mathbb{Q}) = \Gamma(i(s_j)\mathfrak{d}, \mathfrak{c})$$

where, for each integral ideal \mathfrak{c} and fractional ideal \mathfrak{b} of E ,

$$\Gamma(\mathfrak{b}, \mathfrak{c}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G}(\mathbb{Q}) \cap \mathbf{G}(\mathbb{A})^+ : a, d \in \mathcal{O}, b \in \mathfrak{b}^{-1}, c \in \mathfrak{c}\mathfrak{b}, ad - bc \in \mathcal{O}^{\times} \right\}.$$

Write

$$Y_j := \Gamma_j \backslash \mathfrak{H}^n.$$

Let $X_0(\mathfrak{c})$ (resp. X_j) denote the Bailey-Borel compactification of $Y_0(\mathfrak{c})$ (resp. Y_j). Thus Y_j is a dense, Zariski-open subset of the normal complex projective algebraic variety X_j and $X_j - Y_j$ is a finite collection of (closed) points.

We now recall the definition of a classical Hilbert modular form for Γ_j . Let $(\mathrm{GL}_2^+(\mathbb{R}))^n$ act on \mathfrak{H}^n by

$$(2.5) \quad \alpha(z) = (\alpha_1, \dots, \alpha_n)(z_1, \dots, z_n) := (\alpha_1 z_1, \dots, \alpha_n z_n)$$

where

$$\alpha_i z_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} z_i := \frac{a_i z_i + b_i}{c_i z_i + d_i}.$$

Furthermore, for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $z = (z_1, \dots, z_n) \in \mathfrak{H}^n$, we write

$$\begin{aligned} z^k &= \prod_{i=1}^n z_i^{k_i} \\ \mathrm{tr}(k) &= \sum_{i=1}^n k_i, & \mathrm{tr}(z) &= \sum_{i=1}^n z_i \\ e(\mathrm{tr}(z)) &= \exp(2\pi i \mathrm{tr}(z)). \end{aligned}$$

The action (2.5) induces an action on functions

$$f : \mathfrak{H}^n \longrightarrow \mathbb{C}$$

by the rule

$$f|_k \alpha(z) := \det(\alpha)^{k/2} (cz + d)^{-k} f(\alpha(z)).$$

We will only be dealing with the case $k = (2, \dots, 2)$ in this paper, and so we let

$$f|\alpha(z) := f|_{(2, \dots, 2)} \alpha(z).$$

Let $\sigma_1, \dots, \sigma_n$ denote the embeddings of E into \mathbb{R} . Define

$$(2.6) \quad M(\Gamma_j) := M_{(2, \dots, 2)}(\Gamma_j)$$

to be the \mathbb{C} -vector space of complex-valued holomorphic functions f on \mathfrak{H}^n satisfying

$$f|\alpha(z) = f(z)$$

for

$$\alpha \in \left\{ \left(\begin{pmatrix} \sigma_1(a) & \sigma_1(b) \\ \sigma_1(c) & \sigma_1(d) \end{pmatrix}, \dots, \begin{pmatrix} \sigma_n(a) & \sigma_n(b) \\ \sigma_n(c) & \sigma_n(d) \end{pmatrix} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_i \right\}$$

that are regular and holomorphic at the cusps of Γ_i . The subspace of forms that vanish at the cusps will be denoted by

$$(2.7) \quad S(\Gamma_j) := S_{(2, \dots, 2)}(\Gamma_j).$$

The space $M(\Gamma_j)$ (resp. $S(\Gamma_j)$) is known as the space of Hilbert modular forms (resp. Hilbert cusp forms) of weight $(2, \dots, 2)$ for Γ_j (for the definitions of regular and vanishing see [F, §I.4]). We note that a Hilbert modular form $f \in M(\Gamma_j)$ can be identified with its Fourier expansion at the cusp ∞ :

$$f(z) = c(0) + \sum_{0 \ll \xi \in i(s_j)} c(\xi) e(\mathrm{tr}(\xi z))$$

(see [Sh2, p. 649]). Here $c(0) \in \mathbb{C}$ is a constant, $0 \ll \xi$ means that ξ is totally positive and $\xi z := (\sigma_1(\xi)z_1, \dots, \sigma_n(\xi)z_n)$.

Due to the decomposition (2.3), it is natural to define the space of Hilbert modular forms on $U_0(\mathfrak{c})$ as

$$(2.8) \quad M(U_0(\mathfrak{c})) := M_{(2, \dots, 2)}(U_0(\mathfrak{c})) = \{\mathbf{f} = (f_1, \dots, f_{h^+}) : f_j \in M(\Gamma_j)\},$$

and the space of Hilbert cusp forms as

$$(2.9) \quad S(U_0(\mathfrak{c})) := S_{(2, \dots, 2)}(U_0(\mathfrak{c})) = \{\mathbf{f} = (f_1, \dots, f_{h^+}) : f_j \in S(\Gamma_j)\}.$$

We will mostly view $M(U_0(\mathfrak{c}))$ as the \mathbb{C} -vector space defined by (2.8), but in §8 we will require the notion of the Fourier expansion of an automorphic form. We now describe these expansions. To ease notation, if $\mathfrak{a} \in \mathbb{A}_E^\times$, we write $\mathfrak{a} = i(\mathfrak{a})$, and if $x \in \mathbb{A}_E$, we denote by x_∞ the image of x under the canonical projection $\mathbb{A}_E \rightarrow \mathbb{A}_E^\infty$ (the product of the completions of E at the infinite places).

Note that every ideal \mathfrak{m} of E can be written as ξs_j^{-1} for a unique j and a totally positive $\xi \in s_j$. Given $\mathbf{f} = (f_1, \dots, f_{h^+}) \in M(U_0(\mathfrak{c}))$ with

$$f_\nu(z) = a_\nu(0) + \sum_{0 \ll \xi \in i(s_\nu)} a_\nu(\xi) e(\mathrm{tr}(\xi z))$$

define

$$(2.10) \quad c(\mathfrak{m}, \mathbf{f}) := \begin{cases} a_j(\xi) \prod_{\nu=1}^n \sigma_\nu(\xi)^{-1} & \text{if } \mathfrak{m} = \xi s_j^{-1} \text{ and } \mathfrak{m} \text{ is integral,} \\ 0 & \text{if } \mathfrak{m} \text{ is not integral.} \end{cases}$$

Furthermore, define

$$\mathbf{1} := (1, \dots, 1) \in \mathbb{Z}^n.$$

Then \mathbf{f} has a Fourier expansion

$$(2.11) \quad \mathbf{f} \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = c_0(y, \mathbf{f})|y| + \sum_{0 \ll \xi \in E} c(\xi y, \mathbf{f})(\xi y_\infty)^{\mathbf{1}} e(\text{tr}(\xi i y_\infty)) \chi_E(\xi x),$$

where $y \in \mathbb{A}_E^\times$, $x \in \mathbb{A}_E$, $y_\infty \gg 0$. Here χ_E is the character of the additive group \mathbb{A}_E/E such that

$$\chi_E(x_\infty) = e(\text{tr}(x_\infty)),$$

the function c_0 on the ideal group of E is defined by

$$c_0(\eta s_j^{-1}, \mathbf{f}) = a_j(0) N(s_j)^{-1} \text{ for } 0 \ll \eta \in E$$

and

$$|y| := |y|_{\mathbb{A}_E}.$$

Finally,

$$\zeta y_\infty := (\sigma_1(\zeta)y_1, \dots, \sigma_n(\zeta)y_n),$$

where y_ν is the component of y_∞ at the archimedean place associated to σ_ν . This construction defines a correspondence between the n -tuples that comprise $M(U_0(\mathfrak{c}))$ and a certain space of “holomorphic automorphic forms” on $\mathbf{G}(\mathbb{A})$. Holomorphicity in this setting is equivalent to vanishing under the Maass lowering operators (see, for example, [Ga, §3.3]).

3. HECKE OPERATORS ON HOLOMORPHIC AUTOMORPHIC FORMS

In the last section we associated a Hilbert modular variety $Y_0(\mathfrak{c}) = \bigcup_{j=1}^{h^+} Y_j$ to each integral ideal $\mathfrak{c} \subset \mathcal{O}$ through a choice of a compact open subgroup $U_0(\mathfrak{c}) \leq \mathbf{G}(\mathbb{A}^f)$ depending on \mathfrak{c} . We now recall the definition of the Hecke operators associated to $M(U_0(\mathfrak{c}))$. These operators act on modular forms in a manner compatible with the action of Hecke correspondences on classes in intersection homology (this compatibility will be made precise in §4). The primary references for this section are [Sh2] and [Sh1].

The Hecke algebra $\mathbb{T}_\mathfrak{c}$ associated to $U_0(\mathfrak{c})$ is the \mathbb{C} -algebra of all formal sums of double cosets

$$WyW,$$

with $y \in V$. In order to define the action on modular forms, we must introduce the algebra consisting of double cosets

$$\Gamma_j \alpha \Gamma_\mu$$

where $\alpha \in \mathbf{G}(\mathbb{Q}) \cap x_j V x_\mu^{-1}$ (see [Sh1, §3.2] for the definition of multiplication in these algebras). Writing a double coset as a disjoint union of right cosets

$$\Gamma_j \alpha \Gamma_\mu = \bigcup_i \Gamma_j \alpha_i,$$

define

$$f_j | \Gamma_j \alpha \Gamma_\mu := \sum_i f_j | \alpha_i$$

for $f_j \in S(\Gamma_j)$.

In order to define the action of WyW on $M(U_0(\mathfrak{c}))$, it will be convenient to introduce the following notation: For each ideal $\mathfrak{m} \subset \mathcal{O}$, define a permutation $\tau_\mathfrak{m}$ of the set of h^+ elements by requiring that

$$(3.1) \quad \mathfrak{m} s_j s_{\tau_\mathfrak{m}(j)}^{-1}$$

is in the ray class of the principal ideals modulo β_∞ for each $1 \leq j \leq h^+$. Now, note that we can find, for each $1 \leq j \leq h^+$, an element $\alpha_j(y) \in x_j V x_{\tau_{\det(y)}(j)}^{-1} \cap \mathbf{G}(\mathbb{Q})$ such that

$$(3.2) \quad WyW = W x_j^{-1} \alpha_j(y) x_{\tau_{\det(y)}(j)} W$$

(see [Sh2, p. 648]). For $\mathbf{f} = (f_1, \dots, f_{h^+}) \in M(U_0(\mathfrak{c}))$ we then define

$$(3.3) \quad \mathbf{f}|WyW = (g_1, \dots, g_{h^+}),$$

where $g_{\tau_{\det(y)}(j)} := f_j |\Gamma_j \alpha_j(y) \Gamma_{\tau_{\det(y)}(j)}$. Moreover, if

$$(3.4) \quad \Gamma_j \alpha_j(y) \Gamma_{\tau_{\det(y)}(j)} = \sum_{\lambda} \Gamma_j \alpha_{j,\lambda}$$

is a decomposition into disjoint right cosets, then we have a decomposition

$$(3.5) \quad WyW = \sum_{\lambda} Wx_j^{-1} \alpha_{j,\lambda} x_{\tau_{\det(y)}(j)}$$

into disjoint right cosets (see [Sh2, p. 648]). This observation is useful in determining how many right cosets are required to define a Hecke operator “componentwise.” In particular, we use (3.2-3.5) implicitly when dealing with the degrees of Hecke operators (defined in (3.9) below).

Finally, for each ideal $\mathfrak{m} \subset \mathcal{O}$ we have the Hecke operator

$$(3.6) \quad T(\mathfrak{m}) := \sum_{\substack{y \in V \\ \det(y) = \mathfrak{m}}} WyW$$

where the prime indicates that the sum is taken over a set of y defining a disjoint (finite) set of double coset representatives for the set of double cosets WyW such that $\det(y) = \mathfrak{m}$.

We now wish to describe the relations that the $T(\mathfrak{m})$ satisfy. In order to do this we require one more family of elements of $\mathbb{T}_{\mathfrak{c}}$. For every ideal $\mathfrak{m} \subset \mathcal{O}$, let $a_{\mathfrak{m}} \in (\mathbb{A}_E^f)^{\times}$ be an idèle whose associated ideal is \mathfrak{m} . Define

$$(3.7) \quad S(\mathfrak{m}) := \begin{cases} W \begin{pmatrix} a_{\mathfrak{m}} & 0 \\ 0 & a_{\mathfrak{m}} \end{pmatrix} W & \text{if } \mathfrak{m} + \mathfrak{c} = \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

The commutative algebra $\mathbb{T}_{\mathfrak{c}}$ is then generated by the $T(\mathfrak{p})$ and $S(\mathfrak{p})$ as \mathfrak{p} runs over the prime ideals of \mathcal{O} . This can be seen by examining the relations that the Hecke operators satisfy:

$$(3.8) \quad T(\mathfrak{m})T(\mathfrak{n}) = \sum_{\mathfrak{m} + \mathfrak{n} \subset \mathfrak{a}} N(\mathfrak{a})S(\mathfrak{a})T(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}),$$

(see [Sh2]). Note that the elements $S(\mathfrak{m})$ for $\mathfrak{m} + \mathfrak{c} = \mathcal{O}$ act trivially on $S(U_0(\mathfrak{c}))$ because the elements of $S(U_0(\mathfrak{c}))$ have trivial character (see [Sh2]).

We will later require the notion of the *degree* of a Hecke operator, which we now briefly recall. For each $y \in V$, define $t(y)$ by

$$WyW := \sum_{i=1}^{t(y)} Wy_i,$$

where the sum is comprised of disjoint right cosets. Following [Sh1, §3.1], define a map

$$\deg : \mathbb{T}_{\mathfrak{c}} \longrightarrow \mathbb{C}$$

by setting

$$\deg(WyW) := t(y),$$

and extending \mathbb{C} -linearly. It is a standard exercise to compute that

$$(3.9) \quad \deg(T(\mathfrak{m})) = \sigma'_{\mathfrak{c}}(\mathfrak{m}).$$

This formula will be important in §7 and §8 below.

The effect of a Hecke operator on Fourier expansions is as follows:

$$(3.10) \quad c(\mathfrak{m}, \mathbf{f}|T(\mathfrak{n})) = \sum_{\substack{\mathfrak{m} + \mathfrak{n} \subset \mathfrak{a} \\ (\mathfrak{a}, \mathfrak{c}) = 1}} N(\mathfrak{a}^{-1}\mathfrak{n})c(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathbf{f}).$$

In particular, if $\mathbf{f} \in M(U_0(\mathfrak{c}))$ is a normalized (i.e. $c(\mathcal{O}, \mathbf{f}) = 1$) eigenform for all $T(\mathfrak{m})$ with eigenvalues $a_{\mathbf{f}}(\mathfrak{m})$, then

$$(3.11) \quad c(\mathfrak{m}, \mathbf{f}) = a_{\mathbf{f}}(\mathfrak{m})N(\mathfrak{m})^{-1}.$$

Later we will want to work with these Hecke operators “componentwise.” We denote by

$$(3.12) \quad T_j(\mathfrak{m}) : S(\Gamma_j) \longrightarrow S(\Gamma_{\tau_{\mathfrak{m}}(j)})$$

the linear map induced by $T(\mathfrak{m})$. That is,

$$(3.13) \quad T_j(\mathfrak{m}) := \sum_i \Gamma_j \alpha_{ji} \Gamma_{\tau_{\mathfrak{m}}(j)}$$

where the α_{ji} are chosen so that the corresponding cosets WyW defined in (3.2) form a complete, disjoint set of representatives for the set of cosets WyW with $\det(y) = \mathfrak{m}$. Normalize the Petersson inner product of two cusp forms $f \in S(\Gamma_j)$, $g \in S(\Gamma_{j'})$ by

$$\langle f, g \rangle_P = \mu(\Gamma \backslash \mathfrak{H}^n)^{-1} \int_{\Gamma \backslash \mathfrak{H}^n} \overline{f(z)} g(z) y^{2(1)} d\mu(z)$$

where $d\mu(z) := \prod_{\nu=1}^n y_{\nu}^{-2} dx_{\nu} dy_{\nu}$ for $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n) \in \mathfrak{H}^n$ and Γ is a congruence subgroup of $\mathbf{G}(\mathbb{A})^+ \cap \mathbf{G}(\mathbb{Q})$ chosen so that f, g are both cusp forms for Γ (such a subgroup Γ always exists). We note that $\mu(\Gamma \backslash \mathfrak{H}^n)$ is finite for any congruence subgroup. We then have, for $f \in S(\Gamma_j)$ and $g \in S(\Gamma_{\mu})$,

$$(3.14) \quad \langle f | \Gamma_j \alpha \Gamma_{\mu}, g \rangle_P = \langle f, g | \Gamma_{\mu} \alpha' \Gamma_j \rangle_P$$

(see [Sh2, p. 652]). Here $\alpha' := (\det \alpha) \alpha^{-1}$ and the action of $\Gamma_{\mu} \alpha' \Gamma_j$ is defined by decomposing the double coset into a sum of disjoint right cosets as in (3.4). For $\mathbf{f} = (f_1, \dots, f_{h^+})$, $\mathbf{g} = (g_1, \dots, g_{h^+}) \in S(U_0(\mathfrak{c}))$, we have the inner product

$$(3.15) \quad \langle \mathbf{f}, \mathbf{g} \rangle_P := \sum_{j=1}^{h^+} \langle f_j, g_j \rangle_P.$$

It follows from (3.14) that the operators $T(\mathfrak{m})$ on $S(U_0(\mathfrak{c}))$ with $\mathfrak{m} + \mathfrak{c} = \mathcal{O}$ are Hermitian with respect to (3.15) (see [Sh2, p. 652]). Moreover, if ${}^*T(\mathfrak{m})$ is the adjoint of $T(\mathfrak{m})$ with respect to $\langle \cdot, \cdot \rangle_P$ as in the introduction, then its restriction to the $\tau_{\mathfrak{m}}(j)$ th component ${}^*T_{\tau_{\mathfrak{m}}(j)}(\mathfrak{m})$ satisfies

$$(3.16) \quad {}^*T_{\tau_{\mathfrak{m}}(j)}(\mathfrak{m}) = \sum_i \Gamma_{\tau_{\mathfrak{m}}(j)} \alpha_{ji}' \Gamma_j$$

where the α_{ji}' are as in (3.13) above.

Remark. Noting that the measure $\mu(Y_j(\mathfrak{c}))$ only depends on \mathfrak{c} and not j (see [Sh2, p. 651]), we have that $\mathbf{g}, \mathbf{f} \in S(U_0(\mathfrak{c}))$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_P$ if and only if

$$(f_1(z)dz, \dots, f_{h^+}(z)dz) \text{ and } (g_1(z)dz, \dots, g_{h^+}(z)dz)$$

are orthogonal with respect to the usual pairing of differential forms on the manifold $Y_0(\mathfrak{c})$ (which turns out to coincide with the L^2 -cohomology pairing in this case, see §6).

4. HECKE OPERATORS AND ZUCKER'S CONJECTURE

In §2 and §3 we recalled the adelic interpretation of Hilbert modular varieties, the Fourier expansions of Hilbert modular forms, and the basic properties of the Hecke operators on such forms. To relate the traces of Hecke operators to intersection homology, we will use the isomorphism between intersection homology and L^2 -cohomology given by Zucker's conjecture (proven by Looijenga and independently by Saper and Stern). The main result of this section states that, in our case of interest, this isomorphism is Hecke-equivariant.

We now set notation so that we can state this Hecke-equivariance property precisely (we roughly follow the model of [Hid2, §6.3]). For each $y \in V$, define

$$(4.1) \quad \Phi_j(y) := \alpha_j(y) \Gamma_{\tau_{\det(y)}(j)} \alpha_j(y)^{-1} \cap \Gamma_j$$

for each j . Set $\Phi_j(y)^{\alpha_j(y)} := \alpha_j^{-1}(y)\Phi_j(y)\alpha_j(y)$,

$$(4.2) \quad \begin{aligned} Y_{\Phi_j(y)} &:= \Phi_j(y) \backslash \mathfrak{H}^n, \\ Y_{\Phi_j(y)^{\alpha_j(y)}} &:= \Phi_j(y)^{\alpha_j(y)} \backslash \mathfrak{H}^n, \end{aligned}$$

and let

$$(4.3) \quad \begin{aligned} \pi_{1,j} : Y_{\Phi_j(y)} &\longrightarrow Y_j \\ \pi_{2,j} : Y_{\Phi_j(y)^{\alpha_j(y)}} &\longrightarrow Y_{\tau_{\det(y)}(j)} \end{aligned}$$

be the canonical projections. We denote by

$$(4.4) \quad \widetilde{\alpha_j(y)} : Y_{\Phi_j(y)} \longrightarrow Y_{\Phi_j(y)^{\alpha_j(y)}}$$

the map induced by multiplication by $\alpha_j(y)^{-1}$. The maps $\pi_{1,j}$, $\pi_{2,j}$, $\widetilde{\alpha_j(y)}$ extend uniquely to continuous maps on the relevant Bailey-Borel compactifications X_j , $X_{\Phi_j(y)}$, and $X_{\Phi_j(y)^{\alpha_j(y)}}$. We denote their extensions by the same symbols by abuse of notation.

Now let IH_\bullet denote intersection homology with middle perversity and complex coefficients (as defined in [GM2]) and let $H_{(2)}^\bullet$ denote L^2 -cohomology with respect to the canonical Bergmann metric on the symmetric spaces under consideration (also with complex coefficients). This metric, known also in this special case of an arithmetic quotient $\Gamma \backslash \mathfrak{H}^n$ as the Poincaré metric, is simply the metric induced by the metric on \mathfrak{H}^n given matrixally by

$$(x_1 + iy_1, \dots, x_n + iy_n) \mapsto \begin{pmatrix} y_1^{-2} & & 0 \\ & \ddots & \\ 0 & & y_n^{-2} \end{pmatrix}.$$

The maps $\pi_{1,j}$, $\pi_{2,j}$ and $\widetilde{\alpha_j(y)}$ are all finite, and hence induced push-forward and pull-back morphisms in intersection homology (see [GM3]), thus it makes sense to define

$$(4.5) \quad [\{ \Phi_j(y) \}_{j=1}^{h^+}] := (\pi_{2,1*} \widetilde{\alpha_1(y)}^* \pi_{1,1}^*, \dots, \pi_{2,h*} \widetilde{\alpha_h(y)}^* \pi_{1,h}^*).$$

The adjoint map ${}^t[\{ \Phi_j(y) \}_{j=1}^{h^+}]$ is defined in the obvious manner, namely by

$$(4.6) \quad {}^t[\{ \Phi_j(y) \}_{j=1}^{h^+}] := (\pi_{1,1*} \widetilde{\alpha_1(y)}^* \pi_{2,1}^*, \dots, \pi_{1,h*} \widetilde{\alpha_h(y)}^* \pi_{2,h}^*).$$

Given Γ_j and α_j as above, let

$$\Gamma_j \alpha_j(y) \Gamma_{\tau_{\det(y)}(j)} = \bigcup_{i_j} \Gamma_j \alpha_{j i_j}$$

be a decomposition into disjoint right cosets. Let z_j be the local coordinate on Y_j induced by the canonical projection $\mathfrak{H}^n \rightarrow Y_j$. For a differential form $\omega = (\omega_1(z_1), \dots, \omega_{h^+}(z_{h^+})) \in H_{(2)}^\bullet(Y_0(\mathfrak{c}))$, define

$$\omega | \{ \Gamma_j \alpha_j(y) \Gamma_{\tau(j)} \}_{j=1}^{h^+} (z_1, \dots, z_{h^+}) := \left(\sum_{i_{\tau^{-1}(1)}} \omega_{\tau^{-1}(1)}(\alpha_{\tau^{-1}(1)} i_{\tau^{-1}(1)} z_1), \dots, \sum_{i_{\tau^{-1}(h^+)}} \omega_{\tau^{-1}(h^+)}(\alpha_{\tau^{-1}(h^+)} i_{\tau^{-1}(h^+)} z_{h^+}) \right).$$

Here we set $\tau := \tau_{\det(y)}$ to ease notation. We define $\omega | \{ \Gamma_{\tau(j)} \alpha_j(y) \Gamma_j \}_{j=1}^{h^+}$ similarly starting with a decomposition

$$\Gamma_{\tau_{\det(y)}(j)} \alpha_j(y) \Gamma_j = \bigcup_{i_j} \Gamma_{\tau_{\det(y)}(j)} \alpha_{j i_j}$$

into disjoint right cosets. We now recall the following result:

Proposition 4.1. *There is an isomorphism $\Lambda : IH_\bullet(X) \rightarrow H_{(2)}^\bullet(X)$ such that the diagram*

$$\begin{array}{ccc} IH_i(X_0(\mathfrak{c})) & \xrightarrow{\Lambda} & H_{(2)}^{2n-i}(Y_0(\mathfrak{c})) \\ \{ \Phi_j(y) \}_{j=1}^{h^+} \Big\| & & \Big\| \{ \Gamma_j \alpha_j(y) \Gamma_{\tau_{\det(y)}(j)} \}_{j=1}^{h^+} \\ IH_i(X_0(\mathfrak{c})) & \xrightarrow{\Lambda} & H_{(2)}^{2n-i}(Y_0(\mathfrak{c})) \end{array}$$

commutes. Here the L_2 -cohomology groups are defined with respect to the Poincaré metric.

Proof. This is a special case of the Zucker conjecture, proven by Looijenga and independently by Saper and Stern (actually, in our case of interest, this is proved in [Zu, §6], and there is a proof complete with a Hecke-equivariance statement in [BL]). \square

Observing that the center of $\mathbf{G}(\mathbb{Q})$ acts trivially on \mathfrak{H}^n , we see that Proposition 4.1 implies that the following diagram commutes:

$$(4.7) \quad \begin{array}{ccc} IH_i(X_0(\mathfrak{c})) & \xrightarrow{\Lambda} & H_{(2)}^{2n-i}(Y_0(\mathfrak{c})) \\ \downarrow \iota[\{\Phi_j(y)\}_{j=1}^{h^+}] & & \downarrow \{\Gamma_{\tau_{\det(y)}(j)}\alpha_j(y)\iota\Gamma_j\}_{j=1}^{h^+} \\ IH_i(X_0(\mathfrak{c})) & \xrightarrow{\Lambda} & H_{(2)}^{2n-i}(Y_0(\mathfrak{c})). \end{array}$$

5. HECKE CYCLES AND A LEFSCHETZ COINCIDENCE FORMULA

In this section we define the Hecke cycles $Z(\mathfrak{m})$ mentioned in the introduction. We then state and prove Theorem 5.1, an important Lefschetz coincidence formula for the intersection numbers of $Z(\mathfrak{m})$ in terms of alternating traces on L^2 -cohomology. Armed with this theorem, the balance of the paper is devoted to the task of interpreting the quantities appearing in the formula in terms of the arithmetic of Hecke operators.

For each j , let $G(\widetilde{\alpha_j(y)}) \subset X_{\Phi(y)_j} \times X_{\Phi(y)_j^{\alpha_j(y)}}$ be the graph of the continuous map

$$\widetilde{\alpha_j(y)} : X_{\Phi(y)_j} \longrightarrow X_{\Phi(y)_j^{\alpha_j(y)}}.$$

In [GM3], Goresky and MacPherson associate an intersection homology class $[G(f)]$ to the graph $G(f) \subset X \times Y$ of any placid map $f : X \rightarrow Y$. It is easy to see that $\widetilde{\alpha_j(y)}$ is finite, hence placid. Therefore, we have a well-defined homology class

$$[G(\widetilde{\alpha_j(y)})] \in IH_{2n}(X_{\Phi(y)_j} \times X_{\Phi(y)_j^{\alpha_j(y)}}).$$

Moreover, the canonical map

$$\pi_j^y := \pi_{1,j}^y \times \pi_{2,j}^y : X_{\Phi(y)_j} \times X_{\Phi(y)_j^{\alpha_j(y)}} \longrightarrow X_j \times X_{\tau_{\det(y)}(j)}$$

is finite, so it makes sense to consider the cycle $\pi_{j*}^y [G(\widetilde{\alpha_j(y)})] \in IH_{2n}(X_j \times X_{\tau_{\det(y)}(j)})$. We set

$$(5.1) \quad Z_j(y) := \frac{1}{[\Gamma_j : \Phi(y)_j]} \pi_{j*}^y [G(\widetilde{\alpha_j(y)})].$$

Let

$$(5.2) \quad \iota_y : \bigoplus_{j=1}^{h^+} IH_{2n}(X_j \times X_{\tau_{\det(y)}(j)}) \hookrightarrow IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c}))$$

be the canonical injection. Moreover, for each double coset WyW , let

$$(5.3) \quad Z(y) = \iota_y(Z_1(y), \dots, Z_{h^+}(y)) \in IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})).$$

We can now finally make the

Definition. For each integral ideal $\mathfrak{m} \subset \mathcal{O}$ define the **Hecke cycle** $Z(\mathfrak{m})$ to be the intersection homology class

$$Z(\mathfrak{m}) := \sum_{y \in I} Z(y) \in IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})).$$

where the index set $I \subset V$ is chosen so that $T(\mathfrak{m}) = \sum_{y \in I} WyW$ (see (3.6)).

Now that we've defined the Hecke cycles appearing in the title of the paper, we might as well recall the intersection product that appears there as well. Let X be a pseudomanifold. Part of the charm of IH_\bullet is the existence of a generalized Poincaré duality pairing

$$(5.4) \quad \cdot : IH_k(X) \times IH_{\dim_{\mathbb{R}} X - k}(X) \longrightarrow H_0(X) \longrightarrow \mathbb{C}$$

that is nondegenerate ([GM1, §3.3]). Here H_\bullet denotes singular cohomology, and the last map, which we will denote by ϵ , is the sum of the augmentation maps on each connected component. We denote the intersection product *without* applying the augmentation by \star :

$$(5.5) \quad \star : IH_k(X) \times IH_{\dim_{\mathbb{R}} X - k}(X) \longrightarrow H_0(X),$$

though we won't use this often. We remark that this pairing is truly a generalization of the typical Poincaré pairing; the two pairings agree when both are defined (i.e. when X is a smooth compact manifold).

By definition, if we can compute $Z(y) \cdot Z(y')$ for arbitrary $y, y' \in V$, we can compute $Z(\mathfrak{m}) \cdot Z(\mathfrak{n})$ for arbitrary ideals $\mathfrak{m}, \mathfrak{n} \subset \mathcal{O}$. The main result of this section is a preliminary formula for these quantities:

Theorem 5.1. *The intersection number $Z(y) \cdot Z(y')$ is equal to the L^2 -Lefschetz coincidence number*

$$\sum_{i=0}^{4n} (-1)^i \text{Tr} \left({}^t \{ \Gamma_j \alpha_j(y) \Gamma_{\tau_{\det(y)}(j)} \}_{j=1}^{h^+} \circ \{ \Gamma_j \alpha_j(y') \Gamma_{\tau_{\det(y')}(j)} \}_{j=1}^{h^+} : H_{(2)}^i(Y_0(\mathfrak{c})) \rightarrow H_{(2)}^i(Y_0(\mathfrak{c})) \right).$$

The following lemma deals with a trivial case:

Lemma 5.2. *If $\det(y)$ and $\det(y')$ are not in the same ray class modulo β_∞ (the product of the infinite primes) then*

$$Z(y) \cdot Z(y') = 0.$$

Hence if \mathfrak{m} and \mathfrak{n} are not in the same ray class modulo β_∞ , then

$$Z(\mathfrak{m}) \cdot Z(\mathfrak{n}) = 0.$$

Proof. Suppose $\tau_{\det(y)}(j) = \tau_{\det(y')}(j)$ for some $1 \leq j \leq h^+$. Then, since $\det(y) s_j s_{\tau_{\det(y)}(j)}^{-1}$ and $\det(y') s_j s_{\tau_{\det(y')}(j)}^{-1}$ are equivalent in the ray class group modulo β_∞ by (3.1), it follows that $\det(y)$ and $\det(y')$ are equivalent in the ray class group modulo β_∞ .

Thus, if $\det(y)$ and $\det(y')$ are not in the same ray class modulo β_∞ , then $\tau_{\det(y)}(j) \neq \tau_{\det(y')}(j)$ for all $1 \leq j \leq h^+$, and hence the images of ι_y and $\iota_{y'}$ are disjoint (see (5.2)). But $Z(y)$ is in the image of ι_y and $Z(y')$ is in the image of $\iota_{y'}$; the lemma follows. \square

Thus we may assume that $\det(y)$ and $\det(y')$ are in the same ray class modulo β_∞ .

We will now state some results we require for the proof of Theorem 5.1 in the nontrivial case. First note that if $\det(y)$ and $\det(y')$ are in the same ray class modulo β_∞ (and trivially in the other case) we have the identity

$$Z(y) \cdot Z(y') = \sum_{j=1}^{h^+} Z_j(y) \cdot Z_j(y')$$

by analysis, similar to that used in the proof of Lemma 5.2, of the direct summands of $IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c}))$ corresponding to connected components. Thus (for the moment at least) we can restrict our attention to the $Z_j(y)$ and $Z_j(y')$. Our immediate task is to give an expression for $Z_j(y)$ in terms of a basis for $IH_\bullet(X_j \times X_{\tau_{\det(y)}(j)})$. We will use the following Künneth formula to write down a particularly nice basis:

Proposition 5.3. [GM2] *If X and Y are complex analytic varieties, then there is a natural isomorphism*

$$IH_\bullet(X \times Y) \cong IH_\bullet(X) \otimes IH_\bullet(Y).$$

Now let e_1, \dots, e_r be a basis for $IH_\bullet(X_j)$ with dual basis e_1^*, \dots, e_r^* and $P := (\epsilon(e_i, e_j))$. Moreover let f_1, \dots, f_s be a basis for $IH_\bullet(X_{\tau_{\det(y)}(j)})$ with dual basis f_1^*, \dots, f_s^* and $Q := (\epsilon(f_i, f_j))$. Finally, let $|v|$ denote the dimension of a homology class v . We have the following:

Lemma 5.4. *Let $G = (G_{ij})$ be the matrix of $\pi_{2,j}^y \widetilde{\alpha_j(y)} \pi_{1,j}^{y*}$ with respect to the bases e_1, \dots, e_r and f_1, \dots, f_s . Then*

$$\begin{aligned} Z_j(y) &= \sum_{i,k} (-1)^{|e_i|(2n-|e_i|)} G_{ij} e_i^* \otimes f_k \\ &= \sum_{i,k} (-1)^{|e_i|(4n)} U_{ij} e_i \otimes f_k^* \end{aligned}$$

where U is the matrix $U = P^{-1}GQ$.

Assuming this lemma, the proof of Theorem 5.1 is easy:

Proof of Theorem 5.1. Notice that the matrix U appearing in Lemma 5.4 has the property that tU is the matrix of the adjoint of $\pi_{2,j*}^y \widetilde{\alpha_j(y)}_* \pi_{1,j}^{y*}$ with respect to the generalized Poincaré pairing written with respect to the bases f_1, \dots, f_s and e_1, \dots, e_r . A simple calculation using the projection formula for intersection homology (see, e.g., Lemma 5.1 of [GG]) proves that $\pi_{1,j*} \widetilde{\alpha_j(y)}^* \pi_{2,j}^{y*}$ coincides with this adjoint (justifying the notation of (4.6)). Thus Lemma 5.4 implies that $Z(y) \cdot Z(y')$ is equal to

$$\sum_{i=0}^{4n} (-1)^i \text{Tr} \left({}^t[\{\Phi_j(y)\}_{j=1}^{h^+}] \circ [\{\Phi_j(y')\}_{j=1}^{h^+}] : IH_i(X_0(\mathfrak{c})) \rightarrow IH_i(X_0(\mathfrak{c})) \right).$$

Noting that $(-1)^i = (-1)^{4n-i}$, the theorem now follows from Proposition 4.1. \square

Lemma 5.4 is very similar to a result in [GM3], which we now state for use in the proof of Lemma 5.4. As in [GM3], let X, Y be complex analytic varieties of dimension r', s' respectively. Moreover let μ_1, \dots, μ_t be a basis for $IH_\bullet(X)$ with dual basis μ_1^*, \dots, μ_t^* and let P' be the matrix $P' := (\epsilon(\mu_i \cdot \mu_j))$ where ϵ is the augmentation. Similarly, let ν_1, \dots, ν_u be a basis for $IH_\bullet(Y)$ with dual basis ν_1^*, \dots, ν_u^* and let $Q' := (\epsilon(\nu_i \cdot \nu_j))$. Let $\phi : X \rightarrow Y$ be a finite (hence placid) map, and let $F = (F_{ij})$ be the matrix of $\phi_* : IH_\bullet(X) \rightarrow IH_\bullet(Y)$ with respect to the bases μ_1, \dots, μ_t and ν_1, \dots, ν_u . With this notation in place, we have the

Proposition 5.5. [GM3] *The homology class of $G(\phi)$ in $IH_\bullet(X \times Y)$ is given by*

$$\begin{aligned} [G(\phi)] &= \sum_{i,j} (-1)^{|\mu_i|(r' - |\mu_i|)} F_{ij} \mu_i^* \otimes \nu_j \\ &= \sum_{i,j} (-1)^{|\mu_i|(r' + s')} T_{ij} \mu_i \otimes \nu_j^* \end{aligned}$$

where T is the matrix $T = P'^{-1} F Q'$.

For sign conventions, see the appendix to [GM3].

We can now prove Lemma 5.4:

Proof of Lemma 5.4. Since $\pi_{1,j}^y$ is finite and is generically a degree $[\Gamma_j : \Phi(y)_j]$ covering map, $\pi_{1,j*}^y$ maps the fundamental intersection homology class of $X_{\Phi(y)_j}$ to $[\Gamma_j : \Phi(y)_j]$ times the fundamental intersection homology class of X_j . Thus, applying the projection formula for intersection homology (see [GM3, §16.3]), it follows that

$$\pi_{1,j*}^y \pi_{1,j}^{y*} : IH_\bullet(X_j) \longrightarrow IH_\bullet(X_j)$$

is just multiplication by $[\Gamma_j : \Phi(y)_j]$.

Now choose a basis e_1, \dots, e_r for $IH_\bullet(X_j)$ with dual basis e_1^*, \dots, e_r^* . Using the projection formula again, we compute:

$$\begin{aligned} (5.6) \quad \epsilon \left(\pi_{1,j*}^y \left(\pi_{1,j}^{y*} e_i^* \star \pi_{1,j}^{y*} e_k \right) \right) &= e_i^* \cdot \pi_{1,j*}^y \pi_{1,j}^{y*} e_k = [\Gamma_j : \Phi(y)_j] e_i^* \cdot e_k \\ &= \delta_{i,k} [\Gamma_j : \Phi(y)_j]. \end{aligned}$$

Here ϵ is the augmentation, \star is as in (5.5), and $\delta_{i,j}$ is the Kronecker δ -function. As mentioned above, $\pi_{1,j}^y$ is generically an orientation preserving degree $[\Gamma_j : \Phi_j(y)]$ covering map; it follows that $\pi_{1,j*}^y : H_0(X_{\Phi(y)_j}) \rightarrow H_0(X_j)$ just maps the fundamental class of $X_{\Phi(y)_j}$ to $[\Gamma_j : \Phi_j(y)]$ times the fundamental class of X_j . Thus the computation (5.6) shows that

$$(5.7) \quad (\pi_{1,j}^{y*} e_i)^* = \pi_{1,j}^{y*} e_i^*.$$

Now, since $\pi_{1,j*}^y \pi_{1,j}^{y*}$ is just multiplication by $[\Gamma_j : \Phi(y)_j]$, it follows in particular that $\pi_{1,j}^{y*}$ is an injection. Thus we may choose a basis μ_1, \dots, μ_t for $IH_\bullet(X_{\Phi(y)_j})$ such that $\mu_i = \pi_{1,j}^{y*} e_i$ for $1 \leq i \leq r$ and $\pi_{1,j*}^y \mu_i = 0$ for $r < i \leq t$. Similarly, we may choose a basis ν_1, \dots, ν_u for $IH_\bullet(X_{\Phi(y)_j}^{\alpha_j(y)})$ with $\nu_i = \pi_{2,j}^{y*} f_i$ for $1 \leq i \leq s$ and $\pi_{2,j*}^y \nu_i = 0$ for $s < i \leq u$. If we write $F = (F_{ij})$ for the matrix of $\alpha(y)_{j*}$ with respect to the bases μ_1, \dots, μ_t and ν_1, \dots, ν_u , Propositions 5.3 and 5.5 yield

$$\begin{aligned}
Z_j(y) &= \frac{1}{[\Gamma_j : \Phi(y)_j]} \pi_{j*}^y \left(\sum_{i=1}^t \sum_{k=1}^u (-1)^{|\mu_i|(2n-|\mu_i|)} F_{ik} \mu_i^* \otimes \nu_k \right) \\
&= \frac{1}{[\Gamma_j : \Phi(y)_j]} \sum_{i=1}^t \sum_{k=1}^u (-1)^{|\mu_i|(2n-|\mu_i|)} F_{ik} \pi_{1,j*}^y \mu_i^* \otimes \pi_{2,j*}^y \nu_k \\
&= \frac{1}{[\Gamma_j : \Phi(y)_j]} \sum_{i=1}^r \sum_{k=1}^s (-1)^{|e_i|(2n-|e_i|)} F_{ik} \pi_{1,j*}^y (\pi_{1,j}^{y*} e_i)^* \otimes \pi_{2,j*}^y \pi_{2,j}^{y*} f_k \\
&\quad + \frac{1}{[\Gamma_j : \Phi(y)_j]} \sum_{i>r \text{ or } k>s} (-1)^{|\mu_i|(2n-|\mu_i|)} F_{ik} \pi_{1,j*}^y \mu_i^* \otimes \pi_{2,j*}^y \nu_k \\
&= \frac{1}{[\Gamma_j : \Phi(y)_j]} \sum_{i=1}^r \sum_{k=1}^u (-1)^{|e_i|(2n-|e_i|)} F_{ik} \pi_{1,j*}^y (\pi_{1,j}^{y*} e_i)^* \otimes \pi_{2,j*}^y \pi_{2,j}^{y*} f_k.
\end{aligned}$$

Using (5.7), we then have

$$\begin{aligned}
(5.8) \quad Z_j(y) &= \frac{1}{[\Gamma_j : \Phi(y)_j]} \sum_{i=1}^r \sum_{k=1}^u (-1)^{|e_i|(2n-|e_i|)} F_{ik} \pi_{1,j*}^y \pi_{1,j}^{y*} e_i^* \otimes \pi_{2,j*}^y \pi_{2,j}^{y*} f_k \\
&= \sum_{i=1}^r \sum_{k=1}^u (-1)^{|e_i|(2n-|e_i|)} [\Gamma_{\tau_{\det(y)}(j)} : \Phi(y)_j^{\alpha(y)_j}] F_{ik} e_i^* \otimes f_k.
\end{aligned}$$

We note that $([\Gamma_{\tau_{\det(y)}(j)} : \Phi(y)_j^{\alpha(y)_j}] F_{ik})_{\substack{1 \leq i \leq r \\ 1 \leq k \leq s}}$ is the matrix of

$$\pi_{2,j*}^y \widetilde{\alpha_j(y)}_* \pi_{1,j}^{y*} : IH_\bullet(X_j) \longrightarrow IH_\bullet(X_{\tau_{\det(y)}(j)})$$

with respect to the bases e_1, \dots, e_r and f_1, \dots, f_s . This proves the first equality in the lemma. The second follows from the first by linear algebra. \square

Thus we have translated the computation of the intersection numbers $Z(\mathbf{m}) \cdot Z(\mathbf{n})$ into the computation of the traces of $\{\Gamma_j \alpha_j(y) \Gamma_{\tau_{\det(y)}(j)}\}_{j=1}^{h^+}$ on L^2 -cohomology. Section 6 recalls a description of this cohomology (due to Harder and Hida) which we will use in §7 to compute certain traces of this form.

6. L^2 -COHOMOLOGY

In the last section, we concluded with Theorem 5.1, a formula for the intersection numbers of Hecke cycles in terms of alternating traces on L^2 -cohomology. To interpret such traces, we must collect some results on the L^2 -cohomology of $Y_0(\mathfrak{c})$ with respect to the Poincaré metric (see §4). In particular, our goal is to use Harder's notion of Eisenstein cohomology ([Har], [Hid3]) to give an explicit basis of $H_{(2)}^\bullet(Y_0(\mathfrak{c}))$. The reader might also compare [BL]. Our treatment of Harder's theory is modeled on [F, §III].

First notice that, given any $\mathbf{F} = (f_1, \dots, f_{h^+}) \in S(U_0(\mathfrak{c}))$, we obtain a differential form

$$(6.1) \quad \Omega_{\mathbf{F}}$$

given on the j th component Y_j by the holomorphic form $f_j(z) dz_1 \wedge \dots \wedge dz_n$. Here $z = (z_1, \dots, z_n)$ is the coordinate induced by the canonical projection $\mathfrak{H}^n \rightarrow Y_j$. It follows from the existence of the Petersson inner product defined in §3 that $\Omega_{\mathbf{F}} \in H_{(2)}^n(Y_0(\mathfrak{c}))$. This yields a linear map $S(U_0(\mathfrak{c})) \rightarrow H_{(2)}^n(Y_0(\mathfrak{c}))$ that is well-known to be injective.

We now show how to construct non-holomorphic forms from $\Omega_{\mathbf{F}}$. Consider the component group

$$\pi_0(\mathbf{G}(\mathbb{R})) = \{(\gamma_1, \dots, \gamma_n) : \gamma_i = \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

For each $b \subset \{1, \dots, n\}$, we have an element $\iota_b = (\gamma_1, \dots, \gamma_n) \in \pi_0(\mathbf{G}(\mathbb{R}))$ such that γ_i is the identity if and only if $i \notin b$. The element ι_b induces an involution

$$\begin{aligned}
(6.2) \quad \iota_b : \mathbf{G}(\mathbb{R})/K_\infty &\longrightarrow \mathbf{G}(\mathbb{R})/K_\infty \\
(z_1, \dots, z_n) &\longmapsto (w_1, \dots, w_n)
\end{aligned}$$

where

$$w_i := \begin{cases} \bar{z}_i & \text{if } i \in b \\ z_i & \text{if } i \notin b. \end{cases}$$

Here we have used the identification $\mathbf{G}(\mathbb{R})/K_\infty = (\mathbb{C} - \mathbb{R})^n$ (see (2.1)). The involution ι_b induces another involution

$$(6.3) \quad \iota_b : Y_0(\mathfrak{c}) \rightarrow Y_0(\mathfrak{c})$$

which we denote by the same symbol by abuse of notation. We then have, for every $b \subset \{1, \dots, n\}$, the forms

$$\iota_b^* \Omega_{\mathbf{F}} \in H_{(2)}^n(Y_0(\mathfrak{c})).$$

Using these forms we can identify a subspace of *cuspidal classes*

$$(6.4) \quad H_{\text{cusp}}^n(Y_0(\mathfrak{c})) := \bigoplus_{b \subset \{1, \dots, n\}} \{\iota_b^* \Omega_{\mathbf{F}} : \mathbf{F} \in S(U_0(\mathfrak{c}))\} \leq H_{(2)}^n(Y_0(\mathfrak{c})).$$

The fact that this is a direct sum decomposition is clear upon considering the variables in which a given differential form in the decomposition is holomorphic or anti-holomorphic. By convention, $H_{\text{cusp}}^m(Y_0(\mathfrak{c})) = 0$ for $m \neq n$.

We now identify a different subspace of the L^2 -cohomology. If $b = \{b_1, \dots, b_r\}$ with $1 \leq b_1 < \dots < b_r \leq n$, let

$$dz_b = dz_{b_1} \wedge \dots \wedge dz_{b_r}$$

and similarly for $d\bar{z}_b$. Let $a = \{1, \dots, n\} - b$, $1 \leq j \leq h^+$ and define the *universal* class η_j^a which is

$$(6.5) \quad y_{b_1}^{-2} dz_{b_1} \wedge d\bar{z}_{b_1} \wedge \dots \wedge y_{b_r}^{-2} dz_{b_r} \wedge d\bar{z}_{b_r}$$

on the j th factor and zero elsewhere (universal because they are invariant under the whole group $\text{GL}_2(\mathbb{R})^n$). We take the convention that $\eta_j^{\{1, \dots, n\}}$ is the function that is identically 1 on the j th component and zero elsewhere. Because $Y_0(\mathfrak{c})$ has finite volume with respect to the Petersson metric, each of these forms defines an L^2 -class, moreover, these classes are all linearly independent. Thus we have identified a subspace of universal classes

$$(6.6) \quad H_{\text{univ}}^{2k}(Y_0(\mathfrak{c})) := \bigoplus_{j=1}^{h^+} \bigoplus_{\substack{a \subset \{1, \dots, n\} \\ \#a = n-k}} \mathbb{C} \eta_j^a \leq H_{(2)}^{2k}(Y_0(\mathfrak{c})).$$

It is easy to see that the subspace of $H_{(2)}^\bullet(Y_0(\mathfrak{c}))$ spanned by the universal classes has zero intersection with the subspace spanned by the cuspidal classes. By convention, $H_{\text{univ}}^m(Y_0(\mathfrak{c})) = 0$ if $m \notin 2\mathbb{Z}$.

The main result of this section tells us that the classes we just introduced form a basis of the of L^2 -cohomology groups of $Y_0(\mathfrak{c})$:

Proposition 6.1. *Choose an ideal $\mathfrak{c} \subset \mathcal{O}$. Then we have a decomposition*

$$H_{(2)}^m(Y_0(\mathfrak{c})) = H_{\text{univ}}^m(Y_0(\mathfrak{c})) \oplus H_{\text{cusp}}^m(Y_0(\mathfrak{c})).$$

Remark. The proof of Proposition 6.1 we give is elementary assuming Theorem 6.2 below. However, as pointed out by the referee, a more efficient (and perhaps more enlightening) proof could be given by using automorphic representation theory to treat L^2 -cohomology directly and then applying some facts about the (\mathfrak{g}, K) -cohomology of admissible representations of $\text{GL}_2(\mathbb{R})$.

In the interest of recalling some relevant results, we defer the proof for a moment. First recall that if X is a projective variety of complex dimension n with isolated singularities and regular set X^{reg} then

$$(6.7) \quad IH_{2n-i}(X) \cong \begin{cases} H^i(X^{\text{reg}}) & \text{if } i < n \\ \text{Im}(H^n(X) \rightarrow H^n(X^{\text{reg}})) & \text{if } i = n \\ H^i(X) & \text{if } i > n, \end{cases}$$

where the H^\bullet denotes singular cohomology (see [SZ, §5.2]). Noting that $X_0(\mathfrak{c})$ is a complex projective variety with isolated singularities, (6.7) in conjunction with Zucker's conjecture (Propositions 4.1) gives us the dimension of $H_{(2)}^m(Y_0(\mathfrak{c}))$ in terms of singular cohomology groups.

Let $\bar{Y}_0(\mathfrak{c})$ denote the Borel-Serre compactification of $Y_0(\mathfrak{c})$ and set $\partial\bar{Y}_0(\mathfrak{c}) = \bar{Y}_0(\mathfrak{c}) - Y_0(\mathfrak{c})$. For generalities on the Borel-Serre compactification, see [Sa], [Go]; in particular compare [Sa, §5.2] and [F, p. 242]. Following Harder, let

$$(6.8) \quad \tilde{H}^\bullet(Y_0(\mathfrak{c})) := \ker(H^\bullet(\bar{Y}_0(\mathfrak{c})) \rightarrow H^\bullet(\partial\bar{Y}_0(\mathfrak{c})))$$

where the homomorphism is induced by inclusion. Then we have the following

Theorem 6.2 (Harder and Hida). *Let $\mathfrak{c} \subset \mathcal{O}$ be an ideal. If $1 < m < 2n$ then*

$$\dim_{\mathbb{C}} \tilde{H}^m(Y_0(\mathfrak{c})) = \dim_{\mathbb{C}} (H_{\text{cusp}}^m(Y_0(\mathfrak{c})) \oplus H_{\text{univ}}^m(Y_0(\mathfrak{c}))).$$

Moreover, for $m < n$,

$$\dim_{\mathbb{C}} H^m(Y_0(\mathfrak{c})) = \dim_{\mathbb{C}} \tilde{H}^m(Y_0(\mathfrak{c}))$$

and for $m = n$,

$$\dim_{\mathbb{C}} H^n(Y_0(\mathfrak{c})) = \dim_{\mathbb{C}} \tilde{H}^n(Y_0(\mathfrak{c})) + \dim_{\mathbb{C}} H^n(\partial\bar{Y}_0(\mathfrak{c})).$$

Proof. Hida, in [Hid3, Proposition 3.1], identifies $H_{\text{cusp}}^m(Y_0(\mathfrak{c}))$ as a direct summand of $H^m(Y_0(\mathfrak{c}))$. This direct summand can be naturally identified with what Harder denotes by $H_{\text{cusp}}^m(S_{U_0(\mathfrak{c})}, \tilde{\mathbb{C}})$ (see [Har, §3.1-3.2] and [Sch, §4.1]). Here $\tilde{\mathbb{C}}$ is the sheaf over $Y_0(\mathfrak{c})$ associated to the trivial representation of \mathbf{G} . This cohomology group can in turn be identified with a direct summand of $\tilde{H}^m(Y_0(\mathfrak{c}))$ (which, in Harder's notation, is $\tilde{H}^m(S_{U_0(\mathfrak{c})}, \tilde{\mathbb{C}})$).

Its orthogonal complement in $\tilde{H}^m(Y_0(\mathfrak{c}))$, in the notation of [Har, p. 65], is isomorphic to the cohomology group $\tilde{H}_{\text{res}}^m(S_{U_0(\mathfrak{c})}, \tilde{\mathbb{C}})$. It is an easy exercise using the descriptions of this group given in Proposition 3.2.4, p. 62, and p. 65 of [Har] to show that the dimension of it is equal to $\dim_{\mathbb{C}} H_{\text{univ}}^m(Y_0(\mathfrak{c}))$. This proves the first statement of the theorem. The second follows from the first and [Har, Theorem 2] after passing to $U_0(\mathfrak{c})$ -fixed cohomology. \square

Now we wish to relate $\dim_{\mathbb{C}} \tilde{H}^m(Y_0(\mathfrak{c}))$ to $\dim_{\mathbb{C}} H_{(2)}^m(Y_0(\mathfrak{c}))$. For this we require the following

Theorem 6.3 (Harder). *Let $\mathfrak{c} \subset \mathcal{O}$ be an ideal. If $n \leq m \leq 2n - 2$ then the homomorphism*

$$H^m(\bar{Y}_0(\mathfrak{c})) \longrightarrow H^m(\partial\bar{Y}_0(\mathfrak{c}))$$

induced by inclusion is surjective.

Proof. This follows from [Har, Theorem 2] after passing to $U_0(\mathfrak{c})$ -fixed cohomology. \square

We require two more preparatory results. For the purpose of stating them, if $\Gamma \leq \mathbf{G}(\mathbb{R})^0$ is a congruence subgroup, let

$$Y_\Gamma := \Gamma \backslash \mathfrak{H}^n$$

be the associated Hilbert modular variety, X_Γ be its Bailey-Borel compactification, \bar{Y}_Γ its Borel-Serre compactification, and $\partial\bar{Y}_\Gamma := \bar{Y}_\Gamma - Y_\Gamma$. We have the following lemmas:

Lemma 6.4. *Let $\Gamma \leq \mathbf{G}(\mathbb{R})^0$ be a congruence subgroup. If $m > 1$, then*

$$\dim_{\mathbb{C}} H^m(\partial\bar{Y}_\Gamma) = \dim_{\mathbb{C}} H^{m+1}(X_\Gamma, Y_\Gamma).$$

Proof. Let $\{\kappa_i\}$ be the cusps of Γ . Choose A_i such that $A_i \kappa_i = \infty$. Define

$$U_C := \{z \in \mathfrak{H}^n : \prod_{i=1}^n y_i \geq C\}$$

and let

$$p : \mathfrak{H}^n \longrightarrow Y_\Gamma$$

be the canonical projection. By reduction theory, for sufficiently large r_i , Γ_{κ_i} acts on

$$D_i := \{A_i^{-1}z : z \in \mathfrak{H}^n \text{ and } \prod_{i=1}^n y_i = r_i\}$$

and the sets

$$p(A_i^{-1}(U_{r_i})) \cup \kappa_i$$

form a set of disjoint (closed) neighborhoods of the cusp κ_i in X_Γ (compare [F, Lemma I.2.8]). Moreover the action of Γ_{κ_i} on D_i for each i yields a homeomorphism

$$\coprod_i \Gamma_{\kappa_i} \backslash D_i \simeq \partial \bar{Y}_\Gamma.$$

Thus it suffices to prove that

$$(6.9) \quad H^{m+1}(X_\Gamma, Y_\Gamma) \cong H^m \left(\coprod_i \Gamma_{\kappa_i} \backslash D_i \right).$$

By excision,

$$H^m(X_\Gamma, Y_\Gamma) \cong \bigoplus_i H^m(p(A_i^{-1}(U_{r_i})) \cup \kappa_i, p(D_i)).$$

Moreover, $p(A_i^{-1}(U_{r_i})) \cup \kappa_i$ is homeomorphic to the cone over $p(D_i)$ (compare [F, p. 144]), and hence is homotopically trivial. Thus

$$H^m(p(A_i^{-1}(U_{r_i})) \cup \kappa_i) = 0$$

for $m > 0$. The long exact sequence of the pair $(p(A_i^{-1}(U_{r_i})) \cup \kappa_i, p(D_i))$ then provides the isomorphism (6.9). \square

Lemma 6.5. *Let $\Gamma \leq \mathbf{G}(\mathbb{R})^0$ be a congruence subgroup. Then there is an exact sequence*

$$\xrightarrow{\partial} H_c^\bullet(Y_\Gamma) \longrightarrow H^\bullet(Y_\Gamma) \xlongequal{\quad} H^\bullet(\bar{Y}_\Gamma) \longrightarrow H^\bullet(\partial \bar{Y}_\Gamma) \xrightarrow{\partial}$$

where H_c^\bullet denotes singular cohomology with compact supports and the unmarked arrows are the canonical homomorphisms.

Proof. If Γ has the property that

$$(6.10) \quad \Gamma / (\Gamma \cap \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in E^\times \right\})$$

is torsion free, then \bar{Y}_Γ is a topological manifold with boundary $\partial \bar{Y}_\Gamma$. Thus the lemma is a special case of a standard fact from the topological theory of manifolds with boundary (see [F, A.III.XIX] or [Sch, §4.2]). If Γ does not have the property that (6.10) is torsion-free, then choose a congruence subgroup $\Gamma' \leq \Gamma$ with $[\Gamma : \Gamma'] < \infty$. The general case then follows from a standard argument using the fact that the canonical projection $p : \bar{Y}_{\Gamma'} \rightarrow \bar{Y}_\Gamma$ is a finite, orientation-preserving map. \square

We can now prove Proposition 6.1:

Proof of Proposition 6.1. Note that for a connected pseudomanifold X of dimension $2n$ we have $IH^0(X) \approx IH^{2n}(X) \approx \mathbb{C}$. Applying Zucker's conjecture (Proposition 4.1) we have proven the proposition in the cases $m = 0$ and $m = 2n$.

Assume $n = 1$. Then there is an ‘‘Eichler-Shimura’’ isomorphism

$$H_{(2)}^1(Y_0(\mathfrak{c})) \cong S(U_0(\mathfrak{c})) \oplus \overline{S(U_0(\mathfrak{c}))}$$

(see [Sa, §2]). A dimension count now shows that the elements of $H_{(2)}^\bullet(Y_0(\mathfrak{c}))$ given in (6.4,6.6) span the space.

Now assume $n > 1$. Consider the long exact sequence of the pair $(\bar{Y}_0(\mathfrak{c}), \partial \bar{Y}_0(\mathfrak{c}))$:

$$\longrightarrow H^{m-1}(\bar{Y}_0(\mathfrak{c})) \xrightarrow{a} H^{m-1}(\partial \bar{Y}_0(\mathfrak{c})) \longrightarrow H^m(\bar{Y}_0(\mathfrak{c}), \partial \bar{Y}_0(\mathfrak{c})) \xrightarrow{b} H^m(\bar{Y}_0(\mathfrak{c})) \longrightarrow$$

If $m > n$, then by Theorem 6.3, the map a is surjective which implies the map b is injective. Thus, by exactness,

$$H^m(\bar{Y}_0(\mathfrak{c}), \partial \bar{Y}_0(\mathfrak{c})) \hookrightarrow \tilde{H}^m(Y_0(\mathfrak{c})).$$

On the other hand, we have a homeomorphism $\bar{Y}_0(\mathfrak{c})/\partial \bar{Y}_0(\mathfrak{c}) \simeq X_0(\mathfrak{c})$, and it follows that $H^m(\bar{Y}_0(\mathfrak{c}), \partial \bar{Y}_0(\mathfrak{c})) \cong H^m(X_0(\mathfrak{c}))$. Applying (6.7), we have that

$$\dim_{\mathbb{C}} H_{(2)}^m(Y_0(\mathfrak{c})) \leq \dim_{\mathbb{C}} \tilde{H}^m(Y_0(\mathfrak{c})).$$

Applying Theorem 6.2, we see that the forms in $H_{(2)}^m(Y_0(\mathfrak{c}))$ we constructed in (6.4,6.6) span the cohomology group. This proves the proposition for $m > n$. If we apply generalized Poincaré duality (5.4) and Zucker's

conjecture (Proposition 4.1) to conclude that $\dim_{\mathbb{C}} H_{(2)}^m(Y_0(\mathfrak{c})) = \dim_{\mathbb{C}} H_{(2)}^{2n-m}(Y_0(\mathfrak{c}))$, we see that the forms we have constructed for $m < n$ span the relevant cohomology groups as well.

Thus we are left with the case $m = n$. Note that the homomorphism $H^n(X_0(\mathfrak{c})) \rightarrow H^n(X_0(\mathfrak{c})^{\text{reg}})$ induced by inclusion factors as $H^n(X_0(\mathfrak{c})) \rightarrow H^n(Y_0(\mathfrak{c})) \rightarrow H^n(X_0(\mathfrak{c})^{\text{reg}})$ where the two homomorphisms are induced by inclusions. It follows that

$$\dim_{\mathbb{C}} \text{Im}(H^n(X_0(\mathfrak{c})) \rightarrow H^n(X_0(\mathfrak{c})^{\text{reg}})) \leq \dim_{\mathbb{C}} \text{Im}(H^n(X_0(\mathfrak{c})) \rightarrow H^n(Y_0(\mathfrak{c}))).$$

Thus, if we show

$$(6.11) \quad \dim_{\mathbb{C}} \text{Im}(H^n(X_0(\mathfrak{c})) \rightarrow H^n(Y_0(\mathfrak{c}))) \leq \dim_{\mathbb{C}} (H_{\text{cusp}}^n(Y_0(\mathfrak{c})) \oplus H_{\text{univ}}^n(Y_0(\mathfrak{c})))$$

then (6.7) together with Zucker's conjecture (Proposition 4.1) will imply that the forms we have constructed in $H_{(2)}^n(Y_0(\mathfrak{c}))$ span the cohomology group, which will finish the proof of the proposition.

We now prove (6.11). The following is a portion of the long exact sequence of the pair $(X_0(\mathfrak{c}), Y_0(\mathfrak{c}))$:

$$(6.12) \quad \longrightarrow H^n(X_0(\mathfrak{c})) \longrightarrow H^n(Y_0(\mathfrak{c})) \xrightarrow{\partial} H^{n+1}(X_0(\mathfrak{c}), Y_0(\mathfrak{c})) \xrightarrow{\alpha} H^{n+1}(X_0(\mathfrak{c})) \longrightarrow \cdot$$

We claim that α is the zero map. Let $\{\kappa_i\}$ be the cusps of $Y_0(\mathfrak{c})$, and let H_c^\bullet denote singular cohomology with compact support. From the long exact sequence

$$\longrightarrow H_c^{k-1}(\{\kappa_i\}) \longrightarrow H_c^k(Y_0(\mathfrak{c})) \longrightarrow H_c^k(X_0(\mathfrak{c})) \longrightarrow H_c^k(\{\kappa_i\}) \longrightarrow$$

(see [DK, §1.4]) we see that

$$H^{n+1}(X_0(\mathfrak{c})) = H_c^{n+1}(X_0(\mathfrak{c})) \cong H_c^{n+1}(Y_0(\mathfrak{c})).$$

Suppose n is even. Then we claim that $H^{n+1}(X_0(\mathfrak{c})) = 0$. By Theorem 6.3 and exactness of the sequence of Lemma 6.5 there is an injection $H_c^{n+1}(Y_0(\mathfrak{c})) \hookrightarrow \tilde{H}^{n+1}(Y_0(\mathfrak{c}))$. If n is even, then $\tilde{H}^{n+1}(Y_0(\mathfrak{c})) = 0$ by Theorem 6.2. Thus $H_c^{n+1}(Y_0(\mathfrak{c})) \cong H^{n+1}(X_0(\mathfrak{c})) = 0$, which together with exactness of (6.12) implies α is the zero map.

If n is odd, then similar reasoning implies that $H^{n+2}(X_0(\mathfrak{c})) = 0$. Thus, looking at the next few terms of (6.12), we have

$$(6.13) \quad \xrightarrow{\alpha} H^{n+1}(X_0(\mathfrak{c})) \xrightarrow{\beta} H^{n+1}(Y_0(\mathfrak{c})) \longrightarrow H^{n+2}(X_0(\mathfrak{c}), Y_0(\mathfrak{c})) \longrightarrow 0 \longrightarrow \cdot$$

Now, $H^{n+1}(X_0(\mathfrak{c})) \cong H_{(2)}^{n+1}(Y_0(\mathfrak{c}))$ by (6.7) and Zucker's conjecture. By the dimension $n+1$ part of the proposition (which we already proved), $\dim_{\mathbb{C}}(H_{(2)}^{n+1}(Y_0(\mathfrak{c}))) = \dim_{\mathbb{C}}(H_{\text{univ}}^{n+1}(Y_0(\mathfrak{c})))$. Moreover, Theorem 6.3 and the definition of $\tilde{H}^{n+1}(Y_0(\mathfrak{c}))$ imply that

$$\dim_{\mathbb{C}} H^{n+1}(Y_0(\mathfrak{c})) = \dim_{\mathbb{C}} H^{n+1}(\bar{Y}_0(\mathfrak{c})) = \dim_{\mathbb{C}} \tilde{H}^{n+1}(Y_0(\mathfrak{c})) + \dim_{\mathbb{C}} H^{n+1}(\partial \bar{Y}_0(\mathfrak{c})).$$

Combining all these dimension formulae with Lemma 6.4, we know the dimension of every group in the long exact sequence (6.13). A simple dimension count implies that β is injective, which implies in turn that α must be the zero map. Note that here, in order to apply Theorem 6.2, we use the fact that $2n-2 \geq n+1$ for odd $n > 1$.

Now that we have established that α is the zero map, we're essentially done. Theorem 6.2 and Lemma 6.4 give us the dimensions of the second and third terms of (6.12), and this is enough to establish the dimension bound (6.11) using the fact that ∂ is surjective by exactness. \square

7. INTERSECTION NUMBERS OF HECKE CYCLES

As always, fix an ideal $\mathfrak{c} \subset \mathcal{O}$. Let $\mathfrak{m}, \mathfrak{n} \subset \mathcal{O}$. Our goal for this section is to compute the intersection numbers $Z(\mathfrak{m}) \cdot Z(\mathfrak{n})$ using the explicit description of the L^2 -cohomology of Hilbert modular varieties given in the previous section together with Theorem 5.1.

Note that $T_j(\mathfrak{m})$ induces maps

$$(7.1) \quad \begin{aligned} T_j(\mathfrak{m}) : H_{(2)}^\bullet(Y_j) &\longrightarrow H_{(2)}^\bullet(Y_{\tau_{\mathfrak{m}}(j)}) \\ \omega &\longmapsto \sum_{\substack{WyW \\ \det(y)=\mathfrak{m}}} \omega \{ \Gamma_j \alpha_j(y) \Gamma_{\tau(j)} \}_{j=1}^{h^+} \end{aligned}$$

$$(7.2) \quad \begin{aligned} {}^*T_j(\mathfrak{m}) : H_{(2)}^\bullet(Y_{\tau_{\mathfrak{m}}(j)}) &\longrightarrow H_{(2)}^\bullet(Y_j) \\ \omega &\longmapsto \sum_{\substack{WyW \\ \det(y)=\mathfrak{m}}} \omega \{ \Gamma_{\tau_{\det(y)}(j)} \alpha_j(y) {}^t\Gamma_j \}_{j=1}^{h^+} \end{aligned}$$

(see §4). Likewise $T(\mathfrak{m})$ induces maps

$$\begin{aligned} T(\mathfrak{m}) : H_{(2)}^\bullet(Y_0(\mathfrak{c})) &\longrightarrow H_{(2)}^\bullet(Y_0(\mathfrak{c})) \\ {}^*T(\mathfrak{m}) : H_{(2)}^\bullet(Y_0(\mathfrak{c})) &\longrightarrow H_{(2)}^\bullet(Y_0(\mathfrak{c})) \end{aligned}$$

that are just $T_j(\mathfrak{m})$ (resp. ${}^*T_j(\mathfrak{m})$) on the j th (resp. $\tau_{\mathfrak{m}}(j)$ th) component.

The main result of this section is the following:

Theorem 7.1. *If $\mathfrak{m}, \mathfrak{n} \subset \mathcal{O}$ are ideals that are in the same ray class modulo β_∞ , then*

$$\sum_i (-1)^i \text{Tr} \left({}^*T(\mathfrak{m}) \circ T(\mathfrak{n}) : H_{(2)}^i(Y_0(\mathfrak{c})) \rightarrow H_{(2)}^i(Y_0(\mathfrak{c})) \right) = 2^n \left((-1)^n \text{Tr}({}^*T(\mathfrak{m}) \circ T(\mathfrak{n})) + h^+ \sigma'_\mathfrak{c}(\mathfrak{m}) \sigma'_\mathfrak{c}(\mathfrak{n}) \right)$$

where the trace on the right is taken with respect to the action of the Hecke operators on $S(U_0(\mathfrak{c}))$.

Theorem 7.1, in conjunction with Theorem 5.1 and Lemma 5.2, proves Theorem 1.1.

Remark. Notice that Theorem 7.1 relates an alternating trace over the entire graded cohomology group $H_{(2)}^\bullet(Y_0(\mathfrak{c}))$ to a trace on the holomorphic subspace of the cohomology. We have such a formula because of the symmetry in mixed Hodge structures on $H_{(2)}^\bullet(Y_0(\mathfrak{c}))$ that exists because of the involutions (6.2). This is also why the 2^n appears. The same Hodge symmetry underlies the proof of Lemma 7.2 below.

We require one more lemma before we can prove Theorem 7.1:

Lemma 7.2. *Let $\mathbf{F} \in S(U_0(\mathfrak{c}))$. For each subset $b \subset \{1, \dots, n\}$, let ι_b be as in (6.2). Then*

$$\begin{aligned} T(\mathfrak{m}) \iota_b^* \Omega_{\mathbf{F}} &= \iota_b^* T(\mathfrak{m}) \Omega_{\mathbf{F}} = \iota_b^* \Omega_{\mathbf{F}|T(\mathfrak{m})} \\ {}^*T(\mathfrak{m}) \iota_b^* \Omega_{\mathbf{F}} &= \iota_b^* ({}^*T(\mathfrak{m}) \Omega_{\mathbf{F}}) = \iota_b^* \Omega_{\mathbf{F}|{}^*T(\mathfrak{m})} \end{aligned}$$

We give the proof of Lemma 7.2 (which is a calculation) after the proof of Theorem 7.1:

Proof of Theorem 7.1. In the notation of (6.6), note that

$$(7.3) \quad \begin{aligned} T(\mathfrak{m}) \eta_j^b &= \sigma'_\mathfrak{c}(\mathfrak{m}) \eta_{\tau_{\mathfrak{m}}(j)}^b \\ {}^*T(\mathfrak{m}) \eta_j^b &= \sigma'_\mathfrak{c}(\mathfrak{m}) \eta_{\tau_{\mathfrak{m}}(j)}^b, \end{aligned}$$

simply because the number of right cosets defining $T_j(\mathfrak{m})$ is $\sigma'_\mathfrak{c}(\mathfrak{m})$ for each $1 \leq j \leq h^+$. The theorem follows from this observation, Lemma 7.2, and the explicit description of the L^2 -cohomology given by (6.4), (6.6) and Proposition 6.1. \square

We now give the proof of Lemma 7.2:

Proof of Lemma 7.2. The equalities $T(\mathfrak{m}) \Omega_{\mathbf{F}} = \Omega_{\mathbf{F}|T(\mathfrak{m})}$ and ${}^*T(\mathfrak{m}) \Omega_{\mathbf{F}} = \Omega_{\mathbf{F}|{}^*T(\mathfrak{m})}$ follows from (7.1) together with the adjoint relation (3.14) for the Petersson inner product. For the other equivariance statements, consider ι_b as a matrix in $\pi_0(\mathbf{G}(\mathbb{R}))$, and write $T_j(\mathfrak{m}) = \sum_k \Gamma_j \alpha_k$. Then the equality $T(\mathfrak{m}) \iota_b^* \Omega_{\mathbf{F}} = \iota_b^* T(\mathfrak{m}) \Omega_{\mathbf{F}}$ follow from the fact that $\iota_b \alpha_k = \alpha_k \iota_b$. Similar remarks imply that ${}^*T(\mathfrak{m})$ and ι_b^* commute with each other. \square

8. CONNECTION WITH FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS

In this section we show that the intersection numbers $Z(\mathfrak{m}) \cdot Z(\mathfrak{n})$ arise in a natural way as the coefficients of Hilbert automorphic forms. We begin by interpreting the $\sigma'_c(\mathfrak{m})\sigma'_c(\mathfrak{n})$ term that appears in the expression for $Z(\mathfrak{m}) \cdot Z(\mathfrak{n})$ given by Theorem 1.1. We recall that, just as in the case of elliptic modular forms, the functions $\sigma'_c(\mathfrak{m})$ are coefficients of Eisenstein series. More precisely, define

$$\zeta_c(s) := \sum_{\substack{\mathfrak{n} \in \mathcal{O} \\ \mathfrak{n} + \mathfrak{c} = \mathcal{O}}} \frac{1}{N(\mathfrak{n})^s}.$$

Then, provided that $n > 1$ or $\mathfrak{c} \neq \mathcal{O}$, the automorphic Fourier expansion

$$(8.1) \quad \mathbf{E}_c \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) := \delta(\mathfrak{c}) 2^{-[E:\mathbb{Q}]} \zeta_c(-1) |y| + \sum_{0 \ll \xi \in E} \sigma'_c(\xi y) |y| e(\mathrm{tr}(\xi i y)) \chi_E(\xi x)$$

defines an Eisenstein series $\mathbf{E}_c \in M(U_0(\mathfrak{c}))$. Here

$$\delta(\mathfrak{c}) := \begin{cases} 1 & \text{if } \mathfrak{c} = \mathcal{O}_E \\ 0 & \text{otherwise.} \end{cases}$$

The calculation of this Fourier expansion is a special case of [Sh2, Prop. 3.4]; to see this let $\mathfrak{a} = \mathfrak{c}$, $\mathfrak{b} = \mathcal{O}$, and η be the trivial character modulo $\mathfrak{c}\beta_\infty$, χ the trivial character modulo β_∞ . See also Corollary 6.2 of [Hid1].

Let z (resp. w) be the local coordinate for X_j (resp. X_k) induced by the projection $\mathfrak{H}^n \rightarrow Y_j$ (resp. $\mathfrak{H}^n \rightarrow Y_k$) and denote the universal class associated to $b \subset \{1, \dots, n\}$ and the j th (resp. k th) component of the first (resp. second) factor of $Y_0(\mathfrak{c}) \times Y_0(\mathfrak{c})$ by $\eta_j^b(z)$ (resp. $\eta_k^b(w)$). Now define $\nu_k \in H_{\mathrm{univ}}^{2n}(Y_0(\mathfrak{c}) \times Y_0(\mathfrak{c}))$ to be the class which is equal to

$$\sum_{j=1}^{h^+} \sum_{b \subset \{1, \dots, n\}} \eta_j^b(z) \wedge \eta_{\tau_{\mathfrak{m}_k}(j)}^a(w)$$

where \mathfrak{m}_k is any ideal in the ray class of s_k^{-1} modulo β_∞ and $a = \{1, \dots, n\} - b$. Here $\tau_{\mathfrak{m}_k}$ is the permutation defined by (3.1), and s_1, \dots, s_{h^+} are the finite idèles we fixed in §2 whose associated fractional ideals form a complete set of representatives for the ray classes modulo β_∞ .

For each $1 \leq k \leq h^+$, there is a linear map

$$(8.2) \quad \begin{aligned} \int \Lambda^{-1}(\cdot) \wedge \nu_k : IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})) &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto \int \Lambda^{-1}(\gamma) \wedge \nu_k. \end{aligned}$$

Here Λ is the Zucker isomorphism of Proposition 4.1. Since we may write $\Lambda^{-1}(\gamma)$ with respect to the basis given by Harder's theorem (Theorem 6.2) and the Künneth formula (Theorem 5.3), it is easy to see that this map is well-defined. By duality (5.4), the linear map (8.2) defines a class

$$Z_k(0) \in IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c})).$$

Remark. It would seem more natural to define $Z_k(0)$ to be $\Lambda(\nu_k)$, given the Zucker isomorphism of Theorem 4 above. Actually, it follows from the results of [GG, Chapter 4 and §6.3] that $Z_k(0) = \Lambda(\nu_k)$, but we won't need this fact.

Now, for each class $\gamma \in IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c}))$, define

$$(8.3) \quad (\gamma \cdot Z(0))(y) := \frac{h^+ \delta(\mathfrak{c}) \zeta_c(-1)}{Z(\mathcal{O}) \cdot Z_t(0)} \sum_{k=1}^{h^+} (\gamma \cdot Z_k(0)) \mathrm{char}_{s_k^{-1}}(y)$$

for $y \in \mathbb{A}_E^\times$, where $\mathrm{char}_{s_k^{-1}}$ is the characteristic function of the ray class of s_k^{-1} modulo β_∞ and $1 \leq t \leq h^+$ is chosen so that s_t^{-1} is in the ray class of the principal ideals modulo β_∞ . This provides the definition of the constant terms of the $\Phi_{\mathfrak{m}, \mathfrak{c}}$ of the introduction. We are now in a position to prove Theorem 1.2, which states that the $\Phi_{\mathfrak{m}, \mathfrak{c}}$ are modular if $S(U_0(\mathfrak{c})) = S^{\mathrm{new}}(U_0(\mathfrak{c}))$:

Proof of Theorem 1.2. We simply construct the form $\Phi_{\mathfrak{m}, \mathfrak{c}}$. Let $\mathbf{F}_1, \dots, \mathbf{F}_d$ be a basis of normalized newforms on $S(U_0(\mathfrak{c}))$ (that is, we stipulate that $c(\mathcal{O}, \mathbf{F}_i) = 1$). For each ideal $\mathfrak{a} \subset \mathcal{O}$, denote by ${}^*a_s(\mathfrak{a})$ be the eigenvalue of ${}^*T(\mathfrak{a})$ associated to \mathbf{F}_s (it is the eigenvalue of $T(\mathfrak{a})$ if $\mathfrak{a} + \mathfrak{c} = \mathcal{O}$ by (3.14)). Finally, for the unique $1 \leq j \leq h^+$ such that s_j^{-1} and \mathfrak{m} are in the same ray class modulo β_∞ , let

$$(8.4) \quad \begin{array}{ccc} \pi_j : & M(U_0(\mathfrak{c})) & \longrightarrow & M(U_0(\mathfrak{c})) \\ & (f_1, \dots, f_{h^+}) & \longmapsto & (0, \dots, 0, f_j, 0, \dots, 0) \end{array}$$

be the canonical projection. Then

$$\Phi_{\mathfrak{m}, \mathfrak{c}} : = (-1)^n 2^n \sum_{s=1}^d {}^*a_s(\mathfrak{m}) \pi_j \mathbf{F}_s + 2^n h^+ \sigma'_\mathfrak{c}(\mathfrak{m}) \pi_j(E_\mathfrak{c})$$

has the desired Fourier coefficients for all integral ideals in the ray class of \mathfrak{m} modulo β_∞ . For all ideals \mathfrak{n} not equivalent to \mathfrak{m} in the ray class group modulo β_∞ , the n th Fourier coefficient of $\Phi_{\mathfrak{m}, \mathfrak{c}}$ is zero, and $Z(\mathfrak{m}) \circ Z(\mathfrak{n}) = 0$ by Lemma 5.2.

We are left with checking that the constant terms of $\Phi_{\mathfrak{m}, \mathfrak{c}}$ and $2^n h^+ \sigma'_\mathfrak{c}(\mathfrak{m}) \pi_j(E_\mathfrak{c})$ coincide. To prove this, it suffices to show that if \mathfrak{m} is in the ray class of s_j^{-1} modulo β_∞ , then

$$(8.5) \quad Z(\mathfrak{m}) \cdot Z_j(0) = \sigma'_\mathfrak{c}(\mathfrak{m}) Z(\mathcal{O}) \cdot Z_t(0).$$

It follows from the explicit description of $H_{(2)}^\bullet(X_0(\mathfrak{c}))$ given by Harder's theorem (Theorem 6.2), the Künneth formula (Proposition 5.3), and Lemma 5.4 that the projection of $\Lambda^{-1}(Z(\mathcal{O}))$ to $H_{\text{univ}}^{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c}))$ is equal to $c\nu_t$ for some $c \in \mathbb{C}^\times$. Notice that

$$(T(\mathfrak{m}))\eta_t^b(z) = \sum_{m=1}^{\sigma'_\mathfrak{c}(\mathfrak{m})} \eta_j^b(\alpha_m z) = \sigma'_\mathfrak{c}(\mathfrak{m}) \eta_j^b(w)$$

for some $\alpha_m \in Y$. It follows that $\Lambda^{-1}(Z(\mathfrak{m})) = \sigma'_\mathfrak{c}(\mathfrak{m}) c\nu_j$. Applying the explicit basis of $H_{(2)}^\bullet(X_0(\mathfrak{c}) \times X_0(\mathfrak{c}))$ obtained by using Harder's theorem (Theorem 6.2) and the Künneth formula (Proposition 5.3) along with the definition of $Z_j(0)$, this implies (8.5). \square

As in the introduction, let $HC(\mathfrak{c}) \leq IH_{2n}(X_0(\mathfrak{c}) \times X_0(\mathfrak{c}))$ be the subspace spanned by the cycles $Z(\mathfrak{m})$ as \mathfrak{m} ranges over the integral ideals of \mathcal{O} . We now prove Theorem 1.4:

Proof of Theorem 1.4. Using the notation of the proof of Theorem 1.2, we have

$$\Psi(Z(\mathfrak{m})) = (-1)^n 2^n \sum_{s=1}^d {}^*a_s(\mathfrak{m}) \pi_j \mathbf{F}_s + h^+ 2^n \sigma'_\mathfrak{c}(\mathfrak{m}) \pi_j(E_\mathfrak{c}).$$

We claim that

$$\Psi(Z(\mathfrak{m})) = (-1)^n 2^n \sum_{s=1}^d {}^*a_s(\mathfrak{m}) \mathbf{F}_s + h^+ 2^n \sigma'_\mathfrak{c}(\mathfrak{m}) \pi_j(E_\mathfrak{c}).$$

It suffices to show that $\text{Tr}({}^*T(\mathfrak{m}) \circ T(\mathfrak{n})) = 0$ if \mathfrak{m} and \mathfrak{n} are not in the same ray class modulo β_∞ . To see this, assume that \mathfrak{m} and \mathfrak{n} are not in the same ray class modulo β_∞ , and write the matrix of ${}^*T(\mathfrak{m}) \circ T(\mathfrak{n})$ (as an endomorphism of $S(U_0(\mathfrak{c}))$) with respect to any basis such that each cusp form in the basis is supported on one component. Then this matrix has zeros along its diagonal, as ${}^*T(\mathfrak{m}) \circ T(\mathfrak{n})$ leaves no component fixed.

Thus we are reduced to showing that as \mathfrak{m} runs over the integral ideals of \mathcal{O} , the forms

$$\sum_{s=1}^d {}^*a_s(\mathfrak{m}) \mathbf{F}_s$$

span $S(U_0(\mathfrak{c}))$ (which equals $S^{\text{new}}(U_0(\mathfrak{c}))$ by assumption). This follows from multiplicity one for Hilbert modular forms (multiplicity one for Hilbert modular forms is proven in [Mi]). \square

ACKNOWLEDGEMENTS

The author would like to thank J. Bruinier, E. Goren, B. Mazur and D. Nadler for several useful conversations, T. Yang for answering many questions, M. Goresky for his lucid explanations of sheaf-theoretic intersection homology, and K. Ono for his constant, enthusiastic support. M. Goresky and K. Ono also deserve additional thanks for their editing help.

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