PARTITION IDENTITIES AND A THEOREM OF ZAGIER

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Abstract. In partition theory and q-series, one often seeks identities between series and infinite products. Using a recent result of Zagier, we obtain such identities for every positive integer m. For example if $m = 1$, then we obtain the classical Eisenstein series identity

$$\sum_{\lambda \geq 1 \text{ odd}} \frac{(-1)^{(\lambda-1)/2} q^\lambda}{(1-q^{2\lambda})} = q \prod_{n=1}^{\infty} \frac{(1-q^{8n})^4}{(1-q^{4n})^2}.$$

If $m = 2$ and $\left(\frac{\cdot}{3}\right)$ denotes the usual Legendre symbol modulo 3, then we obtain

$$\sum_{\lambda \geq 1} \frac{\left(\frac{\lambda}{3}\right) q^\lambda}{(1-q^{2\lambda})} = q \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{6n})^6}{(1-q^{2n})^2(1-q^{3n})^3}.$$

We describe some of the partition theoretic consequences of these identities. For instance, we obtain simple formulas for the number of representations of an integer as a sum of an arbitrary number of triangular numbers.

1. Introduction and Statement of Results

We begin by recalling the following identity of Euler:

$$\text{(Euler)} \qquad \prod_{n=1}^{\infty} (1-q^n) = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2+k)/2}.$$

Note that the left hand side of Euler's identity is related to partitions of integers into distinct parts. More precisely, each partition of n into an odd number of distinct parts adds -1 to the coefficient of q^n and each partition of n into an even number of distinct parts adds $+1$.

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Therefore, this identity, known as Euler's pentagonal number theorem, shows us that the number of partitions of n into an odd number of distinct parts is equal to the number of partitions of n into an even number of distinct parts except when n is a pentagonal number, that is, a number of the form $(3k^2+k)/2$.

Other identities of this sort exist, perhaps the most famous being the Rogers-Ramanujan identities. The following is one of these celebrated identities:

$$\text{(Rogers-Ramanujan)} \qquad \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Combinatorially, this identity establishes that the number of partitions of an integer n into parts that are congruent to $2, 3 \pmod 5$ equals the number of partitions of n in which any two summands differ by at least 2 and all summands exceed 1.

Such results are interesting both intrinsically and for their combinatorial consequences. The purpose of this paper is to use a master theorem of Zagier [Z] previously conjectured by Kac and Wakimoto [K-W] to prove a natural infinite family of similar simple q-series identities relating infinite products to natural partition generating functions (perhaps well known to the experts). Through a combinatorial examination of these identities, we obtain closed explicit formulas for the number of representations of integers as a sum of an arbitrary number of triangular numbers. This problem is closely related to the discovery of formulas for the number of representations of n as a sum of squares and triangular numbers, a subject that has been examined recently by Milne, Ono and Zagier (see [M], [O], [Z]).

Before we state the main theorem we must offer some notation. Throughout, if m is a positive integer, then let $s(m)$ denote the integer

$$s(m) := \left[\frac{m+1}{2}\right]. \tag{1.1}$$

For convenience, define the following set of vectors which we will see determines the summands in the partitions we consider.

Definition 1.1. *If m is a positive integer, then let $S(m)$ denote the set of integral vectors $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_{s(m)})$ for which the following hold:*

(i) If $j > i$, then $\lambda_i > \lambda_j > 0$.
(ii) For every i we have $\lambda_i \equiv m \pmod 2$.
(iii) For every i we have $\lambda_i \not\equiv 0 \pmod{2m+2}$.
(iv) For every $i \neq j$ we have $\lambda_i \not\equiv \pm\lambda_j \pmod{2m+2}$.

Further, define subsets $S_{\pm}(m)$ of $S(m)$ in the following manner:

$$S_+(m) := \{\Lambda \in S(m) \ : \ \text{the number of } \lambda_i \pmod{2m+2} > m+1 \text{ is even.}\}\,, \tag{1.2}$$

$$S_-(m) := \{\Lambda \in S(m) \ : \ \text{the number of } \lambda_i \pmod{2m+2} > m+1 \text{ is odd.}\}\,. \tag{1.3}$$

In what follows, we will see that $S_{\pm}(m)$ determine two different partition functions, one whose parts contain an even number of λ_i such that the λ_i are in the larger half of the residue classes modulo $2m+2$, and the other whose parts contain an odd number of λ_i such that the λ_i are in the larger half of the residue classes modulo $2m+2$. These partition functions arise naturally from the following theorems.

Theorem 1. *If $m \geq 1$ is odd, then*

$$\sum_{\Lambda \in S_+(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_{s(m)}})} - \sum_{\Lambda \in S_-(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_{s(m)}})}$$

$$= q^{(m+1)^2/8} \prod_{n=1}^{\infty} \frac{(1-q^{2(m+1)n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}.$$

Theorem 2. *If $m \geq 1$ is even, then*

$$\sum_{\Lambda \in S_+(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_{s(m)}})} - \sum_{\Lambda \in S_-(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_{s(m)}})}$$

$$= q^{(m^2+2m)/8} \prod_{n=1}^{\infty} \frac{(1-q^{n})}{(1-q^{2n})^2} \frac{(1-q^{2(m+1)n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}.$$

Before we state some partition theoretic interpretations of this set of identities, we must offer some more definitions.

Definition 1.2. *If m is a positive integer, then let $P_{\pm}(n,m)$ denote the number of partitions of n of the form*

$$n = \sum_{i=1}^{s(m)} (2n_i+1)\lambda_i,$$

where $\Lambda = (\lambda_1, \ldots, \lambda_{s(m)}) \in S_{\pm}(m)$ and each $n_i \geq 0$.

In simple terms, the function $P_{\pm}(n,m)$ counts the number of partitions of n into parts, occurring with odd multiplicity, that are elements of a vector $\Lambda \in S_{\pm}(m)$. Obviously,

$$\sum_{\Lambda \in S_{\pm}(m)} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})}}{(1-q^{2\lambda_1})\cdots(1-q^{2\lambda_{s(m)}})} = \sum_{n=1}^{\infty} P_{\pm}(n,m)q^n. \tag{1.4}$$

These generating functions are closely related to the problem of counting the number of representations of an integer as a sum of triangular numbers, numbers of the form $(k^2+k)/2$ with

$k \geq 0$. If k is a positive integer, then let $T(n,k)$ denote the number of representations of n as a sum of k triangular numbers. A well known identity due to Jacobi implies that

$$\sum_{n=0}^{\infty} T(n,k)q^n = \left(\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)}\right)^k. \tag{1.5}$$

In view of (1.4) and (1.5), it is simple to verify the following two corollaries which give partition theoretic formulas for $T(n,k)$.

Corollary 3. *If $k \geq 2$ is even, then*

$$T(n,k) = P_+\left(2kn + \frac{k^2}{4}, k-1\right) - P_-\left(2kn + \frac{k^2}{4}, k-1\right).$$

Corollary 4. *If $k \geq 1$ is odd, then*

$$T(n,k) = \sum_{j=0}^{\infty}\left(P_+\left(2kn + \frac{k^2-1}{4} - j^2 - j, k-1\right) - P_-\left(2kn + \frac{k^2-1}{4} - j^2 - j, k-1\right)\right).$$

Remark. Although the sum in Corollary 4 appears to be infinite, it is easy to see that it is indeed finite. After all, there are no partitions of negative integers into positive parts. More importantly, observe that Corollaries 3 and 4 completely characterize the number of representations of every integer n as a sum of an arbitrary number of triangular numbers in terms of the partition functions $P_\pm(n,k)$. One should compare these results with those appearing in [M] where explicit formulas of a different type are obtained for $T(n,k)$ for those k of the form $4s^2$ and $4s^2+4s$. It would be very interesting to obtain a combinatorial proof of these results.

In view of (1.4), Theorems 1 and 2 immediately imply the following corollaries.

Corollary 5. *If $k \geq 2$ is even, then for every non-negative integer n we have*

$$P_+(n,k-1) \geq P_-(n,k-1).$$

Furthermore, if $n \not\equiv k^2/4 \pmod{2k}$, then

$$P_+(n,k-1) = P_-(n,k-1).$$

Corollary 6. *If $k \geq 2$ is even and n is a positive odd integer, then*

$$P_+(n,k) = P_-(n,k).$$

Remark. If $k \geq 4 \pmod{2k}$ is even and $n \equiv k^2/4 \pmod{2k}$, then $P_+(n,k-1) > P_-(n,k-1)$ by Gauss' Eureka Theorem (i.e. $T(n,3) > 0$ for all n). If $k = 2$, then an easy analysis shows that $P_+(n,1) = P_-(n,1)$ for almost all n.

One may also make the observation that the right hand side of the identities of Theorems 1 and 2 are quotients of powers of Dedekind eta-functions. In Section 3 we use a well known theorem on such eta-products (see [N1], [N2]) to show that the generating function for $P_+(n,m) - P_-(n,m)$ is a holomorphic integer weight modular form. Then, using a powerful result of Serre [S], we prove the following corollary:

Corollary 7. *If k is a positive integer and M is any integer, then*

$$P_+(n,k) \equiv P_-(n,k) \pmod{M}$$

for a set of positive integers n with arithmetic density 1.

Notice that Corollary 7 is an analog of Euler's pentagonal number theorem mentioned above. Though the result is not nearly as precise as Euler's, it does tell us that $P_+(n,k)$ and $P_-(n,k)$ are congruent modulo M for almost all positive n.

2. Proof of Theorems 1 and 2

Our proof of Theorems 1 and 2 relies on the Kac-Wamimoto Conjecture, proved by Zagier. The following two functions appear in the result:

$$\Theta_0(x) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}x)(1-q^{2n-1}x^{-1})$$
$$\Theta_1(x) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-2}x)(1-q^{2n}x^{-1}).$$

Alternative forms of these functions will be derived shortly. To prove our theorems, we use the following form of the conjecture, which Zagier derived as an intermediate step in the proof (see [Z]):

Theorem 2.1. *(Zagier) If $m \in \mathbb{Z}^+, s(m) = \left[\frac{m+1}{2}\right]$ and $x_1, x_2, \ldots, x_{m+1} \in \mathbb{C}$ are distinct complex numbers such that $|q| < |x_i/x_j| < |q|^{-1}$, then*

$$\sum_{\substack{\lambda_1>\lambda_2>\cdots>\lambda_{s(m)}>0 \\ \lambda_i \equiv m \pmod 2}} \frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})(1-q^{\lambda_2})\ldots(1-q^{\lambda_{s(m)}})} \times (\mathrm{Alt})_{m+1}\left(\prod_{i=1}^{s(m)}\left(\frac{x_i}{x_{m+2-i}}\right)^{\lambda_i/2}\right) \tag{2.1}$$
$$= \theta(\tau)^{2s(m)} \prod_{1\le i<j\le m+1} F(x_j/x_i),$$

where

$$F(x) := q^{1/4}x^{-1/2}\frac{\Theta_1(x)}{\Theta_0(x)}$$

and Alt_{m+1} *denotes the alternating sum over all permutations of $x_1, \ldots, x_{m+1}$, and*

$$\theta(\tau) = q^{1/8}\Delta(q) = q^{1/8}\prod_{n=1}^{\infty}(1-q^{2n})^2/(1-q^n).$$

The identities given in Theorems 1 and 2 are derived directly from Theorem 2.1. To prove this, we first simplify the infinite products on the right side of the equation by substituting roots of cyclotomic polynomials for the x_i, and then use combinatorial arguments to reduce the complicated alternating sum to known values.

Proof of Theorems 1 and 2. Our first task is to simplify the right side of (2.1). The functions Θ_0 and Θ_1 are rewritten so that the exponents on all of the q terms are similar:

$$\Theta_0(x) = \prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^{2n-1}x)(1-q^{2n-1}x^{-1}) \cdot \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^{2n-1})} \tag{2.2}$$

$$= \prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^{2n-1}x)(1-q^{2n-1}x^{-1}) \cdot \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^{n})},$$

$$\Theta_1(x) = (1-x)\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n}x)(1-q^{2n}x^{-1}). \tag{2.3}$$

Before fully expanding the product of $F(x_j/x_i)$, we consider the terms $(x_j/x_i)^{-1/2}$. Since we will be substituting complex values for the x_i, the branching of the square root function necessitates a careful treatment of these terms. Fortunately, when all such terms are moved to the left side and factored into the expanded alternating sum, the exponents all become integral.

Consider the product of all of the Θ_1,

$$\prod_{1\le i<j\le m+1} \Theta_1(x_j/x_i) = \tag{2.4}$$

$$\prod_{1\le i<j\le m+1}\left((1-x_j/x_i)\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n}(x_j/x_i))(1-q^{2n}(x_j/x_i)^{-1})\right).$$

The expansion of the Θ_0 are similar, and all of the x_i appear with integral exponents. Thus we can make the substitution $x_i = \zeta_{m+1}^{i-1}$, $1 \le i \le m+1$, where ζ_{m+1} is a primitive $(m+1)-$st root of unity. The fraction (x_j/x_i) now reduces to ζ_{m+1}^{j-i}, and since the set $\{1, \zeta_{m+1}, \zeta_{m+1}^2, \dots, \zeta_{m+1}^m\}$ contains all $(m+1)$-st roots of unity, the product

$$(1-q^{2n})(1-q^{2n}\zeta_{m+1})\cdots(1-q^{2n}\zeta_{m+1}^m)$$

reduces to $(1-q^{2n(m+1)})$. Note that in the product of the Θ_1 there are $m+1$ copies of the term $\prod_{n=1}^{\infty}(1-q^{2n}\zeta_{m+1}^k)$ for all $1 \le k \le m$. This is because $x_{j'}/x_{i'} = \zeta_{m+1}^k$ whenever $j' \equiv i' + k \pmod{m+1}$, and there are $m+1$ choices for i', ignoring the restrictions on the sum. If $j' > i'$, then let $j = j'$ and $i = i'$; the term occurs as (x_j/x_i). When $j' < i'$, let $j = i'$ and $i = j'$; the term occurs as $(x_i/x_j)^{-1}$ since the higher subscript is always the numerator. Thus (2.4) becomes

$$\prod_{1\le i<j\le m+1} \Theta_1(\zeta_{m+1}^{j-i})$$

$$= \prod_{1\le i<j\le m+1}(1-\zeta_{m+1}^{j-i})\prod_{n=1}^{\infty}(1-q^{2n})^{(m+1)m/2}(1-q^{2n}\zeta_{m+1})^{m+1}\cdots(1-q^{2n}\zeta_{m+1}^m)^{m+1}$$

$$= \prod_{k=1}^{m}(1-\zeta_{m+1}^k)^{m+1-k}\prod_{n=1}^{\infty}(1-q^{2n})^{(m+1)(m-2)/2}(1-q^{2n(m+1)})^{m+1}.$$

Similarly, with powers of ζ_{m+1} substituted in for the x_i, the expansion of the Θ_0 product is

$$\prod_{1\le i<j\le m+1} \Theta_0(\zeta_{m+1}^{j-i})$$
$$= \prod_{n=1}^{\infty}(1-q^{2n-1})^{(m+1)m/2}(1-q^{2n-1}\zeta_{m+1})^{m+1}\cdots(1-q^{2n-1}\zeta_{m+1}^{m})^{m+1}$$
$$\times \prod_{n=1}^{\infty}\frac{(1-q^{2n})^{(m+1)m}}{(1-q^n)^{(m+1)m/2}}$$
$$= \prod_{n=1}^{\infty}(1-q^{2n-1})^{(m+1)(m-2)/2}(1-q^{(2n-1)(m+1)})^{m+1}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{(m+1)m}}{(1-q^n)^{(m+1)m/2}}.$$

After all of these simplifications (including the removal of the fractional powers), the right side of (2.1) becomes

$$\theta(\tau)^{2s(m)}\prod_{1\le i<j\le m+1} q^{1/4}\frac{\Theta_1(x_j/x_i)}{\Theta_0(x_j/x_i)}$$
$$= q^{(m^2+m+2s(m))/8}\prod_{k=1}^{m}(1-\zeta_{m+1}^{k})^{m+1-k}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{4s(m)}}{(1-q^n)^{2s(m)}}$$
$$\times\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{(m+1)(m-2)/2}(1-q^{2n(m+1)})^{m+1}}{(1-q^{2n-1})^{(m+1)(m-2)/2}(1-q^{(2n-1)(m+1)})^{m+1}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{(m+1)m}}{(1-q^n)^{(m+1)m/2}}}$$
$$= q^{(m^2+m+2s(m))/8}\prod_{k=1}^{m}(1-\zeta_{m+1}^{k})^{m+1-k}$$
$$\times\prod_{n=1}^{\infty}\frac{(1-q^n)^{(m+1)m/2-2s(m)}}{(1-q^{2n})^{(m+1)(m+2)/2-4s(m)}(1-q^{2n-1})^{(m+1)(m-2)/2}}\prod_{n=1}^{\infty}\frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}.$$

If m is odd, then $2s(m)=m+1$, so the above product is equal to

$$q^{(m+1)^2/8}\prod_{k=1}^{m}(1-\zeta_{m+1}^{k})^{m+1-k}\prod_{n=1}^{\infty}\left(\frac{(1-q^n)}{(1-q^{2n})(1-q^{2n-1})}\right)^{(m+1)(m-2)/2} \tag{2.5}$$
$$\times\prod_{n=1}^{\infty}\frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}$$
$$= q^{(m+1)^2/8}\prod_{k=1}^{m}(1-\zeta_{m+1}^{k})^{m+1-k}\prod_{n=1}^{\infty}\frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}},$$

which is nearly the form seen in the right side Theorem 1. And when m is even, $2s(m)=m$, so the product is equal to

$$(2.6)\quad q^{(m^2+2m)/8}\prod_{k=1}^{m}(1-\zeta_{m+1}^{k})^{m+1-k}\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-q^{2n})^2}\left(\frac{(1-q^n)}{(1-q^{2n})(1-q^{2n-1})}\right)^{(m+1)(m-2)/2}$$
$$\times\prod_{n=1}^{\infty}\frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}$$
$$= \quad q^{(m^2+2m)/8}\prod_{k=1}^{m}(1-\zeta_{m+1}^{k})^{m+1-k}\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-q^{2n})^2}\frac{(1-q^{(m+1)2n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}}.$$

This resembles the right side of Theorem 2.

Note that when (2.5) and (2.4) are expanded into an infinite sum, the first nonzero term is $q^{(m+1)^2/8}$ when m is odd, $q^{(m^2+2m)/8}$ when m is even, and has coefficient

$$\prod_{k=1}^{m}(1-\zeta_{m+1}^{k})^{m+1-k}$$

in either case.

Now consider the left side of (2.1),

$$\sum_{\substack{\lambda_1>\lambda_2>\cdots>\lambda_{s(m)}>0\\ \lambda_i\equiv m \pmod 2}}\frac{q^{(\lambda_1+\cdots+\lambda_{s(m)})/2}}{(1-q^{\lambda_1})(1-q^{\lambda_2})\ldots(1-q^{\lambda_{s(m)}})}\times(\mathrm{Alt})_{m+1}\left(\prod_{i=1}^{s(m)}\left(\frac{x_i}{x_{m+2-i}}\right)^{\lambda_i/2}\right)\bar{X},$$

where $\bar{X}=\prod_{1\le i<j\le m+1}(x_j/x_i)^{1/2}$ is the product of all of the radical terms transferred from the right side. Before substituting cyclotomic roots for the x_i, we ensure that all of the exponents are integral. Note that $\bar{X}=\prod_{k=1}^{m+1}(x_k)^{(2k-m-2)/2}$. If m is even, then each of the λ_i are even so all of the powers are already integral. If m is odd, then all of the λ_i are also odd. Each of the $m+1$ variables appears in each term of the alternating sum, so once again all of the exponents simplify to integers. Thus we can substitute $x_k=\zeta_{m+1}^{k-1}$.

For all but the simplest cases, the alternating sum becomes too unwieldy to calculate directly. Fortunately, we can still evaluate it by comparing the series expansion of the left and right sides of (2.1). Consider the term of minimum degree in the left side sum. Since the λ_i form a decreasing, nonnegative sequence, there is a unique vector of minimum sum; namely, $\Lambda'=(m,m-2,\ldots,1)$ when m is odd and $\Lambda'=(m,m-2,\ldots,2)$ when m is even. Therefore the minimum exponent is $(\lambda'_1+\cdots+\lambda'_{s(m)})/2$, where $\Lambda'=(\lambda'_1,\ldots,\lambda'_{s(m)})$. This sum evaluates to $\frac{1}{2}(\frac{m+1}{2})^2=\frac{(m+1)^2}{8}$ when m is odd and $\frac{1}{2}(2(\frac{m}{2}(\frac{m}{2}+1)/2))=\frac{m^2+2m}{8}$. These are precisely the exponents of least degree on the right hand side of (2.5) and (2.6).

The series expansion of the left side must correspond to the expansion of the right side, so the coefficient on the first terms are equal. Thus,

$$(\mathrm{Alt})_{m+1}\left(\prod_{i=1}^{s}\left(\frac{x_i}{x_{m+2-i}}\right)^{\lambda'_i/2}\right)\bar{X}=\prod_{k=1}^{m}(1-\zeta_{m+1}^{k})^{m+1-k}.$$

Now we show that if the value of the alternating sum for some Λ is nonzero, then it can be easily related to the value for Λ'.

We need only consider Λ from the residue classes modulo $2m+2$, for if $\lambda_i \equiv \lambda_j \pmod{2m+2}$, and thus $\lambda_i = \lambda_j + r(2m+2)$ for some integer r, then

$$(\zeta_{m+1}^k)^{\lambda_i/2} = (\zeta_{m+1}^k)^{r(2m+2)/2}(\zeta_{m+1}^k)^{\lambda_j/2} = (\zeta_{m+1}^k)^{\lambda_j/2},$$

for ζ_{m+1}^k is an $(m+1)$-th root of unity. Thus we can replace λ_i by λ_j in the alternating sum without affecting its value.

Consider the case where $\lambda_i \equiv 0 \pmod{2m+2}$ for some $1 \le i \le s(m)$. Under the identity permutation, $\lambda_i/2$ is the exponent on the term (x_i/x_{m+2-i}). If the exponent is zero, then this term is always 1. Thus it is irrelevant which variable is the numerator and which is the denominator, so the sum is unaffected when we transpose x_i and x_{m+2-i}. However, adding this single transposition also changes the parity of every permutation in the sum. Therefore every summand in the alternating sum has a complementary term that is equal in magnitude and opposite in sign. Thus the sum is zero for Λ of this form.

Now consider the case $\lambda_i \equiv \lambda_j \pmod{2m+2}$ for some $1 \le i, j \le s(m)$. Under the identity permutation, these two values appear in the terms $(x_i/x_{m+2-i})^{\lambda_i/2}$ and $(x_j/x_{m+2-j})^{\lambda_j/2}$. Since the exponents are equal, these two terms are combined to give $((x_ix_j)/(x_{m+2-i}x_{m+2-j}))^{\lambda_i/2}$. Clearly the x_i and x_j can be interchanged without affecting the sum. Thus, as before, every summand has a complement. Therefore the sum is zero for these Λ as well.

Note that if we have an alternating sum where the identity permutation gives the term

$$(x_1/x_{m+1})^{\lambda_1/2}(x_2/x_m)^{\lambda_2/2}\cdots(x_i/x_{m+2-i})^{\lambda_i/2}\cdots(x_{s(m)}/x_{m+2-s(m)})^{\lambda_{s(m)}/2},$$

then we can construct an equivalent alternating sum in which the identity permutation gives

$$(x_1/x_{m+1})^{\lambda_1/2}(x_2/x_m)^{\lambda_2/2}\cdots(x_{m+2-i}/x_i)^{(2m+2-\lambda_i)/2}\cdots(x_{s(m)}/x_{m+2-s(m)})^{\lambda_{s(m)}/2}.$$

This allows us to show that if $\lambda_i \equiv -\lambda_j \pmod{2m+2}$, then the alternating sum is zero. These two exponents appear on the terms $(x_i/x_{m+2-i})^{\lambda_i/2}$ and $(x_j/x_{m+2-j})^{\lambda_j/2}$ under the identity permutation. As discussed, we can replace the second term by $(x_{m+2-j}/x_j)^{(2m+2-\lambda_j)/2} = (x_{m+2-j}/x_j)^{\lambda_i/2}$. As in the case where two exponents were equal, we combine the terms and interchange x_i and x_{m+2-j}. In this manner we again find a complement of opposite sign and equal magnitude for every summand. Thus, the alternating sum is zero for Λ of this form.

The only remaining cases are those in which there is exactly one λ_i contained in each pair of residue classes $\{k, 2m+2-k\}$. Recall the flipping process described above. If the identity permutation gives us the term

$$(x_1/x_{m+1})^{\lambda_1/2}(x_2/x_m)^{\lambda_2/2}\cdots(x_i/x_{m+2-i})^{\lambda_i/2}\cdots(x_{s(m)}/x_{m+2-s(m)})^{\lambda_{s(m)}/2},$$

then there is an equivalent alternating sum whose identity term is

$$(x_1/x_{m+1})^{\lambda_1/2}(x_2/x_m)^{\lambda_2/2}\cdots(x_{m+2-i}/x_i)^{(2m+2-\lambda_i)/2}\cdots(x_{s(m)}/x_{m+2-s(m)})^{\lambda_{s(m)}/2}.$$

Now if we transpose x_i and x_{m+2-i} we obtain an alternating sum equal in magnitude and opposite in sign with identity permutation

$$(x_1/x_{m+1})^{\lambda_1/2}(x_2/x_m)^{\lambda_2/2}\cdots(x_i/x_{m+2-i})^{-\lambda_i/2}\cdots(x_{s(m)}/x_{m+2-s(m)})^{\lambda_{s(m)}/2}.$$

For a given m, each λ_i has the same parity as m, so at most half of the residue classes modulo $2m+2$ can be achieved. Since we need not consider Λ in which additive inverses are present, we can again halve the number of distinct classes that can occur. But our known alternating sum for Λ' has $\left[\frac{m+1}{2}\right]$ distinct residue classes, none of which are additive inverses. Thus we can determine the value of the alternating sum for all Λ by multiplying a factor of -1 for each inverse exponent. Note that in Λ', $1 \leq \lambda_i \leq m$ for every $1 \leq i \leq s(m)$. Thus the inverses modulo $2m+2$ satisfy $m+2 \leq -\lambda_i \leq 2m+2$.

Hence our definition of $S_{\pm}(m)$. If $\Lambda \notin S_{\pm}(m)$, then

$$(\mathrm{Alt})_{m+1}\left(\prod_{i=1}^{s}\left(\frac{x_i}{x_{m+2-i}}\right)^{\lambda_i/2}\right)\bar{X} = 0,$$

and thus is ignored in the sum.

If $\Lambda \in S_{\pm}(m)$, and there are r inverse exponents, then

$$(\mathrm{Alt})_{m+1}\left(\prod_{i=1}^{s}\left(\frac{x_i}{x_{m+2-i}}\right)^{\lambda_i/2}\right)\bar{X} = (-1)^r\prod_{k=1}^{m}(1-\zeta_{m+1}^k)^{m+1-k}.$$

So if there are an even number of inverses in Λ, then the sum is the same as that for Λ', but if there are an odd number, the sum is negated. Dividing both sides of our identity by $\prod_{k=1}^{m}(1-\zeta_{m+1}^k)^{m+1-k}$ gives Theorems 1 and 2.

Q.E.D.

Remark. It is also possible to let $x_1 = \zeta_{m+1}\ldots x_{m+1} = \zeta_{m+1}^m$ in (2.1) and obtain identities similar to those in Theorem 1 and 2. The infinite product in these other identities is essentially the same, but the definition of the restricted partition functions is slightly different.

3. Modular forms and the proof of Corollary 7

For a positive integer N, define the subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ as follows:

$$\Gamma_0(N) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;:\; ad-bc=1, c\equiv 0 \pmod{N}\right\}. \tag{3.1}$$

Now let $f(z)$ be a meromorphic function on the upper half of the complex plane. We say that $f(z)$ is a modular function of weight k with respect to $\Gamma_0(N)$ if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all z in the upper half of the complex plane and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Further, if $f(z)$ is holomorphic on the upper half of the complex plane and at the cusps of $\Gamma_0(N)$, it is a modular form of weight k with respect to $\Gamma_0(N)$. Finally, let χ be a Dirichlet character modulo N. We say that $f(z)$ is a modular form with character if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z), \tag{3.2}$$

again, for z in the upper half of the complex plane with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. The finite dimensional space of such modular forms is denoted by $M_k(\Gamma_0(N), \chi)$.

If we let $q = e^{2\pi i z}$, we may construct a Fourier expansion for such a holomorphic modular form:

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n$$

In fact, we may identify any form with its Fourier expansion.

In this paper we will be studying modular forms that are products of Dedekind eta-functions, defined by the infinite product

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{3.3}$$

A function $f(z)$ is called an eta-product if it can be expressed as a finite product of the form

$$f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z). \tag{3.4}$$

where N and each r_δ are integers. Such functions have many unique properties. In the following result ([N1] and [N2]), Newman proves that certain eta-products fulfill the functional equation (3.2) for modular forms with character:

Theorem 3.1. *(Newman) If $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ is an eta-product for which*

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24} \tag{3.5}$$

and

$$\sum_{\delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24} \tag{3.6}$$

then $f(z)$ satisfies the functional equation (3.2) for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$*, where* $k = \frac{1}{2}\sum_{\delta|N} r_\delta$*. Here χ is the character defined by:*

$$\chi(d) = \left(\frac{(-1)^k s}{d}\right) \text{ and } s = \prod_{\delta|N} \delta^{r_\delta}.$$

If an eta-product satisfies the functional equation (3.4), we still must demonstrate holomorphicity at the cusps of $\Gamma_0(N)$ to prove it is a holomorphic modular form. The following observation is well known (for example, see Ligozat [L]):

Proposition 3.2. *Let c, d, and N be positive integers with $d|N$ and $(c,d) = 1$. With the notation as above, if the eta-product $f(z)$ satisfies (3.5) and (3.6), then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{1}{24}\sum_{\delta|N} \frac{N(d,\delta)^2 r_\delta}{\left(d, \frac{N}{d}\right) d\delta}.$$

Using Theorem 3.1 and Proposition 3.2 we obtain the following:

Theorem 3.3. *If m is even, then*

$$\sum_{n=0}^{\infty} \left(\left(P_+(n,m) - P_+(n,m)\right) q^{n/2} \in M_{m/2}(\Gamma_0(2(m+1)), \chi) \tag{3.7}$$

where $\chi(d) = \left(\frac{(-1)^{m/2}(m+1)}{d}\right)$.

If m is odd, then

$$\sum_{n=0}^{\infty} \left(\left(P_+(n,m) - P_+(n,m)\right) q^{n} \in M_{(m+1)/2}(\Gamma_0(4(m+1)), \chi) \tag{3.8}$$

where $\chi(d) = \left(\frac{(-1)^{(m+1)/2}(m+1)}{d}\right)$.

Proof of Theorem 3.3. We have, for even m,

$$\begin{aligned} \sum_{n=0}^{\infty} \left(P_+(n,m) - P_-(n,m)\right) q^{n/2} &= q^{(m^2+2m)/8} \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})^2} \frac{(1-q^{2(m+1)n})^{2m+2}}{(1-q^{(m+1)n})^{m+1}} \\ &= \prod_{n=1}^{\infty} \frac{q^{1/24}(1-q^n)}{q^{4/24}(1-q^{2n})^2} \frac{q^{4(m+1)^2/24}(1-q^{2(m+1)n})^{2m+2}}{q^{(m+1)^2/24}(1-q^{(m+1)n})^{m+1}} \\ &= \frac{\eta(z)}{\eta^2(2z)} \frac{\eta^{2m+2}(2(m+1)z)}{\eta^{m+1}((m+1)z)}. \end{aligned} \tag{3.9}$$

Let $f(z) = \frac{\eta(z)}{\eta^2(2z)} \frac{\eta^{2m+2}(2(m+1)z)}{\eta^{m+1}((m+1)z)}$ and write $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ as in Theorem 3.1 and Proposition 3.2. We have

$$\sum_{\delta|N} \delta r_\delta = 1(1) - 2(2) + 4(m+1)^2 - (m+1)^2 = -3 + 3(m+1)^2.$$

Note that $3(m+1)^2 \equiv 3 \pmod{24}$ if and only if $(m+1)^2 \equiv 1 \pmod 8$. One can easily check that this congruence is true for all even m.

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta = 2(m+1)\left(\frac{1}{1} - \frac{2}{2} + \frac{2(m+1)}{2(m+1)} - \frac{(m+1)}{(m+1)}\right) = 0 \equiv 0 \pmod{24}.$$

So $f(z)$ satisfies (3.5) and (3.6). Therefore, by Theorem 3.1, $f(z)$ satisfies (3.2) with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2(m+1))$, $k = \frac{m}{2}$, and $\chi(d) = \left(\frac{(-1)^{m/2}(m+1)^{m+1}}{d}\right)$.

Now we must check the holomorphicity of $f(z)$ at the cusps of $\Gamma_0(2(m+1))$. By Proposition 3.2 the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is

$$\frac{1}{24} \sum_{\delta|N} \frac{N(d,\delta)^2 r_\delta}{\left(d, \frac{N}{d}\right) d\delta}.$$

Therefore, it suffices for our purposes to show that the following is non-negative:

$$\frac{(d,1)^2(1)}{1} - \frac{(d,2)^2(2)}{2} + \frac{(d,2(m+1))^2(2(m+1))}{2(m+1)} - \frac{(d,(m+1))^2((m+1))}{(m+1)}.$$

If $2 \nmid d$, then

$$(d,1)^2 - (d,2)^2 + (d,2(m+1))^2 - (d,(m+1))^2 = 0.$$

If $2|d$, then

$$(d,1)^2 - (d,2)^2 + (d,2(m+1))^2 - (d,(m+1))^2 = 1 - 4 + 4(d,(m+1))^2 - (d,(m+1))^2 \geq 0.$$

Therefore, the order of vanishing of $f(z)$ at the cusps of $\Gamma_0(2(m+1))$ is nonnegative, so $f(z)$ is holomorphic on the entire upper half of the complex plane.

We have, for odd m,

$$\sum_{n=0}^{\infty} (P_+(n,m) - P_-(n,m))\, q^n = q^{(m+1)^2/4} \prod_{n=1}^{\infty} \frac{(1-q^{4(m+1)n})^{2m+2}}{(1-q^{2(m+1)n})^{m+1}}$$

$$= \prod_{n=1}^{\infty} \frac{q^{8(m+1)^2/24}(1-q^{4(m+1)n})^{2(m+1)}}{q^{2(m+1)^2/24}(1-q^{2(m+1)n})^{m+1}}$$

$$= \frac{\eta^{2(m+1)}(2(m+1)z)}{\eta^{m+1}((m+1)z)}.$$

Let $g(z) = \frac{\eta^{2(m+1)}(2(m+1)z)}{\eta^{m+1}((m+1)z)}$ and write $g(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$. We have

$$\sum_{\delta|N} \delta r_\delta = 8(m+1)^2 - 2(m+1)^2 = 6(m+1)^2 \equiv 0 \pmod{24}$$

for odd m.

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta = 4(m+1)\left(\frac{4(m+1)}{2(m+1)} - \frac{2(m+1)}{(m+1)}\right) = 0 \equiv 0 \pmod{24}.$$

So $g(z)$ satisfies (3.5) and (3.6), with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4(m+1))$, $k = \frac{m+1}{2}$, and $\chi(d) = \left(\frac{(-1)^{(m+1)/2}(m+1)^{m+1}}{d}\right)$.

Now we must check the holomorphicity of $g(z)$ at the cusps of $\Gamma_0(4(m+1))$. Arguing as above, it suffices for our purposes to show that the following is non-negative:

$$\frac{(d, 4(m+1))^2(2(m+1))}{4(m+1)} - \frac{(d, 2(m+1))^2((m+1))}{2(m+1)}.$$

Note that $\left(\frac{(d,4(m+1))^2}{2} - \frac{(d,2(m+1))^2}{2}\right) \geq 0$, so the order of vanishing of $g(z)$ is non-negative for all the cusps of $\Gamma_0(4(m+1))$. Therefore, $g(z)$ is holomorphic on the entire upper half of the complex plane.

Q.E.D.

To prove Corollary 7, we recall a theorem of Serre [S].

Theorem 3.4. *(Serre) Let $f(z)$ be a holomorphic modular form of positive integer weight k with character χ on $\Gamma_0(N)$ with Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \tag{3.10}$$

where $a(n)$ are algebraic integers in some number field and N is a positive number. If M is a positive integer, then there exists a constant α such that there are $O\left(\frac{x}{log^\alpha x}\right)$ integers $n \leq x$ such that $a(n)$ is not divisible by M.

Proof of Corollary 7. By Theorem 3.3, for every positive integer k the generating function

$$\sum_{n=0}^{\infty} \left(P_+(n,k) - P_-(n,k)\right) q^n$$

is a holomorphic integer weight modular form. Therefore, Theorem 3.4 immediately implies Corollary 7.

Q.E.D.

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