Energy conservation, counting statistics, and return to equilibrium

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Abstract. We study a microscopic Hamiltonian model describing an N-level quantum system $\mathcal{S}$ coupled to an infinitely extended thermal reservoir $\mathcal{R}$. Initially, the system $\mathcal{S}$ is in an arbitrary state while the reservoir is in thermal equilibrium at inverse temperature β. Assuming that the coupled system $\mathcal{S}+\mathcal{R}$ is mixing with respect to the joint thermal equilibrium state, we study the Full Counting Statistics (FCS) of the energy transfers $\mathcal{S}\to\mathcal{R}$ and $\mathcal{R}\to\mathcal{S}$ in the process of return to equilibrium. The first FCS describes the increase of the energy of the system $\mathcal{S}$. It is an atomic probability measure, denoted $\mathbb{P}_{\mathcal{S},\lambda,t}$, concentrated on the set of energy differences $\mathrm{sp}(H_\mathcal{S})-\mathrm{sp}(H_\mathcal{S})$ ($H_\mathcal{S}$ is the Hamiltonian of $\mathcal{S}$, t is the length of the time interval during which the measurement of the energy transfer is performed, and λ is the strength of the interaction between $\mathcal{S}$ and $\mathcal{R}$). The second FCS, $\mathbb{P}_{\mathcal{R},\lambda,t}$, describes the decrease of the energy of the reservoir $\mathcal{R}$ and is typically a continuous probability measure whose support is the whole real line. We study the large time limit $t\to\infty$ of these two measures followed by the weak coupling limit $\lambda\to 0$ and prove that the limiting measures coincide. This result strengthens the first law of thermodynamics for open quantum systems. The proofs are based on modular theory of operator algebras and on a representation of $\mathbb{P}_{\mathcal{R},\lambda,t}$ by quantum transfer operators.

1 Introduction

The 0^{th}-law of thermodynamics asserts that a large system, left alone and under normal conditions, approaches an equilibrium state characterized by a few macroscopic parameters such as temperature and density (see [Ca] or any book on thermodynamics). In particular, a small system coupled to a large (i.e., infinitely extended) reservoir at temperature T is expected to reach its equilibrium state at the same temperature, irrespective of its initial state. This specific part of the 0^{th}-law is often called *return to equilibrium*. From a mechanical point of view, return to equilibrium holds if the interaction is sufficiently dispersive, which translates into ergodic properties of the dynamics.

In this paper we consider a quantum system consisting of an N-level system $\mathcal{S}$ coupled to a reservoir $\mathcal{R}$. We assume that the joint system $\mathcal{S}+\mathcal{R}$ has the property of return to equilibrium. The precise mathematical formulation of this property is given in Section 2 (Assumption **(M)**). Although notoriously difficult to prove, return to equilibrium has been established for several physically relevant models (spin-boson model, spin-fermion model, electronic black box model, locally interacting fermionic systems) which are discussed in [AMa, AJPP1, AJPP2, BFS, BM, dRK, DJ, FMU, FMSU, JOP1, JOP2, JP1, JP5].

We consider the exchange of energy between the system $\mathcal{S}$ and the reservoir $\mathcal{R}$. Let λ be a parameter describing the strength of the coupling between $\mathcal{S}$ and $\mathcal{R}$ and denote by $\Delta\mathcal{Q}_\mathcal{S}(\lambda,t)$ the increase of energy of the system $\mathcal{S}$

and by $\Delta\mathcal{Q}_{\mathcal{R}}(\lambda,t)$ the decrease of energy of the reservoir $\mathcal{R}$ during the time period from 0 to t. Set

$$\Delta\mathcal{Q}_{\mathcal{S}}=\lim_{\lambda\to 0}\lim_{t\to\infty}\Delta\mathcal{Q}_{\mathcal{S}}(\lambda,t),\qquad \Delta\mathcal{Q}_{\mathcal{R}}=\lim_{\lambda\to 0}\lim_{t\to\infty}\Delta\mathcal{Q}_{\mathcal{R}}(\lambda,t).$$

As a consequence of energy conservation, we expect that

$$\Delta\mathcal{Q}_{\mathcal{S}}=\Delta\mathcal{Q}_{\mathcal{R}}. \tag{1.1}$$

This is well understood and can be proven using Araki's perturbation theory, see Section 3.

The main goal of this paper is to refine the above result by considering the Full Counting Statistics (abbreviated FCS) of the energy transfer between $\mathcal{S}$ and $\mathcal{R}$. There has been much interest in Full Counting Statistics since the seminal paper [LL]. We refer the reader to [JOPP] for additional information and references to the vast literature on the FCS in quantum statistical mechanics.

We consider the probability distribution $\mathbb{P}_{\mathcal{S},\lambda,t}$ of the measured increase of energy of the system $\mathcal{S}$ obtained by performing a first measurement at time 0 and a second one at time t. While it is straightforward to define $\mathbb{P}_{\mathcal{S},\lambda,t}$, the analogous definition of the probability measure $\mathbb{P}_{\mathcal{R},\lambda,t}$ for the reservoir must be given in terms of a relative modular operator due to the fact that the reservoir is infinitely extended (Definition 4.12). We devote Section 5 to motivating this definition by showing that the correct physical interpretation is recovered in the case of a confined reservoir.

Our main result (Theorem 4.2) is the following: under suitable (and in a certain sense minimal) regularity assumptions the weak limit

$$\mathbb{P}_{\mathcal{R}}=\lim_{\lambda\to 0}\lim_{t\to\infty}\mathbb{P}_{\mathcal{R},\lambda,t}$$

exists and

$$\mathbb{P}_{\mathcal{S}}=\lim_{\lambda\to 0}\lim_{t\to\infty}\mathbb{P}_{\mathcal{S},\lambda,t}=\mathbb{P}_{\mathcal{R}}.$$

Noting that $\Delta\mathcal{Q}_{\mathcal{S}}/\Delta\mathcal{Q}_{\mathcal{R}}$ is the first moment of $\mathbb{P}_{\mathcal{S}}/\mathbb{P}_{\mathcal{R}}$, Theorem 4.2 is a somewhat surprising strengthening of the energy conservation law (1.1).

Non-equilibrium open quantum systems describing a finite-level system $\mathcal{S}$ coupled to several independent reservoirs in thermal equilibrium at distinct temperatures have different physics and are characterized by steady state energy fluxes across $\mathcal{S}$. An extension of Theorem 4.2 to such situations requires an approach that differs both technically and conceptually and is discussed in the forthcoming paper [JPPP].

The paper is organized as follows. In Section 2 we outline the mathematical framework used in this paper and state our assumptions. A review of the conservation law (1.1) is given in Section 3. In Section 4 we define the FCS of the energy transfers for the system and reservoir, $\mathbb{P}_{\mathcal{S},\lambda,t}$ and $\mathbb{P}_{\mathcal{R},\lambda,t}$, and state our main result. The definition of $\mathbb{P}_{\mathcal{R},\lambda,t}$ is motivated in Section 5. Section 6 is devoted to the proofs.

Throughout the paper we use the algebraic formalism of quantum statistical mechanics which is conceptually and technically adapted for the problem under study. The Full Counting Statistics has attracted considerable attention in recent experimental and theoretical physics literature, and we have attempted to make the paper accessible to readers interested in FCS but not familiar with the algebraic formalism. For this reason we give detailed proofs and collect some well known constructions and results in Appendix A. This material is standard and can be found in [BR1, BR2]. Modern expositions of the algebraic formalism can also be found in [Pi, DJP] and pedagogical expositions in [JOPP, Th]. In particular, the exposition in [JOPP] is geared toward the application of this algebraic framework to the study of FCS.

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2 Mathematical setting and assumptions

We consider a small system $\mathcal{S}$ coupled to an infinitely extended reservoir $\mathcal{R}$.

The system $\mathcal{S}$ is completely determined by a finite dimensional Hilbert space $\mathcal{H}_\mathcal{S}$, a Hamiltonian $H_\mathcal{S}$ (a self-adjoint operator on $\mathcal{H}_\mathcal{S}$) and a density matrix $\rho_\mathcal{S}$ (a positive operator on $\mathcal{H}_\mathcal{S}$ such that $\mathrm{tr}(\rho_\mathcal{S}) = 1$).

The associated dynamical system is $(\mathcal{O}_\mathcal{S}, \tau_\mathcal{S}, \rho_\mathcal{S})$, where[1] $\mathcal{O}_\mathcal{S} = \mathcal{B}(\mathcal{H}_\mathcal{S})$ is the C^*-algebra of observables of $\mathcal{S}$,

$$\tau_\mathcal{S}^t(A) = \mathrm{e}^{\mathrm{i}tH_\mathcal{S}} A \mathrm{e}^{-\mathrm{i}tH_\mathcal{S}}$$

is the Heisenberg dynamics on $\mathcal{O}_\mathcal{S}$ induced by the Hamiltonian $H_\mathcal{S}$, and[2] $\rho_\mathcal{S}(A) = \mathrm{tr}(\rho_\mathcal{S} A)$ is the initial state of $\mathcal{S}$. We do not require $\rho_\mathcal{S}$ to be faithful. We denote by $\delta_\mathcal{S}(\,\cdot\,) = \mathrm{i}[H_\mathcal{S}, \,\cdot\,]$ the generator of $\tau_\mathcal{S}$.

The reservoir $\mathcal{R}$ is described by a C^*-dynamical system $(\mathcal{O}_\mathcal{R}, \tau_\mathcal{R}, \omega_\mathcal{R})$ in thermal equilibrium at inverse temperature $\beta > 0$. $\mathcal{O}_\mathcal{R}$ is the C^*-algebra of observables of $\mathcal{R}$ and $\mathbb{R} \ni t \mapsto \tau_\mathcal{R}^t$ a strongly continuous group of $*$-automorphisms of $\mathcal{O}_\mathcal{R}$ describing the time evolution of $\mathcal{R}$ in the Heisenberg picture. We denote by $\delta_\mathcal{R}$ its generator, $\tau_\mathcal{R}^t = \mathrm{e}^{t\delta_\mathcal{R}}$. Finally, $\omega_\mathcal{R}$ is a $(\tau_\mathcal{R}, \beta)$-KMS state on $\mathcal{O}_\mathcal{R}$.

The uncoupled joint system $\mathcal{S} + \mathcal{R}$ is described by the C^*-dynamical system $(\mathcal{O}, \tau_0, \omega)$ where

$$\mathcal{O} = \mathcal{O}_\mathcal{S} \otimes \mathcal{O}_\mathcal{R}, \quad \tau_0 = \tau_\mathcal{S} \otimes \tau_\mathcal{R}, \quad \omega = \rho_\mathcal{S} \otimes \omega_\mathcal{R}.$$

We also introduce the (τ_0, β)-KMS state

$$\omega_0 = \omega_\mathcal{S} \otimes \omega_\mathcal{R},$$

where

$$\omega_\mathcal{S} = \frac{\mathrm{e}^{-\beta H_\mathcal{S}}}{\mathrm{tr}(\mathrm{e}^{-\beta H_\mathcal{S}})}$$

is the thermal equilibrium state of $\mathcal{S}$. In the sequel, when tensoring with the identity and whenever the meaning is clear within the context we will omit the identity part. With this convention, the generator of τ_0 is given by $\delta_0 = \delta_\mathcal{S} + \delta_\mathcal{R}$.

Let

$$\delta_\lambda = \delta_0 + \mathrm{i}\lambda[V, \,\cdot\,],$$

where V is a self-adjoint element of $\mathcal{O}$ and λ is a real coupling constant. The interacting dynamics is $\tau_\lambda^t = \mathrm{e}^{t\delta_\lambda}$ and the coupled joint system is described by the C^*-dynamical system $(\mathcal{O}, \tau_\lambda, \omega)$.

Remark 2.1 With only minor changes all our results and proofs extend to cases where the reservoir $\mathcal{R}$ is described by a W^*-dynamical system $(\mathcal{O}_\mathcal{R}, \tau_\mathcal{R}, \omega_\mathcal{R})$, and its coupling to $\mathcal{S}$ by an unbounded perturbation V satisfying the general assumptions of [DJP]. The details of this generalization can be found in [Pa].

To give a precise formulation of the property of return to equilibrium, we recall the following result from Araki's perturbation theory of the KMS-structure (see for example [BR2, DJP]). Let $(\mathfrak{H}, \pi, \Omega_0)$ be a GNS representation of $\mathcal{O}$ induced by the state ω_0. A positive linear functional ζ on $\mathcal{O}$ is ω_0-normal if it is given by

$$\zeta(A) = \mathrm{tr}(\rho_\zeta \pi(A))$$

for some positive trace class operator ρ_ζ on $\mathfrak{H}$ (see Definition A.13). We denote by $\mathcal{N}$ be the set of all ω_0-normal positive linear functionals on $\mathcal{O}$.

Theorem 2.2 (1) *There exists a unique (τ_λ, β)-KMS state in $\mathcal{N}$ which we denote by ω_λ.*

(2) *The set of all ω_λ-normal positive linear functionals on $\mathcal{O}$ coincides with $\mathcal{N}$.*

(3)
$$\lim_{\lambda \to 0} \omega_\lambda = \omega_0 \tag{2.2}$$
holds in the norm topology of the dual space of $\mathcal{O}$.

[1]Throughout the paper $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded operators on a Hilbert space $\mathcal{H}$.

[2]If $\dim \mathcal{H} < \infty$, we shall identify positive linear functionals on $\mathcal{B}(\mathcal{H})$ and positive elements of $\mathcal{B}(\mathcal{H})$ according to $\zeta(A) = \mathrm{tr}(\zeta A)$.

Definition 2.3 The C^*-dynamical system $(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ is called *ergodic* if, for all states $\zeta \in \mathcal{N}$ and all $A \in \mathcal{O}$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \zeta(\tau_\lambda^s(A))\mathrm{d}s = \omega_\lambda(A). \tag{2.3}$$

It is *mixing* if

$$\lim_{t\to\infty} \zeta(\tau_\lambda^t(A)) = \omega_\lambda(A). \tag{2.4}$$

For obvious reasons, ergodicity/mixing of $(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ is often called *the property of return to equilibrium.* (See [Rob] for foundational work on the subject and [BFS, dRK, DJ, FM, JP1, JP2, JP4] for references and additional information.)

Our main dynamical assumption is:

> **Assumption (M)** There exists $\lambda_0 > 0$ such that the C^*-dynamical system $(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ is mixing for $0 < |\lambda| < \lambda_0$.

We shall also need the following regularity assumption:

> **Assumption (A)** $V \in \mathrm{Dom}\,(\delta_\mathcal{R})$.

Remark 2.4 For motivation and clarification purposes we shall sometimes consider a *confined* reservoir described by a finite dimensional Hilbert space $\mathcal{H}_\mathcal{R}$ and a Hamiltonian $H_\mathcal{R}$. In this case $\mathcal{O}_\mathcal{R} = \mathcal{B}(\mathcal{H}_\mathcal{R})$, $\delta_\mathcal{R}(\,\cdot\,) = \mathrm{i}[H_\mathcal{R}, \,\cdot\,]$, and $\omega_\mathcal{R}(A) = \mathrm{tr}(\omega_\mathcal{R} A)$ where

$$\omega_\mathcal{R} = \frac{\mathrm{e}^{-\beta H_\mathcal{R}}}{\mathrm{tr}(\mathrm{e}^{-\beta H_\mathcal{R}})}.$$

Moreover, $\mathcal{O} = \mathcal{B}(\mathcal{H})$ with $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R}$, $\tau_\lambda^t(A) = \mathrm{e}^{\mathrm{i}tH_\lambda} A \mathrm{e}^{-\mathrm{i}tH_\lambda}$ with $H_\lambda = H_\mathcal{S} + H_\mathcal{R} + \lambda V$, and the (τ_λ, β)-KMS state ω_λ is given by the density matrix

$$\omega_\lambda = \frac{\mathrm{e}^{-\beta H_\lambda}}{\mathrm{tr}(\mathrm{e}^{-\beta H_\lambda})}.$$

The GNS representation $(\mathfrak{H}, \pi, \Omega_0)$ can be realized in the following way (see [JOPP]). The Hilbert space $\mathfrak{H}$ is $\mathcal{O}$ equipped with the inner product $\langle X|Y\rangle = \mathrm{tr}(X^*Y)$. For $A \in \mathcal{O}$ the map $\pi(A) \in \mathcal{B}(\mathfrak{H})$ is given by

$$\pi(A)X = AX.$$

Finally, the cyclic vector is $\Omega_0 = \omega_0^{1/2}$. More generally, any positive linear functional on $\mathcal{O}$ can be written as

$$A \mapsto \mathrm{tr}(\zeta A) = \langle \zeta^{1/2}|\pi(A)\zeta^{1/2}\rangle,$$

where ζ is a positive element of $\mathcal{O} = \mathfrak{H}$.

3 Review of the first law of thermodynamics

If Assumption **(M)** holds, then

$$\lim_{t\to\infty} \omega(\tau_\lambda^t(A)) = \omega_\lambda(A)$$

holds for all $A \in \mathcal{O}$. We will study the energy transfer between $\mathcal{S}$ and $\mathcal{R}$ during the state transition $\omega \to \omega_\lambda$. The energy increase of $\mathcal{S}$ over the time interval $[0, t]$ is

$$\Delta\mathcal{Q}_\mathcal{S}(\lambda, t) = \omega(\tau_\lambda^t(H_\mathcal{S})) - \omega(H_\mathcal{S}) = \int_0^t \omega(\tau_\lambda^s(\Phi_\mathcal{S}))\mathrm{d}s, \tag{3.5}$$

where

$$\Phi_{\mathcal{S}} = \delta_\lambda(H_{\mathcal{S}}) = -\lambda\delta_{\mathcal{S}}(V)$$

is the observable describing the energy flux toward $\mathcal{S}$. For $0 < |\lambda| < \lambda_0$, the mixing property yields

$$\Delta\mathcal{Q}_{\mathcal{S}}(\lambda) = \lim_{t\to\infty} \Delta\mathcal{Q}_{\mathcal{S}}(\lambda, t) = \omega_\lambda(H_{\mathcal{S}}) - \omega(H_{\mathcal{S}}),$$

and Eq. (2.2) further gives

$$\Delta\mathcal{Q}_{\mathcal{S}} = \lim_{\lambda\to 0} \Delta\mathcal{Q}_{\mathcal{S}}(\lambda) = \omega_{\mathcal{S}}(H_{\mathcal{S}}) - \rho_{\mathcal{S}}(H_{\mathcal{S}}).$$

In what follows we assume that Assumption **(A)** holds. The observable describing the energy flux out of $\mathcal{R}$ is

$$\Phi_{\mathcal{R}} = \lambda\delta_{\mathcal{R}}(V),$$

and the decrease of energy of $\mathcal{R}$ over the time interval $[0, t]$ is

$$\Delta\mathcal{Q}_{\mathcal{R}}(\lambda, t) = \int_0^t \omega(\tau_\lambda^s(\Phi_{\mathcal{R}}))\mathrm{d}s. \tag{3.6}$$

To motivate this definition, consider a confined reservoir $\mathcal{R}$. In this case, according to Remark 2.4, the decrease of the energy of $\mathcal{R}$ is given by

$$\omega(H_{\mathcal{R}}) - \omega(\tau_\lambda^t(H_{\mathcal{R}})) = \int_0^t \omega(\tau_\lambda^s(\Phi_{\mathcal{R}}))\mathrm{d}s,$$

with

$$\Phi_{\mathcal{R}} = -\delta_\lambda(H_{\mathcal{R}}) = -\mathrm{i}[H_\lambda, H_{\mathcal{R}}] = \mathrm{i}[H_{\mathcal{R}}, \lambda V] = \lambda\delta_{\mathcal{R}}(V).$$

Returning to the general case, since $\Phi_{\mathcal{R}} - \Phi_{\mathcal{S}} = \lambda\delta_\lambda(V)$, we have

$$\Delta\mathcal{Q}_{\mathcal{R}}(\lambda, t) = \Delta\mathcal{Q}_{\mathcal{S}}(\lambda, t) + \lambda\omega(\tau_\lambda^t(V) - V). \tag{3.7}$$

For $0 < |\lambda| < \lambda_0$, the mixing property implies

$$\Delta\mathcal{Q}_{\mathcal{R}}(\lambda) = \lim_{t\to\infty} \Delta\mathcal{Q}_{\mathcal{R}}(\lambda, t) = \Delta\mathcal{Q}_{\mathcal{S}}(\lambda) + \lambda(\omega_\lambda(V) - \omega(V)),$$

and Eq. (2.2) shows that

$$\Delta\mathcal{Q}_{\mathcal{R}} = \lim_{\lambda\to 0} \Delta\mathcal{Q}_{\mathcal{R}}(\lambda)$$

satisfies

$$\Delta\mathcal{Q}_{\mathcal{S}} = \Delta\mathcal{Q}_{\mathcal{R}}. \tag{3.8}$$

Relation (3.8) is a mathematical formulation of the first law of thermodynamics (energy conservation) for the joint system $\mathcal{S} + \mathcal{R}$ in the process of return to equilibrium described by the above double limit (first $t \to \infty$ and then $\lambda \to 0$).

4 The first law and full counting statistics

Our main goal is to refine the previous result by considering the Full Counting Statistics of the energy transfer between $\mathcal{S}$ and $\mathcal{R}$. We start with the small system $\mathcal{S}$. Let[3]

$$H_{\mathcal{S}} = \sum_{e\in\mathrm{sp}(H_{\mathcal{S}})} eP_e$$

[3] $\mathrm{sp}(A)$ denotes the spectrum of the operator A.

be the spectral resolution of $H_{\mathcal{S}}$. Suppose that at time $t = 0$, when the system is in the state $\omega = \rho_{\mathcal{S}} \otimes \omega_{\mathcal{R}}$, a measurement of $H_{\mathcal{S}}$ is performed. The outcome e is observed with probability

$$\omega(P_e) = \rho_{\mathcal{S}}(P_e).$$

After the measurement the state of the system is

$$\frac{P_e \rho_{\mathcal{S}} P_e}{\rho_{\mathcal{S}}(P_e)} \otimes \omega_{\mathcal{R}}.$$

This state evolves in time with the dynamics τ_λ. A second measurement of $H_{\mathcal{S}}$ at time t gives e' with probability

$$\left(\frac{P_e \rho_{\mathcal{S}} P_e}{\rho_{\mathcal{S}}(P_e)} \otimes \omega_{\mathcal{R}}\right)(\tau_\lambda^t(P_{e'})).$$

Hence, $(P_e \rho_{\mathcal{S}} P_e \otimes \omega_{\mathcal{R}})(\tau_\lambda^t(P_{e'}))$ is the joint probability distribution of the two measurements. The respective FCS is the atomic probability measure on $\mathbb{R}$ defined by

$$\mathbb{P}_{\mathcal{S},\lambda,t}(S) = \sum_{e'-e \in S} (P_e \rho_{\mathcal{S}} P_e \otimes \omega_{\mathcal{R}})(\tau_\lambda^t(P_{e'})). \tag{4.9}$$

This measure is concentrated on the set of energy differences $\mathrm{sp}(H_{\mathcal{S}}) - \mathrm{sp}(H_{\mathcal{S}})$ and $\mathbb{P}_{\mathcal{S},\lambda,t}(S)$ is the probability that the measured increase of the energy of $\mathcal{S}$ in the above protocol takes value in the set $S \subset \mathbb{R}$. The measure $\mathbb{P}_{\mathcal{S},\lambda,t}$ contains full information about the statistics of energy transfer to $\mathcal{S}$ over the time period $[0, t]$.

We denote by $\langle \cdot \rangle_{\mathcal{S},\lambda,t}$ the expectation with respect to $\mathbb{P}_{\mathcal{S},\lambda,t}$ (and similarly for other measures that will appear later). If $\rho_{\mathcal{S}}$ and $H_{\mathcal{S}}$ commute, then an elementary computation gives

$$\langle \varsigma \rangle_{\mathcal{S},\lambda,t} = \int_{\mathbb{R}} \varsigma \mathrm{d}\mathbb{P}_{\mathcal{S},\lambda,t}(\varsigma) = \Delta \mathcal{Q}_{\mathcal{S}}(\lambda, t).$$

We are interested in the limiting values of $\mathbb{P}_{\mathcal{S},\lambda,t}$ as $t \to \infty$ and $\lambda \to 0$. Assumption **(M)** implies that for $0 < |\lambda| < \lambda_0$,

$$\mathbb{P}_{\mathcal{S},\lambda}(S) = \lim_{t\to\infty} \mathbb{P}_{\mathcal{S},\lambda,t}(S) = \sum_{e'-e \in S} \omega_\lambda(P_{e'}) \rho_{\mathcal{S}}(P_e). \tag{4.10}$$

If, instead of mixing, we assume that $(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ is ergodic for $0 < |\lambda| < \lambda_0$, then (4.10) holds with $\mathbb{P}_{\mathcal{S},\lambda,t}$ replaced with

$$\frac{1}{t}\int_0^t \mathbb{P}_{\mathcal{S},\lambda,s} \mathrm{d}s.$$

Obviously,

$$\langle \varsigma \rangle_{\mathcal{S},\lambda} = \sum_{e',e} (e' - e) \omega_\lambda(P_{e'}) \rho_{\mathcal{S}}(P_e) = \omega_\lambda(H_{\mathcal{S}}) - \rho_{\mathcal{S}}(H_{\mathcal{S}}) = \Delta \mathcal{Q}_{\mathcal{S}}(\lambda).$$

Relation (2.2) further gives

$$\mathbb{P}_{\mathcal{S}}(S) = \lim_{\lambda\to 0} \mathbb{P}_{\mathcal{S},\lambda}(S) = \sum_{e'-e \in S} \omega_{\mathcal{S}}(P_{e'}) \rho_{\mathcal{S}}(P_e).$$

In particular

$$\langle \varsigma \rangle_{\mathcal{S}} = \Delta \mathcal{Q}_{\mathcal{S}}.$$

Note that the limiting FCS $\mathbb{P}_{\mathcal{S}}$ is the law of $\varsigma = E' - E$ where E and E' are independent random variables such that

$$\mathrm{Prob}[E = e] = \rho_{\mathcal{S}}(P_e), \qquad \mathrm{Prob}[E' = e'] = \omega_{\mathcal{S}}(P_{e'}).$$

For later reference, we note that the characteristic function of $\mathbb{P}_{\mathcal{S}}$ is

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma\varsigma}\mathrm{d}\mathbb{P}_{\mathcal{S}}(\varsigma) = \sum_{e,e'\in\mathrm{sp}(H_{\mathcal{S}})} \omega_{\mathcal{S}}(\mathrm{e}^{\mathrm{i}\gamma e'}P_{e'})\rho_{\mathcal{S}}(\mathrm{e}^{-\mathrm{i}\gamma e}P_e) = \omega_{\mathcal{S}}\left(\mathrm{e}^{\mathrm{i}\gamma H_{\mathcal{S}}}\right)\rho_{\mathcal{S}}\left(\mathrm{e}^{-\mathrm{i}\gamma H_{\mathcal{S}}}\right). \tag{4.11}$$

We now turn to the Full Counting Statistics for the reservoir $\mathcal{R}$. Recall that $(\mathfrak{H}, \pi, \Omega_0)$ denotes a GNS representation of $\mathcal{O}$ induced by the thermal equilibrium state of the decoupled system $\omega_0 = \omega_{\mathcal{S}} \otimes \omega_{\mathcal{R}}$. Let $\mathfrak{M} = \pi(\mathcal{O})''$ be the associated enveloping von Neumann algebra and $\mathcal{P}$ the natural cone of the pair $(\mathfrak{M}, \Omega_0)$. We denote by Ω the unique vector representative of the initial state $\omega = \rho_{\mathcal{S}} \otimes \omega_{\mathcal{R}}$ in $\mathcal{P}$ and by $\Delta_{\zeta|\xi}$ the relative modular operator of the two positive linear functionals $\zeta, \xi \in \mathcal{N}$. As usual, we set $\Delta_\zeta = \Delta_{\zeta|\zeta}$. Finally, let $\eta = \mathbb{1} \otimes \omega_{\mathcal{R}}$. The following definition will be motivated in Section 5.

Definition 4.1 The FCS of the decrease of the energy of $\mathcal{R}$ is the spectral measure $\mathbb{P}_{\mathcal{R},\lambda,t}$ of the self-adjoint operator

$$\frac{1}{\beta}\log\Delta_{\eta\circ\tau_\lambda^{-t}|\eta} \tag{4.12}$$

for the vector Ω.

We note that since $\log\Delta_\eta\Omega = 0$ and (see [JP6, Section 3])

$$\log\Delta_{\eta\circ\tau_\lambda^{-t}|\eta} = \log\Delta_\eta + \beta\int_0^t \tau_\lambda^s(\Phi_{\mathcal{R}})\mathrm{d}s,$$

Eq. (3.6) implies

$$\langle\varsigma\rangle_{\mathcal{R},\lambda,t} = \Delta\mathcal{Q}_{\mathcal{R}}(\lambda, t). \tag{4.13}$$

The two measures $\mathbb{P}_{\mathcal{S},\lambda,t}$ and $\mathbb{P}_{\mathcal{R},\lambda,t}$ are of course very different. The first one is supported on the discrete set $\mathrm{sp}(H_{\mathcal{S}}) - \mathrm{sp}(H_{\mathcal{S}})$ while, for an infinitely extended reservoir, the second one is typically a continuous measure whose support is the whole real line. On the other hand, the first law gives

$$\lim_{\lambda\to0}\lim_{t\to\infty}\langle\varsigma\rangle_{\mathcal{R},\lambda,t} = \lim_{\lambda\to0}\lim_{t\to\infty}\langle\varsigma\rangle_{\mathcal{S},\lambda,t},$$

and one may ask about the relation between the measures $\mathbb{P}_{\mathcal{S},\lambda,t}$ and $\mathbb{P}_{\mathcal{R},\lambda,t}$ in the double limit $t \to \infty$, $\lambda \to 0$. Our main result is:

Theorem 4.2 *Suppose that Assumptions* **(M)** *and* **(A)** *hold and that* $0 < |\lambda| < \lambda_0$. *Then the weak limits*

$$\mathbb{P}_{\mathcal{R},\lambda} = \lim_{t\to\infty}\mathbb{P}_{\mathcal{R},\lambda,t},$$

and

$$\mathbb{P}_{\mathcal{R}} = \lim_{\lambda\to0}\mathbb{P}_{\mathcal{R},\lambda},$$

exist, and

$$\mathbb{P}_{\mathcal{R}} = \mathbb{P}_{\mathcal{S}}.$$

Remark 4.3 (1) For the definition and basic properties of the weak convergence of probability measures we refer the reader to Chapter 1 of [Bi].

(2) The proof of Theorem 4.2 gives more information and in particular provides a formula for the characteristic function of $\mathbb{P}_{\mathcal{R},\lambda}$ in terms of the modular data of the model.

(3) If instead of mixing we assume ergodicity of $(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ for $0 < |\lambda| < \lambda_0$, then Theorem 4.2 holds with $\mathbb{P}_{\mathcal{R},\lambda,t}$ replaced with

$$\frac{1}{t}\int_0^t \mathbb{P}_{\mathcal{R},\lambda,s}\mathrm{d}s.$$

(4) If V is analytic for $\tau_{\mathcal{R}}$ (see Section 2.5.3 in [BR1], or Section A.4), then the proof of Theorem 4.2 considerably simplifies. Moreover, one easily establishes that in addition

$$\lim_{\lambda\to 0}\lim_{t\to\infty}\langle\varsigma^n\rangle_{\mathcal{R},\lambda,t} = \lim_{\lambda\to 0}\lim_{t\to\infty}\langle\varsigma^n\rangle_{\mathcal{S},\lambda,t} \tag{4.14}$$

holds for all integers $n > 0$. Details and additional information can be found in [Pa].

5 Motivation of Definition 4.1

In order to get a physical interpretation of the measure $\mathbb{P}_{\mathcal{R},\lambda,t}$, let us assume that the reservoir $\mathcal{R}$ is confined. It follows from Definition 4.1 that the characteristic function of $\mathbb{P}_{\mathcal{R},\lambda,t}$ is

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\alpha\varsigma}\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma) = \langle\Omega|\Delta^{\mathrm{i}\alpha/\beta}_{\eta\circ\tau_\lambda^{-t}|\eta}\Omega\rangle.$$

For a confined system, the relative modular operator of two positive linear functionals ζ, ξ acts on the GNS Hilbert space $\mathfrak{H}$ as[4]

$$\Delta_{\zeta|\xi}X = \zeta X \xi^{-1}.$$

It follows that

$$\Delta^{\mathrm{i}\gamma}_{\eta\circ\tau_\lambda^{-t}|\eta}X = \mathrm{e}^{\mathrm{i}tH_\lambda}\eta^{\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}tH_\lambda}X\eta^{-\mathrm{i}\gamma}.$$

Let

$$H_{\mathcal{R}} = \sum_{\varepsilon\in\mathrm{sp}(H_{\mathcal{R}})}\varepsilon P_\varepsilon$$

be the spectral resolution of $H_{\mathcal{R}}$. One then easily computes

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\alpha\varsigma}\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma) &= \mathrm{tr}\left(\omega^{1/2}\mathrm{e}^{\mathrm{i}tH_\lambda}\eta^{\mathrm{i}\alpha/\beta}\mathrm{e}^{-\mathrm{i}tH_\lambda}\omega^{1/2}\eta^{-\mathrm{i}\alpha/\beta}\right)\\
&= \mathrm{tr}\left(\left(\mathbb{1}\otimes\omega_{\mathcal{R}}^{\mathrm{i}\alpha/\beta}\right)\mathrm{e}^{-\mathrm{i}tH_\lambda}\left(\rho_{\mathcal{S}}\otimes\omega_{\mathcal{R}}^{1-\mathrm{i}\alpha/\beta}\right)\mathrm{e}^{\mathrm{i}tH_\lambda}\right)\\
&= \sum_{\varepsilon,\varepsilon'\in\mathrm{sp}(H_{\mathcal{R}})}\mathrm{e}^{\mathrm{i}\alpha(\varepsilon-\varepsilon')}\mathrm{tr}\left((\mathbb{1}\otimes P_{\varepsilon'})\mathrm{e}^{-\mathrm{i}tH_\lambda}(\mathbb{1}\otimes P_\varepsilon)(\rho_{\mathcal{S}}\otimes\omega_{\mathcal{R}})\mathrm{e}^{\mathrm{i}tH_\lambda}\right)\\
&= \sum_{\varepsilon,\varepsilon'\in\mathrm{sp}(H_{\mathcal{R}})}\mathrm{e}^{\mathrm{i}\alpha(\varepsilon-\varepsilon')}(\rho_{\mathcal{S}}\otimes P_\varepsilon\omega_{\mathcal{R}}P_\varepsilon)(\tau_\lambda^t(P_{\varepsilon'}))
\end{aligned}$$

from which we can conclude that

$$\mathbb{P}_{\mathcal{R},\lambda,t}(S) = \sum_{\varepsilon-\varepsilon'\in S}(\rho_{\mathcal{S}}\otimes P_\varepsilon\omega_{\mathcal{R}}P_\varepsilon)(\tau_\lambda^t(P_{\varepsilon'})).$$

Comparing this to Eq. (4.9) leads to an interpretation of $\mathbb{P}_{\mathcal{R},\lambda,t}$ analogous to the one of $\mathbb{P}_{\mathcal{S},\lambda,t}$ (except that we sum over $\varepsilon - \varepsilon' \in S$ instead of $\varepsilon' - \varepsilon \in S$ since we are measuring the decrease of the energy of $\mathcal{R}$).

[4] Recall Remark 2.4 and see [JOPP].

Physically relevant infinitely extended reservoirs $\mathcal{R}$ can be obtained as a thermodynamic limit of confined reservoirs $\mathcal{R}_n$. In such cases, and under very general assumptions, the FCS of the infinitely extended system is the weak limit of the FCS of confined systems (see Section 5 in [JOPP]), i.e.,

$$\mathbb{P}_{\mathcal{R},\lambda,t} = \lim_{n\to\infty} \mathbb{P}_{\mathcal{R}_n,\lambda,t}.$$

The measure $\mathbb{P}_{\mathcal{R},\lambda,t}$ contains full information about the statistics of energy transfers out of $\mathcal{R}$ over the time interval $[0,t]$.

6 Proofs

6.1 Notation and preliminaries

According to Remark 2.4, a GNS representation of $\mathcal{O}_\mathcal{S}$ induced by $\omega_\mathcal{S}$ is given by $(\mathfrak{H}_\mathcal{S}, \pi_\mathcal{S}, \Omega_\mathcal{S})$ where the Hilbert space $\mathfrak{H}_\mathcal{S}$ is $\mathcal{O}_\mathcal{S} = \mathcal{B}(\mathcal{H}_\mathcal{S})$ equipped with the inner product $\langle X|Y\rangle = \operatorname{tr}(X^*Y)$. Given $A \in \mathcal{O}_\mathcal{S}$, the linear map $\pi_\mathcal{S}(A) \in \mathcal{B}(\mathfrak{H}_\mathcal{S})$ is given by

$$\pi_\mathcal{S}(A)X = AX,$$

and $\Omega_\mathcal{S} = \omega_\mathcal{S}^{1/2} \in \mathfrak{H}_\mathcal{S}$. The corresponding natural cone $\mathcal{P}_\mathcal{S} \subset \mathfrak{H}_\mathcal{S}$ and modular conjugation $J_\mathcal{S} : \mathfrak{H}_\mathcal{S} \to \mathfrak{H}_\mathcal{S}$ are

$$\mathcal{P}_\mathcal{S} = \{X \in \mathfrak{H}_\mathcal{S} \,|\, X \geq 0\}, \quad J_\mathcal{S} X = X^*.$$

Let $L_\mathcal{S} \in \mathcal{B}(\mathfrak{H}_\mathcal{S})$ be defined by

$$L_\mathcal{S} X = [H_\mathcal{S}, X] = (\pi_\mathcal{S}(H_\mathcal{S}) - J_\mathcal{S}\pi_\mathcal{S}(H_\mathcal{S})J_\mathcal{S})X.$$

One easily checks that for all $t \in \mathbb{R}$ and $A \in \mathcal{O}$,

$$\pi_\mathcal{S}(\tau_\mathcal{S}^t(A)) = \mathrm{e}^{\mathrm{i}tL_\mathcal{S}}\pi_\mathcal{S}(A)\mathrm{e}^{-\mathrm{i}tL_\mathcal{S}}, \qquad \mathrm{e}^{-\mathrm{i}tL_\mathcal{S}}\mathcal{P}_\mathcal{S} = \mathcal{P}_\mathcal{S}, \qquad L_\mathcal{S}\Omega_\mathcal{S} = 0.$$

The operator $L_\mathcal{S}$ is the standard Liouvillean of the dynamical system $(\mathcal{O}_\mathcal{S}, \tau_\mathcal{S}, \omega_\mathcal{S})$.

Let $(\mathcal{H}_\mathcal{R}, \pi_\mathcal{R}, \Omega_\mathcal{R})$ be a GNS representation of $\mathcal{O}_\mathcal{R}$ induced by the state $\omega_\mathcal{R}$ and denote by $\mathfrak{M}_\mathcal{R} = \pi_\mathcal{R}(\mathcal{O}_\mathcal{R})''$ the associated enveloping von Neumann algebra. Since $\omega_\mathcal{R}$ is a $(\tau_\mathcal{R}, \beta)$-KMS state, the cyclic vector $\Omega_\mathcal{R}$ is separating for $\mathfrak{M}_\mathcal{R}$. Let $\mathcal{P}_\mathcal{R}$, $J_\mathcal{R}$, $\Delta_\mathcal{R}$ be the natural cone, modular conjugation, and modular operator of the pair $(\mathfrak{M}_\mathcal{R}, \Omega_\mathcal{R})$. As a consequence of Tomita-Takesaki theory (see Sections A.5 and A.6 of the Appendix), the standard Liouvillean of $(\mathcal{O}_\mathcal{R}, \tau_\mathcal{R}, \omega_\mathcal{R})$, i.e., the unique self-adjoint operator $L_\mathcal{R}$ on $\mathfrak{H}_\mathcal{R}$ such that

$$\pi_\mathcal{R}(\tau_\mathcal{R}^t(A)) = \mathrm{e}^{\mathrm{i}tL_\mathcal{R}}\pi_\mathcal{R}(A)\mathrm{e}^{-\mathrm{i}tL_\mathcal{R}}, \qquad \mathrm{e}^{-\mathrm{i}tL_\mathcal{R}}\mathcal{P}_\mathcal{R} = \mathcal{P}_\mathcal{R}, \qquad L_\mathcal{R}\Omega_\mathcal{R} = 0,$$

for all $t \in \mathbb{R}$ and $A \in \mathcal{O}_\mathcal{R}$, is related to the modular operator by

$$L_\mathcal{R} = -\frac{1}{\beta}\log\Delta_\mathcal{R}.$$

Set

$$\begin{aligned} \mathfrak{H} &= \mathfrak{H}_\mathcal{S} \otimes \mathfrak{H}_\mathcal{R}, & \pi &= \pi_\mathcal{S} \otimes \pi_\mathcal{R}, & \Omega_0 &= \Omega_\mathcal{S} \otimes \Omega_\mathcal{R}, \\ \mathfrak{M} &= \pi_\mathcal{S}(\mathcal{O}_\mathcal{S}) \otimes \mathfrak{M}_\mathcal{R}, & \mathcal{P} &= \mathcal{P}_\mathcal{S} \otimes \mathcal{P}_\mathcal{R}, & J &= J_\mathcal{S} \otimes J_\mathcal{R}, \\ & & L_0 &= L_\mathcal{S} + L_\mathcal{R}. & & \end{aligned}$$

The triple $(\mathfrak{H}, \pi, \Omega_0)$ is a GNS representation of $\mathcal{O}$ induced by ω_0. $\mathfrak{M} = \pi(\mathcal{O})''$ and the natural cone and modular conjugation of the pair $(\mathfrak{M}, \Omega_0)$ are $\mathcal{P}$ and J. The operator L_0 is the standard Liouvillean of $(\mathcal{O}, \tau_0, \omega_0)$. Moreover, for any $t \in \mathbb{R}$ and $A \in \mathcal{O}$, one has

$$\pi(\tau_\lambda^t(A)) = \mathrm{e}^{\mathrm{i}t(L_0+\lambda\pi(V))}\pi(A)\mathrm{e}^{-\mathrm{i}t(L_0+\lambda\pi(V))}.$$

Recall that a positive linear functional ζ on $\mathcal{O}$ belongs to $\mathcal{N}$ (i.e., is ω_0-normal) iff there exists a positive trace class operator ρ_ζ on $\mathfrak{H}$ such that, for all $A \in \mathcal{O}$,

$$\zeta(A) = \mathrm{tr}(\rho_\zeta \pi(A)).$$

Such a functional obviously extends to $\mathfrak{M}$, and we denote this extension by the same letter. As a consequence of modular theory (see Section A.3) there exists a unique vector $\Omega_\zeta \in \mathcal{P}$, called the standard vector representative of ζ, such that

$$\zeta(A) = \langle \Omega_\zeta | A \Omega_\zeta \rangle$$

for all $A \in \mathfrak{M}$. The standard vector representative of $\eta = \mathbb{1} \otimes \omega_\mathcal{R}$ is

$$\Omega_\eta = \mathbb{1} \otimes \Omega_\mathcal{R},$$

and the standard vector representative $\Omega = \Omega_\omega$ of the initial state $\omega = \rho_\mathcal{S} \otimes \omega_\mathcal{R}$ is

$$\Omega = \rho_\mathcal{S}^{1/2} \otimes \Omega_\mathcal{R} = \pi(\rho_\mathcal{S}^{1/2} \otimes \mathbb{1})\Omega_\eta. \tag{6.15}$$

For later reference, we recall some results of Araki's perturbation theory of the KMS structure (see [BR2, Chapter 5] and [DJP]). First

$$\Omega_0 \in \mathrm{Dom}\left(\mathrm{e}^{-\frac{\beta}{2}(L_0 + \lambda\pi(V))}\right), \tag{6.16}$$

and the vector

$$\Omega_\lambda = \mathrm{e}^{-\frac{\beta}{2}(L_0 + \lambda\pi(V))}\Omega_0 \tag{6.17}$$

is well-defined. Moreover, the vector-valued function

$$z \mapsto \mathrm{e}^{-z(L_0 + \lambda\pi(V))}\Omega_0 \in \mathfrak{H} \tag{6.18}$$

is analytic inside the strip $0 < \mathrm{Re}\, z < \beta/2$, and norm continuous and bounded on its closure. The map

$$\mathbb{R} \ni \lambda \mapsto \Omega_\lambda \in \mathfrak{H}$$

is real analytic, $\Omega_\lambda \in \mathcal{P}$, and, for $A \in \mathfrak{M}$,

$$\omega_\lambda(A) = \frac{\langle \Omega_\lambda | A \Omega_\lambda \rangle}{\|\Omega_\lambda\|^2}.$$

The standard Liouvillean of $(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ is given by

$$L_\lambda = L_0 + \lambda\pi(V) - \lambda J\pi(V)J,$$

and it satisfies

$$\pi(\tau_\lambda^t(A)) = \mathrm{e}^{\mathrm{i}tL_\lambda}\pi(A)\mathrm{e}^{-\mathrm{i}tL_\lambda}, \qquad \mathrm{e}^{-\mathrm{i}tL_\lambda}\mathcal{P} = \mathcal{P}, \qquad L_\lambda\Omega_\lambda = 0.$$

It is well-known [BR2, JP1, Pi] that the ergodic properties of $(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ can be characterized in terms of the spectral properties of L_λ.[5] More precisely, $(\mathcal{O}, \tau_\lambda, \omega_\lambda)$ is ergodic iff 0 is a simple eigenvalue of L_λ and mixing iff

$$\mathrm{w} - \lim_{|t|\to\infty} \mathrm{e}^{\mathrm{i}tL_\lambda} = \frac{|\Omega_\lambda\rangle\langle\Omega_\lambda|}{\|\Omega_\lambda\|^2}. \tag{6.19}$$

In particular, the last relation holds if the spectrum of L_λ on the orthogonal complement of $\mathbb{C}\Omega_\lambda$ is purely absolutely continuous.

Recall that $\eta = \mathbb{1} \otimes \omega_\mathcal{R}$ with $\omega_\mathcal{R}$ a $(\tau_\mathcal{R}, \beta)$-KMS state. The proof of Theorem 4.2 is centered around the function

$$\mathcal{F}_{\lambda,t}(\alpha) = \langle \Omega | \Delta^\alpha_{\eta\circ\tau_\lambda^{-t}|\eta} \Omega \rangle = \int_\mathbb{R} \mathrm{e}^{\alpha\beta\varsigma} \mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma).$$

[5] This aspect of modular theory is sometimes called Quantum Koopmanism.

By the properties of the weak convergence of measures (see Chapter 1 in [Bi]), Theorem 4.2 is equivalent to the following statements: for $0 < |\lambda| < \lambda_0$ and $\gamma \in \mathbb{R}$, the limit

$$\mathcal{F}_\lambda(\mathrm{i}\gamma/\beta) = \lim_{t\to\infty} \mathcal{F}_{\lambda,t}(\mathrm{i}\gamma/\beta) \tag{6.20}$$

exists and defines a continuous function $\mathbb{R} \ni \gamma \mapsto \mathcal{F}_\lambda(\mathrm{i}\gamma/\beta)$ such that, for $\gamma \in \mathbb{R}$,

$$\lim_{\lambda\to 0} \mathcal{F}_\lambda(\mathrm{i}\gamma/\beta) = \omega_{\mathcal{S}}\left(\mathrm{e}^{\mathrm{i}\gamma H_{\mathcal{S}}}\right) \rho_{\mathcal{S}}\left(\mathrm{e}^{-\mathrm{i}\gamma H_{\mathcal{S}}}\right). \tag{6.21}$$

By (4.11), the right-hand side in (6.21) is the characteristic function of $\mathbb{P}_{\mathcal{S}}$. The existence of the limit (6.20) and the continuity of $\gamma \mapsto \mathcal{F}_\lambda(\mathrm{i}\gamma/\beta)$ are equivalent to the statement that $\mathbb{P}_{\mathcal{R},\lambda,t}$ converges weakly as $t \to \infty$ to a probability measure $\mathbb{P}_{\mathcal{R},\lambda}$ whose characteristic function is $\mathcal{F}_\lambda(\mathrm{i}\gamma/\beta)$. The relation (6.21) is equivalent to the statement that $\mathbb{P}_{\mathcal{R},\lambda}$ converges weakly to $\mathbb{P}_{\mathcal{S}}$ as $\lambda \to 0$.

We finish this section by recalling some basic properties of the relative modular operator $\Delta_{\eta\circ\tau_\lambda^{-t}|\eta}$. First, since $\eta = \mathbb{1} \otimes \omega_{\mathcal{R}} \in \mathcal{N}$, the modular conjugation J_η and the modular operator Δ_η of the pair $(\mathfrak{M}, \Omega_\eta)$ are given by $J_\eta = J$, $\Delta_\eta = \mathrm{e}^{-\beta L_{\mathcal{R}}}$ (see Proposition A.26 and Eq. (A.40)), and

$$J^* J = J^2 = 1, \quad \Delta_\eta \Omega_\eta = \Omega_\eta, \quad J\Delta_\eta^{\mathrm{i}s} = \Delta_\eta^{\mathrm{i}s} J. \tag{6.22}$$

Moreover, by the result of [JP3, Eq. (2.13)],

$$\Delta_{\eta\circ\tau_\lambda^{-t}|\eta} = \mathrm{e}^{\mathrm{i}t(L_0+\lambda\pi(V))}\Delta_\eta \mathrm{e}^{-\mathrm{i}t(L_0+\lambda\pi(V))} = \Gamma_\lambda(t)\Delta_\eta\Gamma_\lambda^*(t), \tag{6.23}$$

where $\Gamma_\lambda(t)$ is the unitary

$$\Gamma_\lambda(t) = \mathrm{e}^{\mathrm{i}t(L_0+\lambda\pi(V))}\mathrm{e}^{-\mathrm{i}tL_0}. \tag{6.24}$$

One easily checks that $\Gamma_\lambda(t)$ satisfies the Cauchy problem

$$\partial_t\Gamma_\lambda(t) = \mathrm{i}\lambda\Gamma_\lambda(t)\pi(\tau_0^t(V)), \quad \Gamma_\lambda(0) = \mathbb{1}.$$

Hence, for any $B \in \mathfrak{M}'$,

$$\partial_t[B, \Gamma_\lambda(t)] = \mathrm{i}\lambda[B, \Gamma_\lambda(t)]\pi(\tau_0^t(V)), \quad [B, \Gamma_\lambda(0)] = 0,$$

and uniqueness of the solution of the Cauchy problem gives that

$$\Gamma_\lambda(t) \in \mathfrak{M}'' = \mathfrak{M} \tag{6.25}$$

for all $\lambda, t \in \mathbb{R}$.

6.2 Proof of Theorem 4.2

We start by establishing a few basic properties of the function

$$\mathcal{F}_{\lambda,t}(\alpha) = \langle\Omega|\Delta^\alpha_{\eta\circ\tau_\lambda^{-t}|\eta}\Omega\rangle = \|\Delta^{\alpha/2}_{\eta\circ\tau_\lambda^{-t}|\eta}\Omega\|^2 = \int_{\mathbb{R}} \mathrm{e}^{\alpha\beta\varsigma}\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma).$$

Lemma 6.1 *For any $\lambda \in \mathbb{R}$ and any $t \in \mathbb{R}$, one has*

$$\mathcal{F}_{\lambda,t}(0) = 1, \qquad 0 \le \mathcal{F}_{\lambda,t}(1) \le \dim \mathcal{H}_{\mathcal{S}}.$$

Proof. The relations $\mathcal{F}_{\lambda,t}(0) = 1$ and $\mathcal{F}_{\lambda,t}(1) \ge 0$ are obvious. By Eq. (6.15) and (6.23), one has

$$\mathcal{F}_{\lambda,t}(1) = \|\Delta^{\frac{1}{2}}_{\eta\circ\tau_\lambda^{-t}|\eta}\Omega\|^2 = \|\Delta_\eta^{\frac{1}{2}}\Gamma_\lambda^*(t)\pi(\rho_{\mathcal{S}}^{\frac{1}{2}} \otimes \mathbb{1})\Omega_\eta\|^2.$$

Using (6.25) and the anti-unitarity of the modular conjugation J we derive

$$\mathcal{F}_{\lambda,t}(1) = \|J\Delta_\eta^{\frac{1}{2}}\Gamma_\lambda^*(t)\pi(\rho_{\mathcal{S}}^{\frac{1}{2}}\otimes\mathbb{1})\Omega_\eta\|^2 = \|\pi(\rho_{\mathcal{S}}^{\frac{1}{2}}\otimes\mathbb{1})\Gamma_\lambda(t)\Omega_\eta\|^2 \le \|\Omega_\eta\|^2 = \dim\mathcal{H}_{\mathcal{S}}.$$

□

For $S\subset\mathbb{R}$, we denote

$$\mathfrak{S}(S) = \{z\in\mathbb{C}\,|\,\mathrm{Re}\,z\in S\}.$$

Lemma 6.2 *For any $\lambda\in\mathbb{R}$ and any $t\in\mathbb{R}$, the function $\alpha\mapsto\mathcal{F}_{\lambda,t}(\alpha)$ is analytic in the strip $\mathfrak{S}(]0,1[)$ and bounded and continuous on its closure. Moreover, the bound*

$$\sup_{t\in\mathbb{R}}|\mathcal{F}_{\lambda,t}(\alpha)| \le 1+(\dim\mathcal{H}_{\mathcal{S}}-1)\mathrm{Re}\,\alpha \tag{6.26}$$

holds for any $\alpha\in\mathfrak{S}([0,1])$.

Proof. For $\alpha\in[0,1]$, the convexity of the exponential function yields

$$\mathrm{e}^{\alpha\beta\varsigma}\le(1-\alpha)+\alpha\mathrm{e}^{\beta\varsigma}$$

so that, by the previous Lemma,

$$\mathcal{F}_{\lambda,t}(\alpha)\le(1-\alpha)\mathcal{F}_{\lambda,t}(0)+\alpha\mathcal{F}_{\lambda,t}(1)\le 1+(\dim\mathcal{H}_{\mathcal{S}}-1)\alpha.$$

The fact that $|\mathrm{e}^z|=\mathrm{e}^{\mathrm{Re}\,z}$ yields the rigidity of $\mathcal{F}_{\lambda,t}$, i.e.,

$$|\mathcal{F}_{\lambda,t}(\alpha)|\le\mathcal{F}_{\lambda,t}(\mathrm{Re}\,\alpha),$$

from which the bound (6.26) follows. Writing $\mathcal{F}_{\lambda,t}(\alpha)$ as the sum of the two Laplace transforms

$$\mathcal{F}_{\lambda,t}^{\pm}(\alpha)=\int_{\mathbb{R}^\pm}\mathrm{e}^{\alpha\beta\varsigma}\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma),$$

we deduce from the previous estimate that $\mathcal{F}_{\lambda,t}^+$ (resp. $\mathcal{F}_{\lambda,t}^-$) is analytic on the strip $\mathfrak{S}(]-\infty,1[)$ (resp. $\mathfrak{S}(]0,\infty[)$). Hence, $\mathcal{F}_{\lambda,t}$ is analytic on $\mathfrak{S}(]0,1[)$. For $\alpha,\alpha'\in\mathfrak{S}(]-\infty,1])$, one has

$$|\mathcal{F}_{\lambda,t}^+(\alpha')-\mathcal{F}_{\lambda,t}^+(\alpha)|\le\int_{\mathbb{R}^+}|\mathrm{e}^{\alpha'\beta\varsigma}-\mathrm{e}^{\alpha\beta\varsigma}|\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma),$$

and the simple estimate

$$|\mathrm{e}^{\alpha'\beta\varsigma}-\mathrm{e}^{\alpha\beta\varsigma}|\le 2\,\mathrm{e}^{\beta\varsigma}$$

allows us to apply the dominated convergence theorem and conclude that $\mathcal{F}_{\lambda,t}^+$ is continuous in $\mathfrak{S}(]-\infty,1])$. One shows in a similar way that $\mathcal{F}_{\lambda,t}^-$ is continuous in $\mathfrak{S}([0,\infty[)$. □

Lemma 6.3 *For any $\lambda\in\mathbb{R}$ and any $\delta\in]0,1[$, one has*

$$\sup_{t\in\mathbb{R},\alpha\in\mathfrak{S}(]0,\delta[)}|\partial_\alpha\mathcal{F}_{\lambda,t}(\alpha)|<\infty.$$

Proof. By Relation (4.13), one has

$$\Delta\mathcal{Q}_{\mathcal{R}}(\lambda,t)=-\int_{\mathbb{R}^-}|\varsigma|\,\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma)+\int_{\mathbb{R}^+}\varsigma\,\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma).$$

The inequality $\mathrm{e}^x\ge x$ and Lemma 6.1 further give

$$\int_{\mathbb{R}^+}\varsigma\,\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma)\le\beta^{-1}\int_{\mathbb{R}^+}\mathrm{e}^{\beta\varsigma}\mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma)\le\beta^{-1}\mathcal{F}_{\lambda,t}(1)\le\beta^{-1}\dim\mathcal{H}_{\mathcal{S}}.$$

It follows that

$$\int_{\mathbb{R}^-} |\varsigma|\, \mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma) \le \beta^{-1} \dim \mathcal{H}_{\mathcal{S}} - \Delta \mathcal{Q}_{\mathcal{R}}(\lambda,t).$$

By Lemma 6.2 and a well known property of Laplace transforms one has, for $\alpha \in \mathfrak{S}(]0,1[)$,

$$\partial_\alpha \mathcal{F}_{\lambda,t}(\alpha) = \beta \int_{\mathbb{R}} \varsigma \mathrm{e}^{\alpha\beta\varsigma} \mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma).$$

Thus, using the elementary estimate $x\mathrm{e}^{ax} \le (1-a)^{-1}\mathrm{e}^{x}$ valid for $a<1$ and $x \in \mathbb{R}$, we further get

$$\begin{aligned}
|\partial_\alpha \mathcal{F}_{\lambda,t}(\alpha)| &\le \beta \int_{\mathbb{R}^-} |\varsigma|\, \mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma) + \int_{\mathbb{R}^+} \beta\varsigma \mathrm{e}^{(\mathrm{Re}\,\alpha)\beta\varsigma} \mathrm{d}\mathbb{P}_{\mathcal{R},\lambda,t}(\varsigma)\\
&\le \dim \mathcal{H}_{\mathcal{S}} - \beta \Delta \mathcal{Q}_{\mathcal{R}}(\lambda,t) + (1-\mathrm{Re}\,\alpha)^{-1}\mathcal{F}_{\lambda,t}(1)\\
&\le (1+(1-\mathrm{Re}\,\alpha)^{-1}) \dim \mathcal{H}_{\mathcal{S}} - \beta \Delta \mathcal{Q}_{\mathcal{R}}(\lambda,t).
\end{aligned}$$

The result now follows from Eq. (3.5) and (3.7) which imply that $|\Delta \mathcal{Q}_{\mathcal{R}}(\lambda,t)| \le 2\|H_{\mathcal{S}} + \lambda V\|$. □

Next, we derive an alternative representation of the function $\mathcal{F}_{\lambda,t}$ on the line $\frac{1}{2} + \mathrm{i}\mathbb{R}$. To this end, we set

$$\widehat{L}_\lambda = L_{\mathcal{R}} + \pi(H_{\mathcal{S}} + \lambda V), \qquad \widehat{\Omega} = \pi(\rho_{\mathcal{S}}^{\frac{1}{2}} \otimes \mathbb{1})\Omega.$$

Lemma 6.4 *For any $\lambda, t, s \in \mathbb{R}$, one has*

$$\mathcal{F}_{\lambda,t}\left(\frac{1}{2} + \mathrm{i}s\right) = \langle \mathrm{e}^{\mathrm{i}\beta s \widehat{L}_\lambda}\widehat{\Omega} | \mathrm{e}^{\mathrm{i}tL_\lambda} \mathrm{e}^{\mathrm{i}\beta s \widehat{L}_\lambda} \Omega_\eta \rangle.$$

Proof. Set $R = \pi(\rho_{\mathcal{S}}^{\frac{1}{2}} \otimes \mathbb{1}) \in \mathfrak{M}$. Eq. (6.15), (6.23) and the relation $\Delta_\eta^{\frac{1}{2}} A\Omega_\eta = JA^*\Omega_\eta$ yield

$$\Delta_{\eta\circ\tau_\lambda^{-t}|\eta}^{\frac{1}{2}+\mathrm{i}s}\Omega = \Gamma_\lambda(t)\Delta_\eta^{\mathrm{i}s}\Delta_\eta^{\frac{1}{2}}\Gamma_\lambda^*(t)R\Omega_\eta = \Gamma_\lambda(t)\Delta_\eta^{\mathrm{i}s}JR\Gamma_\lambda(t)\Omega_\eta. \tag{6.27}$$

Using Eq. (6.22) and the fact that R commutes with $\Delta_\eta^{\mathrm{i}s} = \mathrm{e}^{-\mathrm{i}\beta s L_{\mathcal{R}}}$, we derive

$$\Gamma_\lambda(t)\Delta_\eta^{\mathrm{i}s}JR\Gamma_\lambda(t)\Omega_\eta = \Gamma_\lambda(t)JR\Delta_\eta^{\mathrm{i}s}\Gamma_\lambda(t)\Omega_\eta = \Gamma_\lambda(t)(JRJ)J\Delta_\eta^{\mathrm{i}s}\Gamma_\lambda(t)\Omega_\eta. \tag{6.28}$$

Since $JRJ \in \mathfrak{M}'$, relation (6.25) and the fact that $\Delta_\eta^{-\mathrm{i}s}J\Omega_\eta = \Omega_\eta$ yield

$$\Gamma_\lambda(t)(JRJ)J\Delta_\eta^{\mathrm{i}s}\Gamma_\lambda(t)\Omega_\eta = (JRJ)\Gamma_\lambda(t)J\Delta_\eta^{\mathrm{i}s}\Gamma_\lambda(t)J\Delta_\eta^{-\mathrm{i}s}J\Omega_\eta = (JRJ)\Gamma_\lambda(t)\widetilde{\Gamma}_\lambda(t)\Omega_\eta, \tag{6.29}$$

where

$$\widetilde{\Gamma}_\lambda(t) = J\Delta_\eta^{\mathrm{i}s}\Gamma_\lambda(t)\Delta_\eta^{-\mathrm{i}s}J. \tag{6.30}$$

Since $JRJ\Omega = \widehat{\Omega}$, it follows from (6.27)–(6.29) that

$$\mathcal{F}_{\lambda,t}\left(\frac{1}{2} + \mathrm{i}s\right) = \langle \widehat{\Omega} | \Gamma_\lambda(t)\widetilde{\Gamma}_\lambda(t)\Omega_\eta\rangle. \tag{6.31}$$

For $A = A^* \in \mathcal{B}(\mathfrak{H})$, set $A_t = \mathrm{e}^{\mathrm{i}tL_{\mathcal{R}}}A\mathrm{e}^{-\mathrm{i}tL_{\mathcal{R}}}$ and denote by U_A^t is the solution of the Cauchy problem

$$\partial_t U_A^t = \mathrm{i}U_A^t A_t, \quad U_A(0) = \mathbb{1}.$$

The following properties are easy consequences of the uniqueness of the solution to this problem.

(1) $U_A^t = \mathrm{e}^{\mathrm{i}t(L_{\mathcal{R}}+A)}\mathrm{e}^{-\mathrm{i}tL_{\mathcal{R}}}$.

(2) If $A \in \mathfrak{M}$, then $U_A^t \in \mathfrak{M}$.

(3) If $A, B \in \mathfrak{M}$, then $U_{A-JBJ}^t = U_A^t J U_B^t J$.

(4) $\Delta_\eta^{\mathrm{i}s} U_A^t \Delta_\eta^{-\mathrm{i}s} = U_{A_{-\mathrm{i}\beta s}}^t$

Set $M = \pi(H_\mathcal{S} + \lambda V)$. Since $L_0 = L_\mathcal{R} + \pi(H_\mathcal{S}) - J\pi(H_\mathcal{S})J$ and $L_0 + \lambda\pi(V) = L_\mathcal{R} + M - J\pi(H_\mathcal{S})J$, it follows from (6.24) and (6.30) that

$$\Gamma_\lambda(t) = U_M^t U_{\pi(H_\mathcal{S})}^{t*}, \qquad \widetilde{\Gamma}_\lambda(t) = (JU_{M_{-\mathrm{i}\beta s}}^t J)(JU_{\pi(H_\mathcal{S})}^t J)^*,$$

and hence

$$\Gamma_\lambda(t)\widetilde{\Gamma}_\lambda(t) = (U_M^t J U_{M_{-\mathrm{i}\beta s}}^t J)(U_{\pi(H_\mathcal{S})}^t J U_{\pi(H_\mathcal{S})}^t J)^* = U_M^t J U_{M_{-\mathrm{i}\beta s}}^t J\, \mathrm{e}^{\mathrm{i}tL_\mathcal{R}} \mathrm{e}^{-\mathrm{i}tL_0}.$$

From the fact that $\widehat{L}_\lambda = L_\mathcal{R} + M$ and $L_\lambda = L_\mathcal{R} + M - JMJ$, we deduce

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}tL_\lambda}\mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda} &= (U_M^t J U_M^t J \mathrm{e}^{\mathrm{i}tL_\mathcal{R}}) U_M^{\beta s} \mathrm{e}^{\mathrm{i}\beta s L_\mathcal{R}} \\
&= U_M^t (JU_M^t J)(\mathrm{e}^{\mathrm{i}tL_\mathcal{R}} U_M^{\beta s} \mathrm{e}^{-\mathrm{i}tL_\mathcal{R}}) \mathrm{e}^{\mathrm{i}tL_\mathcal{R}} \mathrm{e}^{\mathrm{i}\beta s L_\mathcal{R}} \\
&= U_M^t (\mathrm{e}^{\mathrm{i}tL_\mathcal{R}} U_M^{\beta s} \mathrm{e}^{-\mathrm{i}tL_\mathcal{R}}) \mathrm{e}^{\mathrm{i}\beta s L_\mathcal{R}} \mathrm{e}^{-\mathrm{i}\beta s L_\mathcal{R}} (JU_M^t J) \mathrm{e}^{\mathrm{i}\beta s L_\mathcal{R}} \mathrm{e}^{\mathrm{i}tL_\mathcal{R}} \\
&= \mathrm{e}^{\mathrm{i}t\widehat{L}_\lambda} \mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda} \mathrm{e}^{-\mathrm{i}tL_\mathcal{R}} (JU_{M_{-\mathrm{i}\beta s}}^t J) \mathrm{e}^{\mathrm{i}tL_\mathcal{R}} \\
&= \mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda} U_M^t J U_{M_{-\mathrm{i}\beta s}}^t J \mathrm{e}^{\mathrm{i}tL_\mathcal{R}} \\
&= \mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda} \Gamma_\lambda(t)\widetilde{\Gamma}_\lambda(t) \mathrm{e}^{\mathrm{i}tL_0}.
\end{aligned}$$

Since $\mathrm{e}^{\mathrm{i}tL_0}\Omega_\eta = \Omega_\eta$, the result follows from Eq. (6.31). □

We are now in position to investigate the behavior of $\mathcal{F}_{\lambda,t}$ in the limits $t \to \infty$ and $\lambda \to 0$.

Lemma 6.5 *Suppose that* $0 < |\lambda| < \lambda_0$. *Then the limit*

$$\mathcal{F}_\lambda(\alpha) = \lim_{t\to\infty} \mathcal{F}_{\lambda,t}(\alpha) \tag{6.32}$$

exists for $\alpha \in \mathfrak{S}([0,1[)$. *The function* $\alpha \mapsto \mathcal{F}_\lambda(\alpha)$ *is analytic on* $\mathfrak{S}(]0,1[)$ *and continuous on* $\mathfrak{S}([0,1[)$. *Moreover,*

$$\mathcal{F}_\lambda\left(\frac{1}{2} + \mathrm{i}s\right) = \frac{1}{\|\Omega_\lambda\|^2}\langle\widehat{\Omega}|\mathrm{e}^{-\mathrm{i}\beta s\widehat{L}_\lambda}\Omega_\lambda\rangle\langle\Omega_\lambda|\mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda}\Omega_\eta\rangle$$

holds for $s \in \mathbb{R}$.

Proof. The previous lemma and Eq. (6.19) yield that for $0 < |\lambda| < \lambda_0$ and $s \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathcal{F}_{\lambda,t}\left(\frac{1}{2} + \mathrm{i}s\right) = \frac{1}{\|\Omega_\lambda\|^2}\langle\widehat{\Omega}|\mathrm{e}^{-\mathrm{i}\beta s\widehat{L}_\lambda}\Omega_\lambda\rangle\langle\Omega_\lambda|\mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda}\Omega_\eta\rangle.$$

Lemma 6.2 and Vitali's convergence theorem imply that the limit (6.32) exists uniformly on compact subsets of $\mathfrak{S}(]0,1[)$, and that the limiting function $\alpha \mapsto \mathcal{F}_\lambda(\alpha)$ is analytic on $\mathfrak{S}(]0,1[)$.

Let $K \subset \mathfrak{S}([0,1[)$ be compact. Then there exist $\delta \in]0,1[$ and $k > 0$ such that $K \subset \widehat{K} = [0,\delta] + \mathrm{i}[-k,k]$. Set

$$C_\delta = \sup_{t\in\mathbb{R}, \alpha\in\mathfrak{S}(]0,\delta[)} |\partial_\alpha \mathcal{F}_{\lambda,t}(\alpha)|.$$

By Lemma 6.3, $C_\delta < \infty$. If $\epsilon > 0$ is small enough, then $r = \epsilon/12C_\delta < \delta$ and the uniform convergence on compacts in (6.32) ensures that there exists $T > 0$ such that $|\mathcal{F}_{\lambda,t}(\alpha) - \mathcal{F}_\lambda(\alpha)| < \epsilon/3$ for any $t > T$ and

$\alpha \in \widehat{K}_r = \{\alpha \in \widehat{K} \,|\, \mathrm{Re}\,\alpha \geq r\}$. Since for any $\alpha \in \widehat{K}$ there exists $\alpha' \in \widehat{K}_r$ such that $|\alpha - \alpha'| < 2r$ and therefore $|\mathcal{F}_{\lambda,t}(\alpha) - \mathcal{F}_{\lambda,t}(\alpha')| < \epsilon/6$, one has

$$|\mathcal{F}_{\lambda,t}(\alpha) - \mathcal{F}_{\lambda,s}(\alpha)| \leq |\mathcal{F}_{\lambda,t}(\alpha) - \mathcal{F}_{\lambda,t}(\alpha')| + |\mathcal{F}_{\lambda,s}(\alpha) - \mathcal{F}_{\lambda,s}(\alpha')| \\ + |\mathcal{F}_{\lambda,t}(\alpha') - \mathcal{F}_{\lambda}(\alpha')| + |\mathcal{F}_{\lambda,s}(\alpha') - \mathcal{F}_{\lambda}(\alpha')| < \varepsilon,$$

for any $\alpha \in \widehat{K}$ and $s, t > T$. It follows that $\mathcal{F}_{\lambda,t}$ converges uniformly in K as $t \to \infty$, the limiting function $\mathcal{F}_\lambda$ being continuous. □

Since the function $\mathbb{R} \ni \alpha \mapsto \mathcal{F}_\lambda(\mathrm{i}\alpha)$ is continuous, there exists a unique Borel probability measure $\mathbb{P}_{\mathcal{R},\lambda}$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\gamma\varsigma} \mathrm{d}\mathbb{P}_{\mathcal{R},\lambda}(\varsigma) = \mathcal{F}_\lambda(\mathrm{i}\gamma/\beta)$$

for any $\gamma \in \mathbb{R}$. An immediate consequence of the last lemma is:

Proposition 6.6 *For $0 < |\lambda| < \lambda_0$,*

$$\lim_{t\to\infty} \mathbb{P}_{\mathcal{R},\lambda,t} = \mathbb{P}_{\mathcal{R},\lambda}.$$

Lemma 6.7 *For $\gamma \in \mathbb{R}$,*

$$\mathcal{F}(\mathrm{i}\gamma/\beta) = \lim_{\lambda\to 0} \mathcal{F}_\lambda(\mathrm{i}\gamma/\beta) = \rho_{\mathcal{S}}\left(\mathrm{e}^{-\mathrm{i}\gamma H_{\mathcal{S}}}\right) \omega_{\mathcal{S}}\left(\mathrm{e}^{\mathrm{i}\gamma H_{\mathcal{S}}}\right).$$

Proof. For $s \in \mathbb{R}$, set

$$\mathcal{G}_\lambda^{(1)}(s) = \langle \widehat{\Omega} | \mathrm{e}^{-\mathrm{i}\beta s \widehat{L}_\lambda} \Omega_\lambda \rangle, \qquad \mathcal{G}_\lambda^{(2)}(s) = \langle \Omega_\lambda | \mathrm{e}^{\mathrm{i}\beta s \widehat{L}_\lambda} \Omega_\eta \rangle.$$

Writing $\widehat{L}_\lambda = L_0 + \lambda\pi(V) - J\pi(H_{\mathcal{S}})J$ and noticing that $L_0 + \lambda\pi(V) = L_{\mathcal{R}} + \pi(H_{\mathcal{S}} + \lambda V) - J\pi(H_{\mathcal{S}})J$ commutes with $J\pi(H_{\mathcal{S}})J$, we obtain

$$\mathcal{G}_\lambda^{(1)}(s) = \langle \widehat{\Omega} | \mathrm{e}^{-\mathrm{i}\beta s J\pi(H_{\mathcal{S}})J} \mathrm{e}^{-\mathrm{i}\beta s(L_0 + \lambda\pi(V))} \Omega_\lambda \rangle.$$

Araki's perturbation theory (recall (6.16)–(6.18)) implies that the function

$$\mathbb{R} \ni s \mapsto \mathcal{G}_\lambda^{(1)}(s) = \langle \widehat{\Omega} | \mathrm{e}^{-\mathrm{i}\beta s J\pi(H_{\mathcal{S}})J} \mathrm{e}^{-\mathrm{i}\beta(s - \mathrm{i}/2)(L_0 + \lambda\pi(V))} \Omega_0 \rangle,$$

has an analytic continuation to the strip $0 < \mathrm{Im}\, s < \frac{1}{2}$ which is bounded and continuous on its closure. Thus, for $\gamma \in \mathbb{R}$, one has

$$\mathcal{G}_\lambda^{(1)}\left(\frac{\gamma}{\beta} + \frac{1}{2}\mathrm{i}\right) = \langle \widehat{\Omega} | \mathrm{e}^{(\frac{\beta}{2} - \mathrm{i}\gamma)J\pi(H_{\mathcal{S}})J} \mathrm{e}^{-\mathrm{i}\gamma(L_0 + \lambda\pi(V))} \Omega_0 \rangle,$$

and it immediately follows that

$$\lim_{\lambda\to 0} \mathcal{G}_\lambda^{(1)}\left(\frac{\gamma}{\beta} + \frac{1}{2}\mathrm{i}\right) = \langle \widehat{\Omega} | J\pi(\mathrm{e}^{(\frac{\beta}{2} + \mathrm{i}\gamma)H_{\mathcal{S}}}) J\Omega_0 \rangle = Z^{-\frac{1}{2}} \rho_{\mathcal{S}}(\mathrm{e}^{-\mathrm{i}\gamma H_{\mathcal{S}}}), \tag{6.33}$$

where $Z = \mathrm{tr}(\mathrm{e}^{-\beta H_{\mathcal{S}}})$. Since $\Omega_\eta = Z^{\frac{1}{2}} J\pi(\mathrm{e}^{\frac{\beta}{2}H_{\mathcal{S}}})J\Omega_0$, one has, with the notation of the proof of Lemma 6.4,

$$\mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda}\Omega_\eta = U_M^{\beta s} \mathrm{e}^{\mathrm{i}\beta s L_{\mathcal{R}}} \Omega_\eta = U_M^{\beta s}\Omega_\eta = Z^{\frac{1}{2}} U_M^{\beta s} J\pi(\mathrm{e}^{\frac{\beta}{2}H_{\mathcal{S}}})J\Omega_0 = Z^{\frac{1}{2}} J\pi(\mathrm{e}^{\frac{\beta}{2}H_{\mathcal{S}}})JU_M^{\beta s}\Omega_0 \\ = Z^{\frac{1}{2}} J\pi(\mathrm{e}^{\frac{\beta}{2}H_{\mathcal{S}}})J\mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda}\mathrm{e}^{-\mathrm{i}\beta s L_{\mathcal{R}}}\Omega_0 = Z^{\frac{1}{2}} J\pi(\mathrm{e}^{\frac{\beta}{2}H_{\mathcal{S}}})J\mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda}\Omega_0,$$

and hence

$$\mathrm{e}^{\mathrm{i}\beta s\widehat{L}_\lambda}\Omega_\eta = Z^{\frac{1}{2}} J\pi(\mathrm{e}^{\frac{\beta}{2}H_{\mathcal{S}}})J\mathrm{e}^{\mathrm{i}\beta s J\pi(H_{\mathcal{S}})J}\mathrm{e}^{\mathrm{i}\beta s(L_0 + \lambda\pi(V))}\Omega_0 \\ = Z^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\beta(s - \mathrm{i}/2)J\pi(H_{\mathcal{S}})J}\mathrm{e}^{\mathrm{i}\beta s(L_0 + \lambda\pi(V))}\Omega_0.$$

Araki's perturbation theory implies that the function

$$\mathbb{R} \ni s \mapsto \mathcal{G}_\lambda^{(2)}(s) = Z^{\frac{1}{2}} \langle \Omega_\lambda | \mathrm{e}^{\mathrm{i}\beta(s-\mathrm{i}/2)J\pi(H_\mathcal{S})J} \mathrm{e}^{\mathrm{i}\beta s(L_0+\lambda\pi(V))}\Omega_0\rangle$$

also has an analytic continuation to the strip $0 < \mathrm{Im}\, s < \frac{1}{2}$ which is bounded and continuous on its closure. For $\gamma \in \mathbb{R}$, one gets

$$\mathcal{G}_\lambda^{(2)}\left(\frac{\gamma}{\beta} + \frac{1}{2}\,\mathrm{i}\right) = Z^{\frac{1}{2}} \langle \Omega_\lambda | J\pi(\mathrm{e}^{-\mathrm{i}\gamma H_\mathcal{S}})J\mathrm{e}^{\mathrm{i}\gamma(L_0+\lambda\pi(V))}\Omega_\lambda\rangle.$$

Since

$$\lim_{\lambda\to 0} \Omega_\lambda = \Omega_0, \tag{6.34}$$

we conclude that

$$\lim_{\lambda\to 0} \mathcal{G}_\lambda^{(2)}\left(\frac{\gamma}{\beta} + \frac{1}{2}\,\mathrm{i}\right) = Z^{\frac{1}{2}} \langle \Omega_0 | J\pi(\mathrm{e}^{-\mathrm{i}\gamma H_\mathcal{S}})J\Omega_0\rangle = Z^{\frac{1}{2}}\omega_\mathcal{S}(\mathrm{e}^{\mathrm{i}\gamma H_\mathcal{S}}). \tag{6.35}$$

Finally, recall that for $s \in \mathbb{R}$,

$$\mathcal{F}_\lambda\left(\frac{1}{2} + \mathrm{i}s\right) = \frac{1}{\|\Omega_\lambda\|^2}\mathcal{G}_\lambda^{(1)}(s)\mathcal{G}_\lambda^{(2)}(s).$$

Analytic continuation of this relation to

$$s = \frac{\gamma}{\beta} + \frac{1}{2}\,\mathrm{i}$$

combined with (6.33)–(6.35) gives the result. □

An immediate consequence of the last lemma (recall (6.21)) is

Proposition 6.8

$$\lim_{\lambda\to 0} \mathbb{P}_{\mathcal{R},\lambda} = \mathbb{P}_\mathcal{S}.$$

This completes the proof of Theorem 4.2.

A Summary of the algebraic framework

For the reader's convenience, we give here a simplified presentation of some well known constructions, restricting ourselves to the material we have been using throughout the paper. When possible, we give reference to precise parts of [BR1, BR2] where detailed proofs can be found.

We first recall some basic definitions.

Definition A.1 (1) A Banach algebra $\mathcal{A}$ equipped with an involution $*$ is called C^*-algebra if

$$\|A^*A\| = \|A\|^2$$

for all $A \in \mathcal{A}$. We always assume a C^*-algebra admits an identity denoted by $\mathbb{1}$.

(2) A linear functional ω on $\mathcal{A}$ is *positive* if $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$.

(3) A positive linear functional on $\mathcal{A}$ is a *state* if $\omega(\mathbb{1}) = 1$.

(4) A state is said to be *faithful* iff $\omega(A^*A) = 0$ implies $A = 0$.

Proposition A.2 *A positive linear functional on a C^*-algebra is automatically continuous.*

Let $\mathcal{A}, \mathcal{B}$ denote C^*-algebras.

Definition A.3 A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called $*$-*morphism* iff, for all $A, B \in \mathcal{A}$,

(1) $\phi(AB) = \phi(A)\phi(B)$,

(2) $\phi(A^*) = \phi(A)^*$.

If, furthermore, ϕ is bijective, it is called $*$-*isomorphism*. A $*$-isomorphism with $\mathcal{A} = \mathcal{B}$ is called a $*$-*automorphism*.

Proposition A.4 *If ϕ is a $*$-morphism then $\|\phi(A)\| \leq \|A\|$. In particular, ϕ is continuous.*

Definition A.5 Let $\mathcal{H}$ be a Hilbert space and $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$. A vector $\Omega \in \mathcal{H}$ is called *cyclic* for $\mathcal{C}$ iff $\mathcal{C}\Omega$ is dense in $\mathcal{H}$, and *separating* iff $C\Omega = C'\Omega$ for some $C, C' \in \mathcal{C}$ implies $C = C'$.

Definition A.6 A *representation* of a C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a complex Hilbert space and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a $*$-morphism. A *cyclic representation* of $\mathcal{A}$ is a triple $(\mathcal{H}, \pi, \Omega)$ such that $(\mathcal{H}, \pi)$ is a representation of $\mathcal{A}$ and the unit vector $\Omega \in \mathcal{H}$ is cyclic for $\pi(\mathcal{A})$.

A.1 Canonical cyclic representation

Given a C^*-algebra $\mathcal{A}$, it is always possible to find a cyclic representation (see [BR1, Thm 2.3.16]).

Theorem A.7 (Gelfand-Naimark-Segal) *Let ω be a state over the C^*-algebra $\mathcal{A}$. There exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathcal{A}$ such that, for all $A \in \mathcal{A}$,*

$$\omega(A) = \langle \Omega_\omega | \pi_\omega(A)\Omega_\omega \rangle.$$

Moreover, this representation is unique up to unitary equivalence.

Definition A.8 The triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ in the previous theorem is called *the canonical cyclic representation* or *the GNS representation* of $\mathcal{A}$ induced by ω.

The representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is constructed as follows. Equip the vector space $\mathcal{A}$ with the positive semi-definite sesquilinear form

$$\langle A|B \rangle = \omega(A^*B). \tag{A.36}$$

By the inequality $\omega(B^*A^*AB) \leq \|A\|^2\omega(B^*B)$, the set

$$\mathcal{I}_\omega = \{A \in \mathcal{A} \,|\, \omega(A^*A) = 0\}$$

is a closed (left) ideal and the quotient $\mathcal{A}/\mathcal{I}_\omega$ equipped with the inner product induced by (A.36) is a pre-Hilbert space, the completion of which is $\mathcal{H}_\omega$. For $A \in \mathcal{A}$, define π_ω by

$$\pi_\omega(A) : X + \mathcal{I}_\omega \mapsto AX + \mathcal{I}_\omega.$$

Clearly, the vector $\Omega_\omega = \mathbb{1} + \mathcal{I}_\omega$ is cyclic for $\pi_\omega(\mathcal{A})$ and satisfies

$$\omega(A) = \langle \Omega_\omega | \pi_\omega(A)\Omega_\omega \rangle$$

for all $A \in \mathcal{A}$.

A.2 von Neumann Algebras

Definition A.9 A C^*-algebra $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is called a *von Neumann algebra* if it is closed with respect to the strong topology of $\mathcal{B}(\mathcal{H})$.

If $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{C}'$ its commutant, i.e., the set

$$\mathcal{C}' = \{A \in \mathcal{B}(\mathcal{H}) \,|\, [A, C] = 0 \text{ for all } C \in \mathcal{C}\}.$$

Here we give some characterization of von Neumann algebras.

Theorem A.10 (von Neumann) *Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a C^*-algebra. The following conditions are equivalent:*

(1) $\mathfrak{M}$ *is a von Neumann algebra.*

(2) $\mathfrak{M} = \mathfrak{M}''$.

(3) $\mathfrak{M}$ *is closed with respect to the weak topology of* $\mathcal{B}(\mathcal{H})$.

A.3 Normal states

Definition A.11 (1) The *σ-weak topology* on $\mathcal{B}(\mathcal{H})$ is the locally convex topology induced by the semi-norms

$$A \mapsto \left| \sum_{n \in \mathbb{N}} \langle \Phi_n | A \Psi_n \rangle \right|,$$

where $\Phi_n, \Psi_n \in \mathcal{H}$ are such that $\sum_n \|\Phi_n\|^2$ and $\sum_n \|\Psi_n\|^2$ are finite.

(2) A linear functional on the von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is *normal* if it is σ-weakly continuous.

Several properties can be used to characterize normal states of a von Neumann algebra, see [BR1, Thm 2.4.21]. We recall the most useful one for this paper.

Theorem A.12 *A state ω on von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ is normal iff there exists a density matrix ρ_ω, i.e., a positive trace class operator on $\mathcal{H}$ with unit trace, such that $\omega(A) = \mathrm{tr}(\rho_\omega A)$ for all $A \in \mathfrak{M}$.*

Let ω be a state on the C^*-algebra $\mathcal{A}$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the induced GNS representation. The von Neumann algebra $\pi_\omega(\mathcal{A})'' \subset \mathcal{B}(\mathcal{H}_\omega)$ is called the *enveloping von Neumann algebra.* $\pi_\omega(\mathcal{A})$ is σ-weakly dense in $\pi_\omega(\mathcal{A})''$ and the state ω has a unique normal extension to $\pi_\omega(\mathcal{A})''$ given by

$$\widehat{\omega}(A) = \langle \Omega_\omega | A \Omega_\omega \rangle. \tag{A.37}$$

Definition A.13 A state η on $\mathcal{A}$ is ω-normal if it extends to a normal state on $\pi_\omega(\mathcal{A})''$, i.e., if there exists a density matrix ρ_η on $\mathcal{H}_\omega$ such that $\eta(A) = \mathrm{tr}(\rho_\eta \pi_\omega(A))$.

A.4 KMS states

Definition A.14 A C^* (resp. W^*) dynamical system is pair $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a C^*-algebra (resp. a von Neumann algebra) and τ is a strongly (resp. σ-weakly) continuous one-parameter group of $*$-automorphisms of $\mathcal{A}$. A state ω on $\mathcal{A}$ is τ-invariant if $\omega \circ \tau^t = \omega$ for all $t \in \mathbb{R}$.

Definition A.15 Let $(\mathcal{A}, \tau)$ be a C^* (resp. W^*) dynamical system. $A \in \mathcal{A}$ is called *analytic* for τ if there exists a function $f : \mathbb{C} \to \mathcal{A}$ such that

(1) $f(t) = \tau^t(A)$ for $t \in \mathbb{R}$,

(2) for any state (resp. any normal state) η on $\mathcal{A}$, the function $z \mapsto \eta(f(z))$ is entire analytic.

The set of analytic elements for τ is denoted by $\mathcal{A}_\tau$.

Theorem A.16 *The set $\mathcal{A}_\tau$ is a dense (resp. σ-weakly dense) $*$-subalgebra of $\mathcal{A}$.*

Definition A.17 Let $(\mathcal{A}, \tau)$ be a C^* (resp. W^*) dynamical system and $\beta \in \mathbb{R}$. A state (resp. a normal state) ω on $\mathcal{A}$ is said to be a (τ, β)-KMS state iff

$$\omega(A\tau^{\mathrm{i}\beta}(B)) = \omega(BA)$$

for all $A, B \in \mathcal{A}_\tau$.

A (τ, β)-KMS state describes a thermal equilibrium state at inverse temperature β. In particular, it is τ-invariant ([BR2, Prop. 5.3.3]). If $\mathcal{A}$ is finite dimensional and $\tau^t(A) = \mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}$ for some self-adjoint Hamiltonian $H \in \mathcal{A}$, then the state

$$\omega(A) = \frac{\mathrm{tr}(\mathrm{e}^{-\beta H}A)}{\mathrm{tr}(\mathrm{e}^{-\beta H})}$$

is the unique (τ, β)-KMS state.
We note also that if $\beta, \tilde{\beta} \in \mathbb{R}\backslash\{0\}$ and ω is (τ, β)-KMS, then it is also $(\tilde{\tau}, \tilde{\beta})$-KMS for the dynamics $\tilde{\tau}^t = \tau^{t\beta/\tilde{\beta}}$.

Proposition A.18 *Let $(\mathcal{A}, \tau)$ be a C^*-dynamical system and ω a (τ, β)-KMS state for some $\beta \in \mathbb{R}$. Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the induced GNS representation.*

(1) *The cyclic vector Ω_ω is separating for the enveloping von Neumann algebra $\pi_\omega(\mathcal{A})''$.*

(2) *The state $\widehat{\omega}$ (the normal extension (A.37) of ω to $\pi_\omega(\mathcal{A})''$) is faithful.*

(3) *If $\beta \neq 0$, there exists a unique W^*-dynamical system $(\widehat{\tau}, \pi_\omega(\mathcal{A})'')$ such that $\widehat{\tau}^t(\pi_\omega(A)) = \pi_\omega(\tau^t(A))$ for all $t \in \mathbb{R}$ and $A \in \mathcal{A}$.*

(4) *$\widehat{\omega}$ is $(\widehat{\tau}, \beta)$-KMS.*

A.5 Modular theory

Given a von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ and a vector $\Omega \in \mathcal{H}$ which is separating for $\mathfrak{M}$, the map

$$S(A\Omega) = A^*\Omega$$

defines an anti-linear operator on the subspace $\mathfrak{M}\Omega$. If Ω is cyclic for $\mathfrak{M}$, then this operator is densely defined.

Proposition A.19 [BR1, Prop. 2.5.9] *If Ω is cyclic and separating for $\mathfrak{M}$, then S has a closed extension $\overline{S}$ with dense domain* $\mathrm{Dom}\,(\overline{S}) \supset \mathfrak{M}\Omega$.

Since $\overline{S}$ is a closed operator it admits a unique polar decomposition given by

$$\overline{S} = J\Delta^{\frac{1}{2}},$$

where J is anti-unitary and $\Delta = \overline{S}^*\overline{S}$ is positive. By definition, one has

$$J\Delta^{\frac{1}{2}}A\Omega = A^*\Omega \tag{A.38}$$

for all $A \in \mathfrak{M}$. Let $\mathcal{P} \subset \mathcal{H}$ denote the closure of the set $\{AJAJ\Omega \,|\, A \in \mathfrak{M}\}$.

Definition A.20 Δ is the modular operator, J the modular conjugation, and $\mathcal{P}$ the natural cone of the pair $(\mathfrak{M}, \Omega)$.

Some basic properties that follow easily from the definition are $J^2 = 1$, $J\Omega = \Omega$, $\Delta\Omega = \Omega$, $J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J$ (see [BR1, Prop. 2.5.11]). Much deeper is the following:

Theorem A.21 (Tomita-Takesaki) *Let Δ and J be the modular operator and modular conjugation of $(\mathfrak{M}, \Omega)$. Then,*

$$J\mathfrak{M}J = \mathfrak{M}',$$

and for all $t \in \mathbb{R}$,

$$\Delta^{\mathrm{i}t}\mathfrak{M}\Delta^{-\mathrm{i}t} = \mathfrak{M},$$

$$\Delta^{\mathrm{i}t}\mathcal{P} = \mathcal{P}.$$

Corollary A.22 *The modular operator of $(\mathfrak{M}, \Omega)$ defines a W^*-dynamical system $(\mathfrak{M}, \sigma)$ given by*

$$\sigma^t(A) = \Delta^{\mathrm{i}t}A\Delta^{-\mathrm{i}t}.$$

Moreover, the state

$$A \mapsto \frac{\langle\Omega|A\Omega\rangle}{\|\Omega\|^2}$$

is KMS for σ at inverse temperature $\beta = -1$.

Definition A.23 σ is the *modular group* of the pair $(\mathfrak{M}, \Omega)$.

Theorem A.24 (Takesaki) *The modular group σ is the unique dynamics on $\mathfrak{M}$ for which the state ω is KMS at inverse temperature $\beta = -1$.*

Definition A.25 A state ω on the C^*-algebra $\mathcal{A}$ is called *modular* if the cyclic vector Ω_ω of the induced GNS representation is separating for the enveloping von Neumann algebra $\pi_\omega(\mathcal{A})''$. In this case, we denote by Δ_ω, J_ω, $\mathcal{P}_\omega$ and σ_ω the modular operator, the modular conjugation, the natural cone and the modular group of the pair $(\pi_\omega(\mathcal{A})'', \Omega_\omega)$.

Proposition A.26 [BR1, Proposition 2.5.30, Theorem 2.5.31] *Let ω be a modular state on the C^*-algebra $\mathcal{A}$. For any ω-normal state ν on $\mathcal{A}$ there exists a unique unit vector $\Omega_\nu \in \mathcal{P}_\omega$ such that $\nu(A) = \langle\Omega_\nu|\pi_\omega(A)\Omega_\nu\rangle$ for all $A \in \mathcal{A}$. Moreover, Ω_ν is separating for $\pi_\omega(\mathcal{A})''$ iff it is cyclic. In that case, the modular conjugation and the natural cone of the pair $(\pi_\omega(\mathcal{A})'', \Omega_\nu)$ satisfy $J_\nu = J_\omega$ and $\mathcal{P}_\nu = \mathcal{P}_\omega$.*

Definition A.27 Ω_ν is the *standard vector representative* of the state ν.

We now state an important consequence of Takesaki's theorem (see [BR2, Thm. 5.3.10]).

Proposition A.28 *Let ω be a modular state on the C^*-algebra $\mathcal{A}$ and $\beta \in \mathbb{R}\setminus\{0\}$. Then $\widehat{\tau}^t = \sigma_\omega^{-t/\beta}$ defines the unique W^*-dynamical system on $\pi_\omega(\mathcal{A})''$ such that the normal extension $\widehat{\omega}$ is $(\widehat{\tau}, \beta)$-KMS.*

By the above proposition, given a (τ, β)-KMS state ω, the relation

$$\pi_\omega(\tau^t(A)) = \sigma_\omega^{-t/\beta}(\pi_\omega(A)) \tag{A.39}$$

identifies τ with the modular dynamics σ_ω.

A.6 The standard Liouvillean

Concerning the implementation of C^*-dynamical systems by unitary groups in GNS representations, one has the following general result (see [BR1, Corollary 2.5.32] and [Pi, Theorem 4.43]).

Proposition A.29 *Let ω be a modular state on the C^*-algebra $\mathcal{A}$ and denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the induced GNS representation. For any strongly continuous one-parameter group τ of $*$-automorphisms of $\mathcal{A}$ there exists a unique self-adjoint operator L on $\mathcal{H}_\omega$ such that, for all $t \in \mathbb{R}$,*

(1)

$$\mathrm{e}^{\mathrm{i}tL}\mathcal{P}_\omega = \mathcal{P}_\omega.$$

(2)

$$\mathrm{e}^{\mathrm{i}tL}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL} = \pi_\omega(\tau^t(A))$$

for all $A \in \mathcal{A}$.

Definition A.30 The operator L is the *standard Liouvillean* of the triple $(\mathcal{A}, \tau, \omega)$.

The standard Liouvillean L also satisfies $[J_\omega, \mathrm{e}^{\mathrm{i}tL}] = 0$, from which one deduces

$$\mathrm{e}^{\mathrm{i}tL}\pi_\omega(\mathcal{A})'\mathrm{e}^{-\mathrm{i}tL} \subset \pi_\omega(\mathcal{A})'.$$

If ω is a (τ, β)-KMS state, the identification (A.39) of τ with the modular group σ_ω given by Takesaki's theorem A.24 translates as

$$L = -\frac{1}{\beta}\log\Delta_\omega. \tag{A.40}$$

A.7 Relative modular operator

Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\Psi, \Phi \in \mathcal{H}$. If Φ is separating for $\mathfrak{M}$, then $S_{\Psi|\Phi}(A\Phi) = A^*\Psi$ defines an anti-linear operator on the subspace $\mathfrak{M}\Phi$.

Proposition A.31 ([Ar1, Ar2]) *If Φ is cyclic and separating for $\mathfrak{M}$, then $S_{\Psi|\Phi}$ has a closed extension $\overline{S}_{\Psi|\Phi}$ with a dense domain $\mathrm{Dom}\,(\overline{S}_{\Psi|\Phi}) \supset \mathfrak{M}\Phi$. If J is the modular conjugation of the pair $(\mathfrak{M}, \Phi)$, then*

$$\overline{S}_{\Psi|\Phi} = J\Delta_{\Psi|\Phi}^{\frac{1}{2}},$$

where $\Delta_{\Psi|\Phi} = \overline{S}^*_{\Psi|\Phi}\overline{S}_{\Psi|\Phi}$ is positive.

Definition A.32 Let ω be a modular state on the C^*-algebra $\mathcal{A}$. For any ω-normal state ν on $\mathcal{A}$, we define the relative modular operator of ν w.r.t. ω by

$$\Delta_{\nu|\omega} = \Delta_{\Omega_\nu|\Omega_\omega},$$

where $\Omega_\omega, \Omega_\nu \in \mathcal{P}_\omega$ are the standard vector representatives of ω and ν.

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