TUBULAR NEIGHBORHOODS OF NODAL SETS AND DIOPHANTINE APPROXIMATION

DMITRY JAKOBSON AND DAN MANGOUBI

Abstract. We give upper and lower bounds on the volume of a tubular neighborhood of the nodal set of an eigenfunction of the Laplacian on a real analytic closed Riemannian manifold M. As an application we consider the question of approximating points on M by nodal sets, and explore analogy with approximation by rational numbers.

1. Introduction and Main Results

Let (M, g) be a real analytic closed Riemannian manifold. In the first part of this paper we give upper and lower bounds on the volume of tubular neighborhoods of nodal sets. Consider the eigenequation

$$\Delta\phi_\mu + \mu^2\phi_\mu = 0,$$

where Δ is the Laplace–Beltrami operator on M. We denote the nodal set $\{\phi_\mu = 0\}$ by $\mathcal{N}_\mu$. Consider the tubular neighborhood of the nodal set

$$T_{\mu,\delta} = \{x \in M \,:\, \mathrm{dist}(x, \mathcal{N}_\mu) < \delta\} \;. \tag{1.1}$$

We prove

Theorem 1.2. *Let (M, g) be a real analytic closed Riemannian manifold. Then there exist $C_1, C_2, C_3 > 0$ such that*

$$C_1\mu\delta \le \mathrm{Vol}(T_{\mu,\delta}) \le C_2\mu\delta,$$

whenever $\mu\delta \le C_3$.

To put Theorem 1.2 in the right context we recall

Theorem 1.3 ([DF88, Theorem 1.2])**.** *Let (M, g) be a closed real analytic Riemannian manifold. Then, there exist $C_4, C_5 > 0$ such that $C_4\mu \le \mathrm{Vol}_{n-1}(\mathcal{N}_\mu) \le C_5\mu$, where Vol_{n-1} is the $(n-1)$-dimensional Hausdorff measure on M.*

From this perspective, we see that Theorem 1.2 describes a regularity property of the nodal set. For example, the upper bound implies that the nodal set does not have too many needles or very narrow branches, while the lower bound says that the nodal set doesn't curve too much.

For the proof of Theorem 1.2 we need to study the behavior of eigenfunctions in all scales $0 < \delta \le 1/\mu$ ($1/\mu$ is called the wavelength). Roughly, we show that for most points x, $\phi_\mu(x)$ is comparable to the average of ϕ_μ on a ball of radius δ centered at x. This study is the content of Sections 2-6, and it extends the work of Donnelly

The first author was supported by NSERC, NATEQ and Dawson fellowship. The second author was supported by EPDI and CRM-ISM fellowship.

and Fefferman in Section 5 of [DF88], where they consider scales *comparable* to the wavelength $C_1/\mu \leq \delta \leq C_2/\mu$.

Donnelly and Fefferman showed that by considering an analytic continuation of ϕ_μ, one can treat our problem by studying polynomials in dimension one, and then applying an induction argument. We adopt this approach here. The key proposition is Proposition 4.4. Most of its proof goes without change from the proof of Proposition 5.11 in [DF88]. We had to adjust the arguments from [DF88] in two main points. The first is the proof in dimension one, where we added the parameter δ to the proofs in [DF88], and showed that everything goes through. The second is in the proof of Proposition 3.7 where the change of variables argument is different and more subtle than the parallel argument in [DF88].

The proof of the lower bound in Theorem 1.2 is given in Section 7. It is based on the behavior of eigenfunctions in scales *comparable* to the wavelength and on the Brunn-Minkowski inequality.

The idea of the proof of the upper bound in Theorem 1.2 was suggested to the authors by C. Fefferman. We give the proof in Section 8. A proof of the upper bound by different methods can be found in [Yom08]. Our proof is based on the upper bound in Theorem 1.3 and our study of eigenfunctions in all scales $\delta \leq C/\mu$.

In Section 10 we consider the special case where $\dim(M) = 2$. We show that the lower bound is true for any *smooth* surface and the upper bound is true for any smooth surface which satisfies Yau's conjecture.

In the second part of the paper we make an attempt to look simultaneously on the ensemble of nodal sets which belong to different eigenvalues. Consider first a simple example: Eigenfunctions on $M = [0, \pi]$ with the standard metric and (say) with Dirichlet boundary conditions. Then

$$\mu_k = k, \ \phi_k(x) = \sin(kx), \ \mathcal{N}_k = \left\{ \frac{\pi j}{k} : 0 \leq j \leq k \right\}.$$

Accordingly, the set $\mathcal{N}_k$ is $\pi/(2k)$-dense in M. Interestingly, a similar result holds on any smooth Riemannian manifold (see e.g. [Brü78]):

Proposition 1.4. *There exists $C > 0$ (which depends only on M, g) such that*

$$B(x, C/\mu) \cap \mathcal{N}(\phi_\mu) \neq \emptyset$$

for any $x \in M$ and $\mu > 0$.

Here $B(x, r)$ denotes the ball of radius r centered at $x \in M$. Thus $\mathcal{N}_\mu$ is C/μ-dense in M.

To study the rate of approximation by $\mathcal{N}_\mu$ as $\mu \to \infty$ in more detail, consider again the case of $M = [0, \pi]$ where approximating by points in $\mathcal{N}_k$ is equivalent (after rescaling by π) to approximating by rationals with denominator k. It is well-known (see e.g. [Khi97]) that the distance from any $x \in [0, 1]$ to the m-th convergent of its continued fraction expansion p_m/q_m is $O(1/q_m^2)$. However, the *denominator* q_m of the m-th continued fraction grows exponentially in m for $x \notin \mathbb{Q}$ ([Khi97]).

Denote by $||x||$ the distance from $x \in \mathbb{R}$ to the nearest integer. The following proposition can be found in [Khi97] and is proved by an application of the Borel-Cantelli Lemma.

Proposition 1.5. *If $\sum_q \psi(q)$ converges, then for Lebesgue-almost all x, there exist only finitely many q such that $||qx|| < \psi(q)$.*

Taking $\psi(q) = C/q^{1+\varepsilon}$ in Proposition 1.5 we conclude that

Corollary 1.6. *Given $C, \varepsilon > 0$, for Lebesgue-almost all $x \in [0,1]$ the inequality*

$$|x - p/q| < C/q^{2+\varepsilon}$$

has finitely many integer solutions (p, q).

Equivalently, almost all $x \in M = [0, \pi]$ *cannot* be approximated by points in $\mathcal{N}_k$ to within $C/k^{2+\varepsilon}$ infinitely often. We prove an analogous statement for any real analytic manifold M.

To characterize the *rate* of approximation by nodal sets, we make the following definition:

Definition 1.7. Given $b > 0$ *(exponent)*, and $C > 0$ *(constant)*, let $M(b, C)$ be the set of all $x \in M$ such that there exists an infinite sequence of eigenvalues $\mu_k \to \infty$ for which

$$B\left(x, \frac{C}{\mu_k^b}\right) \cap \mathcal{N}(\phi_{\mu_k}) \neq \emptyset.$$

For example, Proposition 1.4 implies that $M(1, C) = M$ for some $C > 0$. Also, Corollary 1.6 implies that for $M = [0, \pi]$, we have $\mathrm{Vol}(M(2 + \varepsilon, C)) = 0$ $\forall C, \varepsilon > 0$. We prove

Theorem 1.8. *Let (M, g) be a closed real analytic Riemannian manifold of dimension n. Then for any $C > 0, \varepsilon > 0$,*

$$\mathrm{Vol}(M(n + 1 + \varepsilon, C)) = 0.$$

The proof consists of Theorem 1.2, the Borel–Cantelli Lemma and Weyl's asymptotics of eigenvalues.

1.1. **A Reader's Guide.** In Sections 2-4 we study eigenfunctions in small scales. The key proposition is Proposition 4.4, which roughly shows that for most points x, $\phi_\mu(x)$ is comparable to the average of ϕ_μ on a ball of radius δ. On a first reading one may assume this proposition. In Section 5 we show how Proposition 4.4 implies geometric information on the nodal set and its neighborhood. Section 6 is a technical section which helps us treat the scales $\delta = C_1/\mu$ with C_1 large. The results of Sections 4, 5 and 6 are combined in Section 7 in order to prove the lower bound in Theorem 1.2. Section 8 gives the line of proof of the upper bound in Theorem 1.2. On a first reading one may start with this section and move to sections 4 and 5 when necessary. In Section 9 we combine the upper bound in Theorem 1.2 with Weyl's Law and the Borel-Cantelli Lemma in order to establish Theorem 1.8. In Section 10 we discuss Theorem 1.2 for smooth surfaces. In Section 11 we discuss possible extensions of the approximation result.

1.2. **Acknowledgments.** The idea of the proof of Theorem 1.2 was suggested to the authors by C. Fefferman.

The authors would like to thank an anonymous referee who helped to find a gap in an earlier version of this paper, and a second referee due to him the paper is in a much more readable form. The authors would like to thank M. Sodin for his motivating question about the lower bound on the volume of a nodal tube. The authors would also like to thank also E. Bogomolny, Y. Fyodorov, J. Marklof, I. Polterovich, Z. Rudnick, U. Smilansky, J. Toth, I. Wigman, Y. Yomdin and S. Zelditch for stimulating discussions about this problem.

The first author would also like to thank the organizers of the Workshop on Wavefunctions (Univ. of Bristol, September 2005), and the organizers of the Workshop on Dynamical Systems and Related Topics in honor of Y. Sinai (Univ. of Maryland, march 2006) for their hospitality. A large part of this paper was written while the first author visited Max Planck Institute for Mathematics in Bonn, Germany and the IHES, France; their hospitality is greatly appreciated.

The second author is an EPDI fellow at the IHES, France; Their hospitality and support is gratefully acknowledged. The second author would also like to thank McGill University and the CRM, Montreal where this work begun for their hospitality during his stay in Montreal.

2. Holomorphic Functions in Small Scales - dimension 1

In this section we describe the behavior of holomorphic functions of one variable in small scales. The proofs in this section follow closely the proofs in section 5 of [DF88].

We denote by $B_r \subset \mathbb{C}$ the disk $|z| \leq r$. Suppose F is holomorphic on B_3 and satisfies the following growth assumption:

$$\sup_{B_2} |F| \leq |F(0)| \mathrm{e}^{C_1 \mu} \ . \tag{2.1}$$

Let $I \subset \mathbb{R}$ denote the interval $[-1, 1]$. Let $0 < \delta < 1/\mu$ be given. We decompose I into disjoint subintervals of sizes $\delta < |I_\nu| < 2\delta$. Given $x \in I$, we denote by I_x the subinterval to which x belongs (I_x is defined outside a set of measure 0). We denote by $\mathrm{Av}_{I_x} F$ the average of F on I_x. The main proposition of this section is

Proposition 2.2. *Assume F satisfies (2.1). For all $\varepsilon > 0$ there exists a subset $E_\varepsilon \subseteq I$ of measure $|E_\varepsilon| \leq C_2 \varepsilon \mu \delta$ such that*

$$\frac{1}{C_3(\varepsilon)} \leq \frac{|F(x)|}{\mathrm{Av}_{I_x}|F|} \leq C_3(\varepsilon) \quad \forall x \in I \setminus E_\varepsilon \ ,$$

with $C_3(\varepsilon) = \mathrm{e}^{11/\varepsilon^2}$.

Proposition 2.2 generalizes Proposition 5.1 from [DF88]. The main new point here is the introduction of the parameter δ of the subdivision, while in [DF88] the size of the subdivision is taken to be comparable to $1/\mu$. A minor technical difference is that here we also allow subdivisions with non-fixed size of the subintervals. This will serve us in the change of variable argument in the proof of Proposition 3.7.

The first step we make is a reduction to polynomials. It is shown in Section 5 of [DF88]

Lemma 2.3 ([DF88, Lemma 5.2])**.** *F has at most $C_4 \mu$ zeroes in $B_{3/2}$.*

Denote the set of zeroes of F in B_r by $Z_r(F)$. Fix $r < 3/2$ close to $3/2$ so that F does not have zeroes on $|z| = r$. Let $P(z) := \prod_{\alpha \in Z_r(F)} (z - \alpha)$. P is a polynomial of degree $d \leq C_4 \mu$. Let $f(z) = \log |P(z)|$. The next lemma shows that we can assume $F(z) = P(z)$.

Lemma 2.4. (i)

$$|(\log |F(x)| - \log |F(y)|) - (f(x) - f(y))| \leq C_5 \quad \forall \nu \forall x, y \in I_\nu \ .$$

(ii)

$$C_6 \frac{|P(x)|}{\mathrm{Av}_{I_x}|P|} \leq \frac{|F(x)|}{\mathrm{Av}_{I_x}|F|} \leq C_7 \frac{|P(x)|}{\mathrm{Av}_{I_x}|P|} \ .$$

Proof. Let

$$B_r(z,\alpha) = \frac{(z-\alpha)/r}{1-\bar{\alpha}z/r^2}$$

be the Blaschke factor. We write

$$F(z) = \mathrm{e}^{g(z)} \prod_{\alpha \in Z_r(F)} B_r(z,\alpha) \ .$$

We calculate

$$\log|F(x)| - \log|F(y)| = \Re(g(x)-g(y)) + (\log|P(x)| - \log|P(y)|) - \sum_{\alpha \in Z_r(F)} (\log|r^2-\alpha x| - \log|r^2 - \alpha y|) \ . \tag{2.5}$$

The first term on the right hand side of (2.5) is handled by Lemma 5.3 (iii) of [DF88]:

$$|\Re(g(x)-g(y))| \leq \max_{I_\nu} |\nabla \Re(g)||x-y| \leq C_8 \mu\delta \ .$$

To bound the third term in the right hand side of (2.5) one should only check by direct computation that

$$\sup_{|x|\leq 1} |(\log|r^2-\alpha x|)'| = \sup_{|x|\leq 1} \left| \left(\Re \frac{-\alpha}{r^2-\alpha x} \right) \right| \leq \frac{1}{r-1} = C_9 \ .$$

The conclusion of part (i) of the Lemma follows.

Part (i) says that

$$\forall y \in I_x, \ \mathrm{e}^{-C_5} \frac{|P(y)|}{|P(x)|} \leq \frac{|F(y)|}{|F(x)|} \leq \mathrm{e}^{C_5} \frac{|P(y)|}{|P(x)|} \ .$$

It only remains to integrate over I_x in order to conclude part (ii). □

We now turn to bound $|f(x)-f(y)|$. For each ν we decompose f into a good part and a bad part. Let A_ν be the set of all roots α for which $\mathrm{dist}(\alpha, I_\nu) < \delta$.

$$g_\nu := \textstyle\sum_{\alpha \notin A_\nu} \log|x-\alpha|, \quad b_\nu := \sum_{\alpha \in A_\nu} \log|x-\alpha|.$$

We now define bad subsets $E_{j,\varepsilon}$:

$E_{1,\varepsilon} := \{x \in I : \ |f'(x)| > 1/(\varepsilon\delta)\}$, $\quad E_{2,\varepsilon} := \{x \in I : \ \exists \alpha, |x-\alpha| < \varepsilon\delta\}$,
$E_{3,\varepsilon} := \cup\{I_\nu : |A_\nu| > 1/\varepsilon\}$, $\quad E_{4,\varepsilon} := \cup\{I_\nu : \int_{I_\nu} |g''_\nu(x)|\, dx > 1/(\varepsilon\delta)\}$,
$E_{5,\varepsilon} := E_{1,\varepsilon} \cup E_{2,\varepsilon} \cup E_{3,\varepsilon} \cup E_{4,\varepsilon}$, $\quad E_{6,\varepsilon} := \cup\{I_\nu : |I_\nu \cap E_{5,\varepsilon}|/|I_\nu| > 1/2\}$,
$E_\varepsilon := E_{5,\varepsilon} \cup E_{6,\varepsilon}$.

Lemma 2.6. *Let $x \in I_\nu \setminus (E_{2,\varepsilon} \cup E_{3,\varepsilon})$. Then,*

$$\forall y \in I_\nu, \ b_\nu(x) - b_\nu(y) \geq -\frac{1}{\varepsilon} \log \frac{3}{\varepsilon} \ .$$

Proof. For all $y \in I_\nu$

$$b_\nu(y) = \sum_{\alpha \in A_\nu} \log|y-\alpha| \leq (\log 3\delta)/\varepsilon \ ,$$

while $b_\nu(x) \geq (\log(\varepsilon\delta))/\varepsilon$. □

Lemma 2.7. *Let $x \in I_\nu \setminus (E_{2,\varepsilon} \cup E_{3,\varepsilon})$. Then, $|b'_\nu(x)| \le 1/(\varepsilon^2\delta)$.*

Proof. Since $x \notin E_{2,\varepsilon} \cup E_{3,\varepsilon}$. $|b'_\nu(x)| \le \sum_{\alpha \in A_\nu} \frac{1}{|x-\alpha|} \le (1/\varepsilon) \cdot 1/(\varepsilon\delta)$. ☐

Lemma 2.8. *Let $x \in I_\nu \setminus (E_{1,\varepsilon} \cup E_{2,\varepsilon} \cup E_{3,\varepsilon})$. Then,*

$$|g'_\nu(x)| \le 2/(\varepsilon^2\delta) \ .$$

Proof. Since $x \notin E_{1,\varepsilon}$, $|f'(x)| \le 1/(\varepsilon\delta)$. By Lemma 2.7

$$|b'_\nu(x)| \le 1/(\varepsilon^2\delta) \ .$$

It only remains to observe that $|g'_\nu| \le |f'| + |b'_\nu|$. ☐

Lemma 2.9. *Suppose $I_\nu \not\subseteq E_{5,\varepsilon}$. Then $\max_{I_\nu} |g'_\nu| \le 3/(\varepsilon^2\delta)$.*

Proof. By Lemma 2.8 $\exists x_\nu \in I_\nu$ such that

$$|g'_\nu(x_\nu)| \le 2/(\varepsilon^2\delta) \ .$$

Also, from the definition of $E_{4,\varepsilon}$ and the fundamental theorem of calculus

$$\forall x \in I_\nu \ |g'_\nu(x) - g'_\nu(x_\nu)| \le 1/(\varepsilon\delta) \ .$$

Together we obtain $\forall x \in I_\nu \ |g'_\nu(x)| \le 3/(\varepsilon^2\delta)$. ☐

Lemma 2.10. *Suppose $I_\nu \not\subseteq E_{5,\varepsilon}$. Then*

$$\forall x, y \in I_\nu \ |g_\nu(x) - g_\nu(y)| \le 6/\varepsilon^2 \ .$$

Proof. The proof is an immediate corollary of Lemma 2.9. ☐

Lemma 2.11. *Let $x \in I_\nu \setminus E_{5,\varepsilon}$.*

$$\forall y \in I_\nu, \ f(x) - f(y) \ge -9/\varepsilon^2 \ .$$

Proof. $f(x) - f(y) = (g_\nu(x) - g_\nu(y)) + (b_\nu(x) - b_\nu(y))$. It only remains to combine Lemmas 2.6 and 2.10. ☐

Lemma 2.12. *Let $x \in I_\nu \setminus E_\varepsilon$. Then*

$$\mathrm{e}^{-9/\varepsilon^2}/4 \le \frac{\mathrm{e}^{f(x)}}{\mathrm{Av}_{I_\nu}\mathrm{e}^f} \le 4\mathrm{e}^{9/\varepsilon^2}$$

Proof. On the one hand, Lemma 2.11 gives

$$\frac{\mathrm{e}^{f(x)}}{(\int_{I_\nu} \mathrm{e}^f \ dx)/|I_\nu|} \le \frac{\mathrm{e}^{f(x)}}{(\int_{I_\nu \setminus E_\varepsilon} \mathrm{e}^f \ dx)/|I_\nu|} =$$
$$\frac{\mathrm{e}^{f(x)-\max_{I_\nu} f}}{(\int_{I_\nu \setminus E_\varepsilon} \mathrm{e}^{f(x)-\max_{I_\nu} f} \ dx)/|I_\nu|} \le \frac{\mathrm{e}^{f(x)-\max_{I_\nu} f}}{(|I_\nu \setminus E_\varepsilon|/|I_\nu|)\mathrm{e}^{-9/\varepsilon^2}/2} \le \frac{1}{\mathrm{e}^{-9/\varepsilon^2}/4} = 4\mathrm{e}^{9/\varepsilon^2} \ .$$

On the other hand, Lemma 2.11 also gives

$$\frac{\mathrm{e}^{f(x)}}{(\int_{I_\nu} \mathrm{e}^f \ dx)/|I_\nu|} \ge \frac{\mathrm{e}^{f(x)}}{\mathrm{e}^{\max_{I_\nu} f}} = \mathrm{e}^{f(x)-\max_{I_\nu} f} \ge \mathrm{e}^{-9/\varepsilon^2} \ .$$

☐

We now turn to estimating the size of the bad subset E_ε.

Lemma 2.13. $|E_{1,\varepsilon}| < C_9\varepsilon\mu\delta$

Proof. This follows from the properties of the Hilbert Transform. We imitate the proof of Lemma 5.4 in [DF88] with a little more details.

We recall the definition and some basic properties of the Hilbert Transform. Let $u \in L^2(\mathbb{R})$. Let sgn be the sign function on $\mathbb{R}$. Let $\mathcal{F}$ be the Fourier Transform on $L^2(\mathbb{R})$. Define the Hilbert Transform $\mathcal{H}u$ by

$$\mathcal{F}(\mathcal{H}u) = \frac{i}{2}\text{sgn} \cdot \mathcal{F}(u) \ .$$

From this definition it is clear that $\mathcal{H}$ is a bounded operator on $L^2(\mathbb{R})$. Observe that

$$f'(x) = \sum_{\alpha} \Re\left(\frac{1}{x-\alpha}\right) \ .$$

We may assume $\forall\alpha,\ \Im\alpha \leq 0$. Consider first the case where $\forall\alpha,\ \alpha \notin \mathbb{R}$. Let $q_\alpha(x) = -\Im(1/(x-\alpha))$, and $q = \sum_\alpha q_\alpha$. Then, $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and by Theorem 3 in III.2.3 of [Ste70] $\mathcal{H}q = f'$. From the fact that $\text{sgn}' = 2\delta_0$ and by basic properties of the Fourier Transform one sees that if $u \in L^2(\mathbb{R})$ has a compact support and $x \notin \text{Supp}\, u$, then

$$(\mathcal{H}u)(x) = \int_{\mathbb{R}} \frac{u(y)}{x-y}\, dy$$

(See also exc. 1.9 in [GS94] and Theorem 5 in III.3.3 of [Ste70]). We have verified that the conditions of Theorem 3 in I.5 of [Ste93] are fulfilled for the Hilbert Transform. We conclude that the Hilbert Transform is of weak type (1,1) and we get

$$|\{|f'| > 1/(\varepsilon\delta)\}| \leq C_{10}\varepsilon\delta\|q\|_1 \leq C_{11}\varepsilon\mu\delta \ . \tag{2.14}$$

Finally, we move to the case where $\exists\alpha \in \mathbb{R}$. Define $g_t(x) := f'(x - it)$. A small calculation shows that $g_t \to f'$ in measure as $t \to 0$. Since we can apply the considerations above to g_t we conclude that the assertion in the lemma is true with $C_9 = 2C_{11}$. □

Lemma 2.15.

$$|E_{2,\varepsilon}| \leq C_{12}\varepsilon\mu\delta \ .$$

Proof. Proof is obvious. □

Lemma 2.16.

$$|E_{3,\varepsilon}| \leq C_{13}\varepsilon\mu\delta \ .$$

Proof. This is an immediate corollary of Lemma 2.3. □

Lemma 2.17. *$|E_{4,\varepsilon}| \leq C_{14}\varepsilon\mu\delta$.*

Proof. We observe that

$$g''_\nu(x) = -\sum_{\alpha\notin A_\nu} \Re\left(\frac{1}{(x-\alpha)^2}\right) \ .$$

Hence,

$$\sum_\nu \int_{I_\nu} |g''_\nu(x)|\, dx \leq \sum_\nu \sum_{\alpha\notin A_\nu} \int_{I_\nu} \frac{1}{|x-\alpha|^2}\, dx =$$

$$\sum_\alpha \sum_{\nu,\alpha\notin A_\nu} \int_{I_\nu} \frac{1}{|x-\alpha|^2}\, dx \leq \sum_\alpha \int_{|x-\alpha|>\delta} \frac{1}{|x-\alpha|^2}\, dx \leq \mu/\delta.$$

On the other hand

$$\sum_\nu \int_{I_\nu} |g''_\nu(x)|\, dx \geq \sum_{\nu, I_\nu \subseteq E_{4,\varepsilon}} \int_{I_\nu} |g''_\nu(x)|\, dx \geq \#\{\nu : I_\nu \subseteq E_{4,\varepsilon}\} 1/(\delta\varepsilon)\ .$$

Together, we get that $\#\{\nu : I_\nu \subseteq E_{4,\varepsilon}\} \leq \varepsilon\mu$. Hence, $|E_{\varepsilon,4}| \leq C_{14}\varepsilon\mu\delta$. □

Lemma 2.18. $|E_{6,\varepsilon}| \leq C_{15}\varepsilon\mu\delta$

Proof. Let N be the number of intervals I_ν for which $|I_\nu \cap E_{5,\varepsilon}|/|I_\nu| > 1/2$. We have

$$C_{16}\varepsilon\mu\delta \leq |E_{5,\varepsilon}| = \sum_\nu |I_\nu \cap E_{5,\varepsilon}| \geq N|I_\nu|/2 \geq N\delta/2\ .$$

Hence, $N \leq 2C_{16}\varepsilon\mu$. It follows that $|E_{6,\varepsilon}| \leq 4C_{16}\varepsilon\mu\delta$. □

This completes the proof of Proposition 2.2.

3. Holomorphic Functions in Small Scales - dimension $n > 1$

In this section we prove an analog of Proposition 2.2 in dimension $n > 1$. We adjust the proof of Proposition 5.11 in [DF88].

Let F be a holomorphic function defined in the polydisk $B_3^n = B_3 \times \ldots \times B_3 \subseteq \mathbb{C}^n$. Let $I = [-1, 1] \subseteq \mathbb{R}$ and $Q = I^n$. Assume F satisfies

$$\sup_{B_2^n} |F| \leq |F(0)| \mathrm{e}^{C_1\mu}\ . \tag{3.1}$$

Given $0 < \delta < C_2/\mu$, decompose Q into subboxes Q_ν in the following way: First, we define n decompositions of I into intervals $\{I_l^{(k)}\}$ where $\delta < |I_l^{(k)}| < 2\delta$ $1 \leq k \leq n$ and $1 \leq l \leq N(k)$. Given a multi-index $\nu = (\nu_1, \ldots, \nu_n)$, $1 \leq \nu_k \leq N(k)$, we set $Q_\nu = I_{\nu_1}^{(1)} \times \ldots \times I_{\nu_n}^{(n)}$. Given $x \in Q$, we denote by Q_x the subbox which contains x. We prove

Proposition 3.2. *Let F satisfy (3.1) and $F \geq 0$ on Q. Assume that $F \equiv 1$ on each of the hyperplanes $z_i = 0$. For all $\varepsilon > 0$ there exists a subset $E_\varepsilon \subseteq Q$ of measure $|E_\varepsilon| \leq C_3\varepsilon\mu\delta$ such that*

$$\frac{1}{C_4(\varepsilon)} \leq \frac{F(x)}{\mathrm{Av}_{Q_x} F} \leq C_4(\varepsilon) \quad \forall x \in Q \setminus E_\varepsilon, \tag{3.3}$$

with $C_4(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Proof. For $n = 1$, the proposition reduces to Proposition 2.2. For $n > 1$, let E_ε be the subset of all $x \in Q$ for which the inequalities in (3.3) are not true with $C_4(\varepsilon) = \mathrm{e}^{11n/\varepsilon^2}$. Given $z' = (z_1, \ldots z_{n-1}) \in B_3^{n-1}$ we define

$$F_{z'}(z) = F(z', z)\ .$$

$F_{z'}$ has the following properties:

- $F_{z'}$ is defined in B_3.
- If $z' \in B_2^{n-1}$, then $\sup_{B_2} |F_{z'}| \leq \mathrm{e}^{C_1\mu}$.
- Let $Q' = I^{n-1}$. If $x' \in Q'$ then $F_{x'} \geq 0$ on the interval $[-1, 1]$.
- If $x' \in Q'$ then $F_{x'}(0) = 1$.

We have checked that for $x' \in Q'$, $F_{x'}$ satisfies the conditions in Proposition 2.2. Let $E_\varepsilon(F_{x'}) \subseteq I$ be the corresponding bad subset. We let $E_\varepsilon^{x'} := \{x'\} \times E_\varepsilon(F_{x'})$. Let $E_\varepsilon^{(n)} := \cup_{x' \in Q'} E_\varepsilon^{x'}$. $E_\varepsilon^{(n)}$ might not be measurable, but it intersects every line parallel to the x_n-axis in a measurable set.

Given $1 \le l \le N(n)$, $z' \in B_3^{n-1}$ define,

$$G_l(z') := \mathrm{Av}_{I_l^{(n)}} F_{z'} \ .$$

It is easy to check that

- G_l is defined in B_3^{n-1}.
- $\sup_{B_2^{n-1}} |G_l| \le \mathrm{e}^{C_1 \mu}$.
- $G_l(x') \ge 0$ for $x' \in Q'$.
- $G_l(z') = 1$ whenever one of the coordinates $z_i = 0$.

Thus, by the induction hypothesis applied to G_l and the decomposition $Q'_{\nu'}$ we get a corresponding bad subset $E_\varepsilon(G_l)$. We set $E_\varepsilon^l := E_\varepsilon(G_l) \times I_l^{(n)}$. Let $E'_\varepsilon := \cup_{1 \le l \le N(n)} E_\varepsilon^l$.

Claim 3.4. $E_\varepsilon \subseteq E'_\varepsilon \cup E_\varepsilon^{(n)}$.

Proof. Let $x \in Q_\nu \setminus (E'_\varepsilon \cup E_\varepsilon^{(n)})$. Since $x' \notin E_\varepsilon(G_{\nu_n})$ we have

$$\mathrm{e}^{-11(n-1)/\varepsilon^2} \le \frac{G_{\nu_n}(x')}{\mathrm{Av}_{Q'_{\nu'}} G_{\nu_n}} \le \mathrm{e}^{11(n-1)/\varepsilon^2} \ . \tag{3.5}$$

Since $x_n \notin E_\varepsilon(F_{x'})$ we have

$$\mathrm{e}^{-11/\varepsilon^2} \le \frac{F_{x'}(x_n)}{\mathrm{Av}_{I_{\nu_n}^{(n)}} F_{x'}} \le \mathrm{e}^{11/\varepsilon^2} \ . \tag{3.6}$$

Now recall that

$$F_{x'}(x_n) = F(x), \ \mathrm{Av}_{I_{\nu_n}^{(n)}} F_{x'} = G_{\nu_n}(x'),$$

and observe that $\mathrm{Av}_{Q'_{\nu'}} G_{\nu_n} = \mathrm{Av}_{Q_\nu} F$. To complete the proof of Claim 3.4 we multiply (3.5) by (3.6). ☐

It only remains to check that the size of E_ε is not too big: By Claim 3.4 we know that $E_\varepsilon \setminus E'_\varepsilon$ is a measurable set all of whose intersections with lines parallel to the x_n-axis are measurable sets of sizes $\le C_5 \varepsilon \mu \delta$. By Fubini's Theorem, we get $|E_\varepsilon \setminus E'_\varepsilon| \le C_6 \varepsilon \mu \delta$. $|E'_\varepsilon| \le \sum_l 2|E_\varepsilon(G_l)|\delta \le C_7 N(n) \varepsilon \delta^2 \mu$. But $N(n) \le 2/\delta$. This completes the proof of Proposition 3.2, since $|E_\varepsilon| \le |E_\varepsilon \setminus E'_\varepsilon| + |E'_\varepsilon|$. ☐

We now remove the technical assumption in proposition 3.2. The main proposition of this section is

Proposition 3.7. *Let F satisfy (3.1) and $F \ge 0$ on Q. There exists a cube $R \subseteq Q$ independent of F with the following property: Suppose $\mu\delta < C_8$. We decompose R into boxes R_ν of sides $\delta < l_\nu^{(k)} < 2\delta$. Then, there exists a subset $E_\varepsilon \subseteq R$ of measure $|E_\varepsilon| \le C_9 \varepsilon \mu \delta$ such that*

$$\frac{1}{C_{10}(\varepsilon)} \le \frac{F(x)}{\mathrm{Av}_{R_x} F} \le C_{10}(\varepsilon) \quad \forall x \in R \setminus E_\varepsilon, \tag{3.8}$$

with $C_{10}(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Proof. We construct R in the same way as in [DF88]:

Lemma 3.9 ([DF88, Lemma 5.10])**.** *There exists a map $W : \mathbb{R}^n \to \mathbb{R}^n$ which extends to a map $\hat{W} : \mathbb{C}^n \to \mathbb{C}^n$. with the following properties:*

1. *W is a polynomial map.*
2. *$\hat{W}(B_3^n) \subseteq B_3^n$.*
3. *$\hat{W}(B_2^n) \subseteq B_2^n$.*
4. *$W(Q) \subseteq Q$.*
5. *W maps the hyperplanes $x_i = 0$ to 0.*
6. *W is a local diffeomorphism outside the hyperplanes $x_i = 0$.*

Let $U \subseteq Q$ be an open set which is mapped diffeomorphically onto $W(U)$ and has a positive distance from any hyperplane $x_i = 0$. We let $R \subseteq W(U)$ be any cube with sides parallel to the sides of Q.

Let us now describe the bad subset E_ε. We begin with

Lemma 3.10. *There exists a finite number of subdivisions $\mathcal{D}_i$ of Q into boxes $Q_{\nu,i}$ of sides $\delta < |I_{l,i}^{(k)}| < 2\delta$ such that every set of diameter $< \delta/2$ is contained in a box $Q_{\nu,i}$ for some ν and i.*

The function $\tilde{F} = F \circ \hat{W}/F(0)$ satisfies the conditions of Proposition 3.2. So, given any of the subdivisions $\mathcal{D}_i$ of Lemma 3.10 we can find an exceptional set $\tilde{E}_{\varepsilon,i} \subseteq Q$ corresponding to $\tilde{F}$. Let $\tilde{E}_\varepsilon = \cup_i \tilde{E}_{\varepsilon,i}$. Set $E_\varepsilon^0 = W(\tilde{E}_\varepsilon \cap U) \cap R$.

Call R_α *ε-bad* if $|E_\varepsilon^0 \cap R_\alpha|/|R_\alpha| > 1/2$. Let B_ε be the union of all ε-bad R_α's. Finally, set $E_\varepsilon = E_\varepsilon^0 \cup B_\varepsilon$. We now estimate the size of E_ε.

Lemma 3.11. *$|E_\varepsilon| \leq C_{11}\varepsilon\mu\delta$.*

Proof of Lemma. Since the Jacobian of the map W is bounded on U, and $|\tilde{E}_\varepsilon| \leq C_{12}\varepsilon\mu\delta$ we conclude that $|E_\varepsilon^0| \leq C_{13}\varepsilon\mu\delta$. We estimate $|B_\varepsilon|$:

$$C_{13}\varepsilon\mu\delta \geq |E_\varepsilon^0| \geq \sum_{\text{bad } R_\alpha\text{'s}} |E_\varepsilon^0 \cap R_\alpha| \geq (1/2)\#(\text{bad } R_\alpha\text{'s})|R_\alpha| \geq C_{14}|B_\varepsilon| \ .$$

We got $|E_\varepsilon| \leq |E_\varepsilon^0| + |B_\varepsilon| \leq C_{11}\varepsilon\mu\delta$. ☐

The last step is to check that (3.8) is true: Let R_α be a subbox of R with sides $C_{15}\delta \leq l_\alpha^{(k)} \leq 2C_{15}\delta$, where C_{15} is small enough. Look at $\tilde{R}_\alpha = W^{-1}(R_\alpha)$. Since W^{-1} has a bounded Jacobian on R, $\tilde{R}_\alpha$ is a set of diameter $< \delta/2$. Let $\mathcal{D}$ be one of the subdivisions of Q from Lemma 3.10 whose one of its boxes Q_ν contains $\tilde{R}_\alpha$.

It follows from Proposition 3.2 that $\tilde{F}(y_1)/\tilde{F}(y_2) \leq C_4(\varepsilon)^2 \ \forall y_1, y_2 \in Q_\nu \setminus \tilde{E}_\varepsilon$. Hence, if we let $x_0 \in R_\alpha \setminus E_\varepsilon$ and $y_0 = W^{-1}(x_0)$, then $y_0 \in \tilde{R}_\alpha \setminus \tilde{E}_\varepsilon$ and we obtain

$$\text{(3.12)} \quad \text{Av}_{R_\alpha} F = \frac{1}{|R_\alpha|}\int_{R_\alpha} F(x)\, dx \geq \frac{1}{|R_\alpha|}\int_{R_\alpha \setminus E_\varepsilon} F(x)\, dx = \frac{1}{|R_\alpha|}\int_{\tilde{R}_\alpha \setminus \tilde{E}_\varepsilon} \tilde{F}(y)|J_W|\, dy \geq \frac{1}{C_4(\varepsilon)^2|R_\alpha|}\int_{\tilde{R}_\alpha \setminus \tilde{E}_\varepsilon} \tilde{F}(y_0)|J_W|\, dy = \frac{|R_\alpha \setminus E_\varepsilon|}{C_4(\varepsilon)^2|R_\alpha|}F(x_0) \geq F(x_0)/(2C_4(\varepsilon)^2).$$

On the other hand,

$$\text{Av}_{R_\alpha} F = \frac{1}{|R_\alpha|}\int_{R_\alpha} F(x)\,dx = \frac{1}{|R_\alpha|}\int_{\tilde{R}_\alpha} \tilde{F}(y)|J_W|\,dy \le \frac{1}{|R_\alpha|}\int_{Q_\nu} \tilde{F}(y)|J_W|\,dy \le \frac{C_{16}|Q_\nu|}{|R_\alpha|}\frac{1}{|Q_\nu|}\int_{Q_\nu} \tilde{F}(y)\,dy \le C_{17}\text{Av}_{Q_\nu}\tilde{F} \le C_{17}C_4(\varepsilon)\tilde{F}(y_0) = C_{18}(\varepsilon)F(x_0). \tag{3.13}$$

Inequalities (3.12) and (3.13) complete the proof of Proposition 3.7. □

4. Eigenfunctions in Small Scales on Real Analytic Manifolds

Let ϕ_μ be an eigenfunction. Let V be a small open set in which the metric g can be developed in power series. We identify V with a ball $B(0,\rho_0) \subseteq \mathbb{R}^n$. We prove

Proposition 4.1. *There exists a cube $R \subseteq V$ with the following property: Suppose $\mu\delta < C_1$. We subdivide R into boxes R_ν of sides $\delta < l_\nu^{(k)} < 2\delta$. Then for all $\varepsilon > 0$ there exists a subset $E_\varepsilon \subseteq R$ of measure $|E_\varepsilon| \le C_2\varepsilon\mu\delta$ such that*

$$\frac{1}{C_3(\varepsilon)} \le \frac{\phi_\mu(x)^2}{\text{Av}_{R_x}\phi_\mu^2} \le C_3(\varepsilon) \quad \forall x \in R \setminus E_\varepsilon,$$

with $C_3(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Proof. We consider an analytic continuation of ϕ_μ. In order to avoid confusion we denote by $B^{\mathbb{R}}$, $B^{\mathbb{C}}$ balls in $\mathbb{R}^n, \mathbb{C}^n$ respectively.

Lemma 4.2 ([DF88, Lemma 7.3])**.** *$\phi_\mu|_{B^{\mathbb{R}}(0,\rho_0)}$ has an analytic continuation F defined on $B^{\mathbb{C}}(0,\rho_1)$ for some $\rho_1 < \rho_0$. Moreover, the function F satisfies*

$$\sup_{B^{\mathbb{C}}(0,\rho_1)} |F| \le e^{C_4\mu} \sup_{B^{\mathbb{R}}(0,\rho_0)} |\phi_\mu| \ .$$

The crucial point is that the domain to which the function ϕ_μ can be continued is independent of μ.

Let $\rho_2 = \rho_1/C_5$ with C_5 large so that the polydisk $B^n_{2\rho_2} \subseteq B^{\mathbb{C}}(0,\rho_1)$. We now recall the Donnelly-Fefferman Growth Bound

Theorem 4.3.

$$\sup_{B^{\mathbb{R}}(0,\rho_0)} |\phi_\mu| \le e^{C_6(\rho_0/\rho_2)\mu} \sup_{B(0,\rho_2)} \phi_\mu \ .$$

Lemma 4.2 and Theorem 4.3 give

$$\sup_{B^n_{2\rho_2}} |F| \le e^{C_7\mu} \sup_{B(0,\rho_2)} \phi_\mu \ .$$

Now, shift the coordinate system to be centered on the point $x \in B(0,\rho_2)$ for which $\phi_\mu(x) = \sup_{B(0,\rho_2)} \phi_\mu$. We get that

$$\sup B^n_{\rho_2}|F| \le e^{C_7\mu}|F(0)| \ .$$

Hence, we can conclude the proof by applying Proposition 3.7 to F^2. □

We need a slightly different version of this proposition. We say that R_α touches R_β if they have at least one vertex in common. Each box R_α touches at most 3^n boxes. Let us denote by R_α^* the union of the box R_α with all boxes which touch R_α. There exist 3^n subdivisions $\mathcal{D}_i$ of R such that each box of $\mathcal{D}_i$ is equal to R_α^* for some α. Let $E_\varepsilon = \cup_i E_{\varepsilon,i}$ where $E_{\varepsilon,i}$ is the bad subset corresponding to the subdivision $\mathcal{D}_i$ according to Proposition 4.1. $|E_\varepsilon| \leq C_8\varepsilon\mu\delta$. These considerations prove the following version of Proposition 4.1.

Proposition 4.4. *There exists a cube $R \subseteq V$ with the following property: Suppose $\mu\delta < C_9$. We subdivide R into boxes R_ν of sides $\delta < l_\nu^{(k)} < 2\delta$. There exists a subset $E_\varepsilon \subseteq R$ of measure $|E_\varepsilon| \leq C_8\varepsilon\mu\delta$ such that*

$$\frac{1}{C_{10}(\varepsilon)} \leq \frac{\phi_\mu(x)^2}{\mathrm{Av}_{R_x^*}\phi_\mu^2} \leq C_{10}(\varepsilon) \quad \forall x \in R \setminus E,$$

with $C_{10}(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

5. Good Boxes - Bad Boxes

Let F be a nonnegative function defined on a cube R. Let $\mathcal{D}$ be a subdivision of R. We divide the boxes R_ν into *good* and *bad*. We show that in a vicinity of a good box we have a bounded L^2-growth, and that the geometry is under control. We show that the proportion of bad boxes is small.

We always assume that the sides of all boxes of a subdivision are of comparable sizes. Moreover, we assume that any two boxes R_{ν^1}, R_{ν^2} satisfy

$$\frac{\max \mathrm{side}(R_{\nu^1})}{\min \mathrm{side}(R_{\nu^2})} \leq 5 \ .$$

We recall that R_ν^* denotes the union of R_ν with its $3^n - 1$ neighbors.

Definition 5.1. Let $E \subseteq R$ be such that

$$\frac{1}{A} \leq \frac{F(x)}{\mathrm{Av}_{R_\nu^*}F} \leq A \quad \forall\nu\ \forall x \in R_\nu \setminus E \ .$$

We say that $(F, \mathcal{D}, E, A)$ *is true*.

Definition 5.2. Given $E \subseteq R$, we say that R_ν is *E-good* if $|E \cap R_\nu|/|R_\nu| < \omega_n 10^{-2n}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Otherwise R_ν is called *E-bad*.

The next lemma shows that in the vicinity of any good box we have bounded growth. $2Q$ denotes a box concentric with Q, whose sides are parallel to the sides of Q and twice as large.

Lemma 5.3. *Suppose that $(F, \mathcal{D}, E, A)$ is true. Let R_ν be E-good. Let $B \subseteq R_\nu$ be a ball such that $2B \subseteq R_\nu^*$ and whose radius $r \geq \mathrm{side}(R_\nu)/20$. Then,*

$$\int_B F\ dx \geq C_1 A^{-1} \int_{2B} F\ dx$$

Proof.

$$\int_B F(x)\,dx \geq \int_{B\setminus E} F(x)\,dx \geq A^{-1}\int_{B\setminus E} \mathrm{Av}_{R_\nu^*}F\,dx =$$
$$A^{-1}\frac{|B\setminus E|}{|R_\nu^*|}\int_{R_\nu^*} F\,dx \geq A^{-1}\frac{|B|-|E\cap R_\nu|}{|R_\nu^*|}\int_{2B} F\,dx \geq$$
$$\omega_n(20^{-n}-10^{-2n})A^{-1}\frac{|R_\nu|}{|R_\nu^*|}\int_{2B} F\,dx \geq C_1 A^{-1}\int_{2B} F\,dx\ .$$

□

The next proposition shows that the geometry in good cubes is controlled.

Proposition 5.4. *Suppose* $(\phi_\mu^2, \mathcal{D}, E, A)$ *is true. Let* R_ν *be* E*-good. Let* $B = B(o, 2r) \subseteq R_\nu$ *be a ball of radius* $2r$*, with* $r > \mathrm{side}(R_\nu)/20$*. Let* $B^+ = B\cap\{\phi_\mu > 0\}$*. Similarly, define* B^-*. Suppose* $\phi_\mu(o) = 0$*. Then*

$$\frac{1}{C_2(\log A)^{n-1}} \leq \frac{\mathrm{Vol}(B^+)}{\mathrm{Vol}(B^-)} \leq C_2(\log A)^{n-1}\ .$$

Proof. After rescaling the ball B to the unit ball, ϕ_μ becomes a solution φ of an elliptic equation

$$-\partial_i(a^{ij}\partial_j\varphi) - (2r\mu)^2 q\varphi = 0\ . \tag{5.5}$$

It's important to observe that the coefficients are bounded independently of μ, and the zero order coefficient is small. Thus, by Lemma 5.3 and by elliptic regularity

$$\sup_{B_{3/4}} |\varphi| \leq C_3\|\varphi\|_{L^2(B_1)} \leq C_4 A^{1/2}\|\varphi\|_{L^2(B_{1/2})} \leq C_5 A^{1/2}\sup_{B_{1/2}}|\varphi|\ . \tag{5.6}$$

Recall now

Theorem 5.7 ([Man08, Theorem 4.7])**.** *Let* φ *satisfy equation (5.5) in the unit ball* B_1*. Suppose* $\varphi(0) = 0$ *and satisfies (5.6). Then*

$$\frac{\mathrm{Vol}(B_1\cap\{\varphi > 0\})}{\mathrm{Vol}(B_1)} \geq \frac{C_6}{(\log A)^{n-1}}\ . \tag{5.8}$$

By symmetry, we have a lower bound also on $\mathrm{Vol}(B_1\cap\{\varphi<0\})/\mathrm{Vol}(B_1)$. Thus, we get upper and lower bounds on the ratio between the volumes of the positivity and the negativity sets of φ. □

The last lemma in this section shows that the proportion of bad cubes is small.

Lemma 5.9.

$$\frac{\#(E\text{-}bad\ boxes)}{\#(all\ boxes)} \leq C_7|E|\ .$$

Proof.

$$|E| \geq \sum_{\text{bad } R_\nu\text{'s}} |E\cap R_\nu| \geq \omega_n 10^{-n}\#(\text{bad } R_\nu\text{'s})|R_\nu| \geq C_8\frac{\#(\text{bad } R_\nu\text{'s})}{\#(\text{all boxes})}\ .$$

□

6. Refinements of subdivisions

In this section we analyze what happens when we pass from a fine subdivision $\mathcal{D}_1$ to a subdivision $\mathcal{D}$ of whose $\mathcal{D}_1$ is a refinement. We show, roughly, that if a box of $\mathcal{D}$ is composed of smaller good boxes then it is also good. The results in this section are applied in the the proof of the lower bound in Theorem 1.2.

Let $\mathcal{D}^1$ be a subdivision of a cube R obtained by a refinement of $\mathcal{D}$. If the sides of every box in $\mathcal{D}$ are partitioned into $\leq M$ intervals, we write $[\mathcal{D} : \mathcal{D}^1] \leq M$. Let $\mathcal{D}^2$ be the subdivision of R which is formed by taking the centers of the boxes in $\mathcal{D}^1$.

Throughout this section F is a nonnegative function defined on R. In the next proposition we use the terminology from Definitions 5.1 and 5.2.

Proposition 6.1. *Suppose that $(F, \mathcal{D}^1, E^1, A)$ and $(F, \mathcal{D}^2, E^2, A)$ are true. Let B be the union of all boxes R_ν of $\mathcal{D}$ for which R_ν^* contains an E^1-bad box of $\mathcal{D}^1$ or an E^2-bad box of $\mathcal{D}^2$. Assume $[\mathcal{D} : \mathcal{D}^1] \leq M$. Then, $(F, \mathcal{D}, E^1 \cup E^2 \cup B, C_1 A^{8M+1})$ is true.*

Proof. Let us denote the boxes of $\mathcal{D}$ by R_ν, and the boxes of $\mathcal{D}^1$ by R_α. Let $E = E^1 \cup E^2 \cup B$.

Lemma 6.2. *For all $x \in R_\nu \setminus E$, $y \in R_\nu^* \setminus E$ there exists a sequence of $4M$ points $x = x_1, \ldots, x_{4M} = y$ in $R_\nu^* \setminus E$ such that any consecutive pair x_k, x_{k+1} is contained in a box of $\mathcal{D}^1$ or is contained in a box of $\mathcal{D}^2$.*

Proof of Lemma. The only point to observe is that if $R_\nu \not\subseteq E$ then all boxes R_α contained in R_ν^* are E^1-good. So $|E^1 \cap R_\alpha|/|R_\alpha| < \omega_n 10^{-2n} < 2^{-n}/2$. Similarly for E^2. □

Let $x \in R_\nu \setminus E$. For any $y \in R_\nu^* \setminus E$, let $x_1, \ldots, x_{4M}$ be a sequence of points as in Lemma 6.2. Since $(F, \mathcal{D}^1, E^1, A)$ and $(F, \mathcal{D}^2, E^2, A)$ are true we have $F(x_k)/F(x_{k+1}) \leq A^2$, and we get $F(x)/F(y) \leq A^{8M}$.

Now,

$$\frac{1}{|R_\nu^*|}\int_{R_\nu^*} F(y)\,dy \geq \frac{1}{|R_\nu^*|}\int_{R_\nu^*\setminus E} F(y)\,dy \geq \frac{|R_\nu^* \setminus E|}{|R_\nu^*|}\frac{F(x)}{A^{8M}} \geq 3^{-n}(1 - 2\omega_n 10^{-2n})\frac{F(x)}{A^{8M}} \ .$$

The last inequality is true since R_ν contains no E^1-bad boxes neither E^2-bad boxes.

Conversely, let $x \in R_\nu$ and let J be the set of α's for which $R_\alpha \subseteq R_\nu^*$. For all $\alpha \in J$, let $x_\alpha \in R_\alpha \setminus (E^1 \cup E^2)$. Such points exist, since $R_\nu \not\subseteq B$. Then,

$$\frac{1}{|R_\nu^*|}\int_{R_\nu^*} F(y)\,dy = \frac{1}{|R_\nu^*|}\sum_{\alpha\in J}\int_{R_\alpha} F(y)\,dy \leq \frac{1}{|R_\nu^*|}\sum_{\alpha\in J} AF(x_\alpha)|R_\alpha^*| \leq 3^n A^{8M+1} F(x) \ .$$

□

7. Proof of the Lower Bound in Theorem 1.2

First, we prove the following proposition which is announced in the introduction of [DF88].

Proposition 7.1. *There exists a finite collection of balls $B_i = B(x_i, r)$ centered at x_i of radius $r = C_1/\mu$ which satisfy*

(i) $\phi_\mu(x_i) = 0$,
(ii) *their doubles $2B_i = B(x_i, 2r)$ are pairwise disjoint,*
(iii) *Denote by B_i^+ the set $\{\phi_\mu > 0\} \cap B_i$. Similarly, we define B_i^-. Then*

$$C_2 < \frac{\mathrm{Vol}(B_i^+)}{\mathrm{Vol}(B_i^-)} < C_3 \ ,$$

(iv) $\sum_i \mathrm{Vol}(B_i) > C_4 \,\mathrm{Vol}(M)$.

Proof. It is enough to prove the proposition in a coordinate neighborhood V. It is well known that there exists a constant C_5 such that every cube of side C_5/μ contains a zero of ϕ_μ (see [Brü78]). We can decompose V into small cubes R_ν whose side is of size $\delta = 3C_5/\mu$. We call this subdivision $\mathcal{D}$. Each cube R_ν contains a zero x_ν of ϕ_μ in its middle third. We now take a refinement $\mathcal{D}^1$ of $\mathcal{D}$: We partition each side of a cube R_ν into M intervals of equal sizes, where $\mu\delta/M$ is small enough in order to apply Proposition 4.4. We deduce that $(\phi_\mu^2, \mathcal{D}^1, E_\varepsilon^1, C_6(\varepsilon))$ is true (cf. definitions 5.1 & 5.2). If $\mathcal{D}^2$ is the subdivision obtained by taking the centers of cubes belonging to $\mathcal{D}^1$, then the same proposition gives that $(\phi_\mu^2, \mathcal{D}^2, E_\varepsilon^2, C_6(\varepsilon))$ is true. Let B be as in Proposition 6.1, and let $E = E_\varepsilon^1 \cup E_\varepsilon^2 \cup B$. Then, $(\phi_\mu^2, \mathcal{D}, E, C_7(\varepsilon))$ is true.

For each E-good cube R_ν we pick a ball $B_\nu \subset R_\nu$ whose center is x_ν and whose radius $= \delta/6$. By Proposition 5.4

$$\frac{1}{C_8(\varepsilon)} \le \frac{\mathrm{Vol}(B_\nu^+)}{\mathrm{Vol}(B_\nu^-)} \le C_8(\varepsilon) \ .$$

The crucial point is to estimate the number of E-good cubes. By Lemma 5.9, the proportion of E-good cubes is $\ge (1 - C_9|E|)$ (which can be negative). It only remains to estimate $|E|$: $|E_\varepsilon^1| \le C_{10}\varepsilon\mu\delta$, $|E_\varepsilon^2| \le C_{11}\varepsilon\mu\delta$ and

$$|B| \le \delta^n 3^n (\#(E_\varepsilon^1)\text{-bad cubes} + \#(E_\varepsilon^2)\text{-bad cubes}) \le C_{12}(|E_\varepsilon^1| + |E_\varepsilon^2|) \le C_{13}\varepsilon\mu\delta \ .$$

So $|E| \le C_{14}\varepsilon\mu\delta$. To conclude, we take ε small enough in order that the proportion of good cubes is $\ge 70\%$. ☐

Proof of Theorem 1.2 - Lower Bound. The next proposition gives a lower bound in a good ball.

Proposition 7.2. *Let $B(x, r)$ be one of the balls described above. Then we have $\mathrm{Vol}(T_{\mu,\delta} \cap 2B) \ge C_{15} r^{n-1}\delta$, whenever $\mu\delta < C_{16}$.*

Proof. Let $(B^+)_\delta$ be a δ-neighborhood of B^+, and similarly for $(B^-)_\delta$. Since $T_{\mu,\delta} \cap 2B \supseteq (B^+)_\delta \cap (B^-)_\delta$, it is clear that

$$\mathrm{Vol}(T_{\mu,\delta} \cap 2B) \ge \mathrm{Vol}(B^+)_\delta + \mathrm{Vol}(B^-)_\delta - \mathrm{Vol}(B(x, r+\delta)) \ .$$

Assume first that the metric g is flat on $2B$. By the Brunn-Minkowski Inequality [Fed69, §3.2.41] we know

$$\mathrm{Vol}(B^+)_\delta \ge \mathrm{Vol}(B^+) + n\omega_n^{1/n}\delta\,\mathrm{Vol}(B^+)^{1-1/n} \ ,$$

where ω_n is the volume of the n-dimensional unit ball. We have the same inequality for $(B^-)_\delta$. Set $\mathrm{Vol}(B^+) = \alpha \,\mathrm{Vol}(B)$, and $\mathrm{Vol}(B^-) = (1-\alpha)\,\mathrm{Vol}(B)$. We have

$$\begin{aligned}(7.3)\quad \mathrm{Vol}(T_{\mu,\delta} \cap 2B) \geq \mathrm{Vol}(B) - \mathrm{Vol}(B(x, r+\delta)) +& \\ n\omega_n^{1/n}\delta\,\mathrm{Vol}(B)^{1-1/n}\left(\alpha^{1-1/n} + (1-\alpha)^{1-1/n}\right) \geq& \\ \omega_n(r^n - (r+\delta)^n) + n\omega_n r^{n-1}\delta\left(\alpha^{1-1/n} + (1-\alpha)^{1-1/n}\right)&\ .\end{aligned}$$

At this point one observes that when α is bounded away from 0 and 1 we have $\alpha^{1-1/n} + (1-\alpha)^{1-1/n} > 1 + C_{17}$. So, if we take $\delta/r = C_{18}\mu\delta$ small enough then the last expression in (7.3) is positive and we obtain

$$\mathrm{Vol}(T_{\mu,\delta} \cap 2B) \geq C_{19} n\omega_n r^{n-1}\delta\ .$$

Finally, since the metric g is comparable to a flat metric on a small ball, we have a similar inequality also for g. □

To conclude the proof of the lower bound in Theorem 1.2 we observe that due to Proposition 7.1 (iv) the number of balls in Proposition 7.1 is $> C_{20}\mu^n$. So, by Proposition 7.2 $\mathrm{Vol}(T_{\mu,\delta}) > C_{21}\delta/\mu^{n-1} \cdot \mu^n = C_{22}\mu\delta$. □

8. Proof of the Upper Bound in Theorem 1.2

In this section we estimate from above the volume of a tubular neighborhood of the nodal set. The proof is based on the study in Section 4 of eigenfunctions in small scales.

Let $\mathcal{V} = \{V_k\}$ be a covering of M by small open sets. Let $R_k \subseteq V_k$ be a cube preferred by Proposition 4.4. The next lemma shows that it is enough to estimate the volume of $T_{\mu,\delta}$ in preferred cubes.

Lemma 8.1. *There exists a covering $\mathcal{V} = \{V_k\}$ on M with the following properties*

(a) *$\mathcal{V}$ is a finite covering.*
(b) *the metric g can be developed in power series in each chart V_k.*
(c) *$M = \cup_k R_k$ for some choice of cubes $R_k \subseteq V_k$ preferred by Proposition 4.4.*

We defer the proof of this Lemma to Section 8.1.

Now, let $R \subseteq V$ be a preferred cube. We decompose it into boxes R_ν, where the sides of R_ν are of sizes $\delta < l_\nu^{(k)} < 2\delta$. We will denote this subdivision by $\mathcal{D}$.

Definition 8.2. We call R_ν a *nodal* box if $\mathcal{N}_\mu \cap R_\nu \neq \emptyset$.

Let us denote the set of nodal boxes R_ν by Nod. Recall that R_ν^* denotes the union of R_ν with its $3^n - 1$ neighbors.

Lemma 8.3. $T_{\mu,\delta} \subseteq \cup_{R_\nu \in \mathsf{Nod}} R_\nu^*$.

It remains to estimate the number of nodal boxes. Fix $\varepsilon = 1$. Proposition 4.4 tells us that $(\phi_\mu^2, \mathcal{D}, E, C_1)$ is true (cf. Def. 5.1 & 5.2).

Lemma 8.4. *The number of E-good nodal cubes is $\leq C_2 \,\mathrm{Vol}_{n-1}(\mathcal{N}_\mu)/\delta^{n-1}$.*

Proof. We begin by

Claim 8.5. *Let R_ν be an E-good nodal cube. Then*

$$(8.6)\qquad \mathrm{Vol}_{n-1}(\mathcal{N}_\mu \cap R_\nu^*) \geq C_3\delta^{n-1}\ .$$

Proof of Claim. First we see from the Brunn-Minkowski Inequality as in Proposition 7.2 that

$$\liminf_{t\to 0} \frac{\mathrm{Vol}(T_{\mu,t})}{t} \geq C_3\delta^{n-1}. \tag{8.7}$$

Since $\mathcal{N}_\mu$ is an analytic set, it is rectifiable ([Fed69, Theorem 3.4.8 (13)]) and thus ([Fed69, Theorem 3.2.39]), the limit in (8.7) exists and equals $\mathrm{Vol}_{n-1}(\mathcal{N}_\mu \cap R^*_\nu)$. □

Summing up (8.6) over all good nodal cubes we arrive at

$$(8.8)\quad 3^n \,\mathrm{Vol}_{n-1}(\mathcal{N}_\mu) \geq \sum_\nu \mathrm{Vol}_{n-1}(\mathcal{N}_\mu \cap R^*_\nu) \geq \sum_{\text{good nodal } R_\nu\text{'s}} \mathrm{Vol}_{n-1}(\mathcal{N}_\mu \cap R^*_\nu) \geq C_4\#(\text{good nodal } R_\nu\text{'s})\delta^{n-1}\ .$$

□

Proof of Theorem 1.2. By Lemma 5.9 we know that the number of E-bad nodal cubes is $\leq C_5\mu/\delta^{n-1}$. By Lemma 8.4 and Theorem 1.3 the number of E-good nodal cubes is $\leq C_6\mu/\delta^{n-1}$. Together, we get that the number of nodal cubes is $\leq C_7\mu/\delta^{n-1}$. By Lemma 8.3 $\mathrm{Vol}(T_{\mu,\delta}) \leq C_8\#(\mathsf{Nod})\delta^n \leq C_9\mu\delta$. □

8.1. **Proof of Lemma 8.1.** The following lemma is clear by compactness of M.

Lemma 8.9. *There exists $\rho_0 > 0$ such that for all p, the metric g can be developed in power series in $B(p,\rho_0)$.*

Let $\rho_1 = \rho_0/C$ with C large enough.

Lemma 8.10. *Every ball $B(p,\rho_1)$ contains a preferred cube R which contains p.*

Proof. We identify $B(p,\rho)$ with the Euclidean ball $B(0,\rho)$ by working in geodesic coordinates. Suppose that the point $x_0 \in R \subseteq B(0,\rho_0)$. Let $x_1 \in B(0,\rho_0)$ with $|x_1| = |x_0| =: r$. From proposition 4.4 we know that R is independent of μ. By symmetry considerations, or just by examining the proof of Proposition 3.7 we see that any orthogonal transformation in $B(0,\rho_0)$ takes R to another preferred cube.

Now, given p, let q be any point on M such that $\mathrm{dist}(p,q) = r$. The geodesic ball $B(q,\rho_0)$ contains a preferred cube R_1 which contains p. Take a cube R in $R_1 \cap B(p,\rho_1)$ which contains p. □

Proof of Lemma 8.1. By lemma 8.10 we can cover M by preferred cubes. Then by compactness of M we can extract a finite covering by preferred cubes. □

9. Approximation by Nodal Sets

Proof of Theorem 1.8. The proof proceeds similarly to the proof of Corollary 1.6. Fix $C,\varepsilon > 0$. Let $T_{k,\delta}$ be the tubular neighborhood of $\mathcal{N}(\phi_k)$ of radius $\delta_k = C/\mu_k^{n+1+\varepsilon}$. By Theorem 1.2 $\mathrm{Vol}(T_{k,\delta_k}) \leq C/\mu_k^{n+\varepsilon}$. We conclude that

$$\sum_k \mathrm{Vol}\,(T_{k,\delta_k}) \leq C\sum_k \mu_k^{-n-\varepsilon}. \tag{9.1}$$

By Weyl's Law [Wey12, Hör68] we know that

$$\mu_k \asymp Ck^{1/n}.$$

Hence

$$\sum_k \mathrm{Vol}\,(T_{k,\delta_k}) \leq C \sum_k k^{-1-\varepsilon/n}$$

is finite. So, by the Borel-Cantelli Lemma (see e.g. [Fel68]) we obtain

$$\mathrm{Vol}(\cap_{j=1}^{\infty} \cup_{k=j}^{\infty} T_{k,\delta_k}) = 0 \ .$$

□

10. Dimension two

Theorem 10.1. *Let (Σ, g) be a smooth (i.e. C^∞) closed Riemannian surface. Then there exist $C_1, C_2 > 0$ such that*

$$C_1 \mu\delta \leq \mathrm{Vol}(T_{\mu,\delta}) \leq C_2 \,\mathrm{length}(\mathcal{N}_\mu)\delta \ .$$

In particular, Theorem 1.2 is true for surfaces which satisfy Yau's conjecture.

We recall from [DF90] that for any smooth surface $\mathrm{length}(\mathcal{N}_\mu) \leq C_3\mu^{3/2}$. Hence, if we modify the proof of Theorem 1.8 according to Theorem 10.1 we obtain

Proposition 10.2. *Let (Σ, g) be a closed compact surface with a* smooth *metric g. Then we have* $\mathrm{Vol}(M(7/2+\varepsilon, C)) = 0$ *for all $C, \varepsilon > 0$.*

10.1. **Lower Bound in Theorem 10.1.** This is basically Brüning's argument. We can cover a fixed portion of Σ with pairwise disjoint balls $B_i = B(x_i, r)$ of radius $r = c/\mu$ and such that $\phi_\mu(x_i) = 0$. The set $\mathcal{N}_\mu \cap B(x_i, r)$ is of length $\geq r$. Moreover, in local coordinates it has a projection of length $\geq cr$ on one of the axes. This implies that $T_{\mu,\delta} \cap B(x_i, r)$ has area $\geq cr\delta$. Summing up over all the balls B_i we obtain

$$\mathrm{Vol}(T_{\mu,\delta}) \geq c_1\mu^2 \cdot c_2\delta/\mu = c_3\mu\delta \ .$$

10.2. **Upper Bound in Theorem 10.1- First Proof.** Let an eigenfunction ϕ_μ have nodal domains $\Omega_1, \ldots, \Omega_{N(\mu)}$. Given $\partial\Omega_j \subset \mathcal{N}_\mu$, let $L_j(t)$ denote the *interior parallel* of $\partial\Omega_k$ at the distance t inside Ω_j. It is clear that

$$\mathrm{area}(A_\mu) = \sum_{j=1}^{N} \int_{t=0}^{\delta} \mathrm{length}(L_j(t))\, dt. \tag{10.3}$$

The following inequality can be found in [Sav01, Proposition A.1.iv]:

$$\mathrm{length}(L_j(t)) \leq \mathrm{length}(\partial\Omega_j) + R(\Omega_j) \max\left\{ \int_{\Omega_j} K^+ - 2\pi\chi(\Omega_j), 0 \right\}. \tag{10.4}$$

Here K^+ denotes the positive part of the Gauss curvature, $\chi(\Omega_j)$ is proportional to the number $m_j = m_j(\mu)$ of connected components of $\partial\Omega_j$, and $R(\Omega_j)$ denotes the inner radius of Ω_j. We substitute (10.4) into (10.3) and sum over $1 \leq j \leq N$. By Proposition 1.4 we know that $R(\Omega_j) \leq C/\mu$. We get the estimate

$$\frac{\mathrm{area}(A_\mu)}{\delta} \leq 2 \cdot \mathrm{length}(\mathcal{N}_\mu) + \frac{C\int_M K^+}{\mu} + \frac{4\pi C}{\mu} \sum_{j=1}^{N(\mu)} m_j(\mu) \tag{10.5}$$

As $\mu_j = \mu \to \infty$, the second term goes to zero. It remains to estimate the third term. One can construct a connected graph on M whose edges will include all arcs of $\mathcal{N}_\mu$, and show using Euler's formula that

$$\sum_{j=1}^{N} m_j \leq 2(N+g-1),$$

where g denotes the genus of the surface M. Also, by Courant's nodal domain theorem

$$N = N(\mu_k) \leq k+1.$$

We recall that by [Wey12, Hör68] in dimension two $\mu_k \asymp C\sqrt{k}$, hence $N(\mu_k) \leq C\mu_k^2$. It follows that the third term in the right-hand side of (10.5) is less than $C\mu$. Substituting everything back into (10.5) and recalling that $\text{length}(\mathcal{N}_\mu) \geq C\mu$ (see [Brü78]) we get the desired estimate. □

10.3. **Upper Bound in Theorem 10.1- Second Proof.** This proof was communicated by M. Sodin.

It suffices to give a proof for the neighborhood of $\mathcal{N}_\mu$ of size $\delta/3$. We cover M with cubes of side $C\delta$ (*large cubes*), as well as by cubes of side $C\delta/3$ (*small cubes*). One can easily arrange that each cube intersects a bounded number of other cubes. For every small cube, there exists a unique concentric large cube whose side is three times larger. To estimate the area of $T_{\mu,\delta}$, it suffices to estimate the volume of the union B_j of all small cubes which intersect the nodal set $\mathcal{N}_\mu$. Indeed, if $x \in T_{\mu,\delta}$, then $\mathcal{N}_\mu$ intersects either the small cube containing x, or one of the 8 neighboring small cubes, so the volume of $T_{\mu,\delta}$ is at most $9 \cdot \text{vol}(B_j)$.

We distinguish several cases

i) $\mathcal{N}_\mu$ intersects a small cube Q, but any connected component of $\mathcal{N}_\mu \cap Q$ doesn't intersect the boundary of the big concentric cube Q'.
ii) $\mathcal{N}_\mu$ intersects a small cube Q, and there exists a connected component of $\mathcal{N}_\mu \cap Q$ that intersects the boundary of the big concentric cube Q'.

In case (i) there is at least one nodal domain contained in Q', so by the Faber-Krahn Inequality (see [EK96, Ch. 7, Th. 1]) we get that the area of this nodal domain is $> C/\mu^2$. By the Isoperimetric Inequality, the length of $\mathcal{N}_\mu \cap Q'$ is at least $C/\mu \geq C\delta$.

In case (ii), the length of $\mathcal{N}_j \cap Q'$ is at least $\delta/3$.

Hence, we conclude that the number of Q' for which Q satisfies case (i) or case (ii) is $\ll \text{length}(\mathcal{N}_\mu)/\delta$. Accordingly, the sum of the areas of those cubes is

$$\ll \text{length}(\mathcal{N}_\mu)/\delta \cdot \delta^2 \leq C\,\text{length}(\mathcal{N}_\mu)\delta. \tag{10.6}$$

□

11. Discussion

For a given M it seems interesting to find

$$E(M) := \sup\{b : \text{vol}(M(b,C)) > 0 \ \textit{for some}\ C > 0\}.$$

Theorem 1.8 implies that on real-analytic n-dimensional manifolds, $E(M) \leq n+1$. In dimension one, it follows from the theory of continued fractions that $E(M) = 2$ for $M = [0,\pi]$. In fact, $M(2,\pi) = M$ while $\text{Vol}(M(2+\varepsilon, C)) = 0\ \forall \varepsilon > 0$.

The same result likely holds for *separable systems* (Examples include surfaces of revolution, Liouville tori and *quantum completely integrable* systems [TZ02]). In such systems one can separate variables and choose a basis of eigenfunctions that (in appropriate coordinates) have the form $\phi(x_1,\ldots,x_n)=\prod\psi_j(x_j)$, where ψ_j are solutions of 2nd order differential equations. Accordingly, $\mathcal{N}(\phi)$ forms a "grid" of hypersurfaces determined by zeros of ψ_j-s, and approximation by $\mathcal{N}(\phi)$ reduces to a series of one-dimensional problems.

As a model example we consider an n-dimensional cube

$$M(n)=\prod_{j=1}^{n}[0,\pi/\alpha_j],$$

with Dirichlet boundary conditions, where for simplicity we assume $\{\alpha_j^2\}_{j=1}^n$ are linearly independent over $\mathbb{Q}$. Then the eigenvalues have the form $\sum_{j=1}^n\alpha_j^2m_j^2$ (where $m_j\in\mathbb{N}$) and are simple, while the corresponding eigenfunctions have the form

$$\phi(m_1,\ldots,m_n;x_1,\ldots,x_n)=\prod_{j:\,m_j\neq 0}\sin(m_j\alpha_jx_j).$$

Proposition 11.1. *$E(M(n))=2$ for all n.*

Proof of Proposition 11.1.

We first make a change of variables $y_j=\pi\alpha_jx_j$. This change of variables will only affect constants in the rate of approximation by nodal sets; it won't affect the exponent. In the rescaled coordinates, nodal sets have the form

$$\mathcal{N}(\phi(m_1,\ldots,m_n))=\cup_{j:m_j\neq 0}\mathcal{A}_j, \tag{11.2}$$

where $\mathcal{A}_j:=\{(y_1,\ldots,y_n):y_j=k_j/m_j,\ \ 0\le k_j\le m_j\}$. We first show that

Claim 11.3. $E(M(n))\ge 2$.

Proof. Let $(y_1,\ldots,y_n)\in M$ be an arbitrary point on M; we have $0\le y_j\le 1$. We can assume without loss of generality that $y_j\notin\mathbb{Q},\forall 1\le j\le n$, since the set of such points has the full measure. Consider next the continued fraction expansion of its first (say) coordinate,

$$y_1=[0;a_1,a_2,\ldots],$$

where we use the notation of [Khi97]. Let $p_k/q_k,\ k=1,2,\ldots$ be the corresponding continued fractions. Then the points $(p_k/q_k,y_2,\ldots,y_n)\in\mathcal{N}(\phi(q_k,0,\ldots,0))$, and the Claim follows from the well-known inequality [Khi97]

$$|y_1-p_k/q_k|<1/q_k^2.$$

□

We next show that

Claim 11.4. $E(M(n))\le 2$.

Proof. It suffices to show that $\mathrm{Vol}(M(2+\varepsilon,C))=0$ for all $C,\varepsilon>0$. Let $\mathbf{y}=(y_1,\ldots,y_n)\in M(2+\varepsilon,C)$. As before, we may assume that $y_j\notin\mathbb{Q}$. We know that there exists a sequence of eigenvalues $\mu_k\to\infty$ such that $d(\mathbf{y},\mathcal{N}(\phi_{\mu_k}))<C/\mu_k^{2+\varepsilon}$. Since all distances on $[0,1]^n$ are equivalent, we may define $d(\mathbf{x},\mathbf{y})=\max_{1\le j\le n}|x_j-y_j|$.

In view of (11.2), it follows that for some $1\le j\le n$ (say, for $j=1$), there exists a sequence of integers $q_k,k=1,2,\ldots$, such that $q_k\to\infty$ and $|y_1-p_k/q_k|<C/q_k^{2+\varepsilon}$

for some $0 \le p_k \le q_k$. The Claim now follows from Corollary 1.6. This also finishes the proof of Proposition 11.1. □

For manifolds with ergodic geodesic flows (e.g. in negative curvature), eigenfunction behavior has been studied using *random wave model* [Ber77]. In addition, *percolation model* [BS02] has been used to study the statistics of nodal domains in chaotic systems. We refer the reader to [FGS04] and references therein for a nice discussion about applicability of those models for studying various questions about eigenfunctions of chaotic systems.

In the opinion of the authors, it would be difficult to use these models directly to predict the "best possible" rate of approximation by nodal sets. The reason is that these models describe a *single* eigenfunction on a scale of C/μ (several wavelengths). However (as shown by the example of $M = [0,\pi]$) for a given $x \in M$ the values of μ giving the best approximation of x by $\mathcal{N}(\phi_\mu)$ can grow exponentially. It thus seems difficult to take into account simultaneous behavior of all eigenfunctions in such a large energy range. However, one can probably expect that $E(M) > 2$ for such manifolds (in contrast to the integrable case), due to irregularity of nodal lines for such systems.

It also seems interesting to study "level sets" $M(b)$ for the approximation exponent b, e.g. defined by

$$M(b) := \cup_C M(b,C) \setminus (\cup_{a<b} \cup_C M(a,C)) \,.$$

Remark 11.5. It should follow from the results of [JL99] that the conclusion of Theorem 1.8 should also hold for *level sets* of eigenfunctions (since the level set of an eigenfunction is a nodal set of a linear combination of that eigenfunction with a constant eigenfunction). It seems interesting to determine which level sets are C/μ-dense (like nodal sets).

References

[Ber77] M. V. Berry, *Focusing and twinkling: critical exponents from catastrophes in non-Gaussian random short waves*, J. Phys. A **10** (1977), no. 12, 2061–2081.

[Brü78] J. Brüning, *Über Knoten von Eigenfunktionen des Laplace-Beltrami-Operators*, Math. Z. **158** (1978), no. 1, 15–21.

[BS02] E. Bogomolny and C. Schmit, *Percolation model for nodal domains of chaotic wave functions*, Phys. Rev. Lett. **88** (2002), 114102.

[DF88] H. Donnelly and C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), no. 1, 161–183.

[DF90] ———, *Nodal sets for eigenfunctions of the Laplacian on surfaces*, J. Amer. Math. Soc. **3** (1990), no. 2, 333–353.

[EK96] Y. Egorov and V. Kondratiev, *On spectral theory of elliptic operators*, Operator Theory: Advances and Applications, vol. 89, Birkhäuser Verlag, Basel, 1996.

[Fed69] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[Fel68] W. Feller, *An introduction to probability theory and its applications. Vol. I*, Third edition, John Wiley & Sons Inc., New York, 1968.

[FGS04] G. Foltin, S. Gnutzmann, and U. Smilansky, *The morphology of nodal lines—random waves versus percolation*, J. Phys. A **37** (2004), no. 47, 11363–11371.

[GS94] A. Grigis and J. Sjöstrand, *Microlocal analysis for differential operators*, London Mathematical Society Lecture Note Series, vol. 196, Cambridge University Press, Cambridge, 1994, An introduction.

[Hör68] L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218.

[JL99] D. Jerison and G. Lebeau, *Nodal sets of sums of eigenfunctions*, Harmonic analysis and partial differential equations (Chicago, IL, 1996), Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1999, pp. 223–239.

[Khi97] A. Ya. Khinchin, *Continued fractions*, russian ed., Dover Publications Inc., Mineola, NY, 1997, With a preface by B. V. Gnedenko, Reprint of the 1964 translation.

[Man08] D. Mangoubi, *Local asymmetry and the inner radius of nodal domains*, Comm. Partial Differential Equations **33** (2008), 1611–1621.

[Sav01] A. Savo, *Lower bounds for the nodal length of eigenfunctions of the Laplacian*, Ann. Global Anal. Geom. **19** (2001), no. 2, 133–151.

[Ste70] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[Ste93] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[TZ02] J. A. Toth and S. Zelditch, *Riemannian manifolds with uniformly bounded eigenfunctions*, Duke Math. J. **111** (2002), no. 1, 97–132.

[Wey12] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), no. 4, 441–479.

[Yom08] Y. Yomdin, *Zero sets of functions and their piecewise-polynomial approximations*, preprint (2008).

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montréal QC H3A 2K6, Canada.
E-mail address: jakobson@math.mcgill.ca

IHÉS, Le Bois-Marie, 35, Route de Chartres, F-91440 Bures-sur-Yvette, France
E-mail address: mangoubi@ihes.fr