How large can the first eigenvalue be on a surface of genus two?

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Abstract

Sharp upper bounds for the first eigenvalue of the Laplacian on a surface of a fixed area are known only in genera zero and one. We investigate the genus two case and conjecture that the first eigenvalue is maximized on a singular surface which is realized as a double branched covering over a sphere. The six ramification points are chosen in such a way that this surface has a complex structure of the Bolza surface. We prove that our conjecture follows from a lower bound on the first eigenvalue of a certain mixed Dirichlet-Neumann boundary value problem on a half-disk. The latter can be studied numerically, and we present conclusive evidence supporting the conjecture.

Keywords: Laplacian, first eigenvalue, surface of genus two, mixed boundary value problem.

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1 Introduction and main results

1.1 Upper bounds on the first eigenvalue

Let M be a closed surface of genus γ and let g be the Riemannian metric on M. Denote by Δ the Laplace-Beltrami operator on M, and by λ_1 the smallest positive eigenvalue of the Laplacian. Let the area $\mathrm{Area}(M)$ be fixed. How large can λ_1 be on such a surface?

Sharp bounds for the first eigenvalue are known only for the sphere ([H], see also [SY]), the projective plane ([LY]), the torus ([Ber], [N]), and the Klein bottle ([JNP], [EGJ]). The present paper is concerned with the surface of genus 2.

Let M be orientable and let $\Pi : M \to \mathbb{S}^2$ be a non-constant holomorphic map (or, conformal branched covering) of degree d. It was proved in [YY] that

$$\lambda_1 \mathrm{Area}(M) \leq 8\pi d \,. \tag{1.1.1}$$

Any Riemann surface of genus γ can be represented as a branched cover over $\mathbb{S}^2$ of degree $d = \left[\dfrac{\gamma+3}{2}\right]$, where $[\cdot]$ denotes the integer part (see [Gun], [GH]). Therefore,

$$\lambda_1 \mathrm{Area}(M) \leq 8\pi \left[\frac{\gamma+3}{2}\right]. \tag{1.1.2}$$

In general, (1.1.2) is not sharp, for example for $\gamma = 1$ ([N]).

Let $M = \mathcal{P}$ be a surface of genus $\gamma = 2$. Then (1.1.2) implies

$$\lambda_1 \mathrm{Area}(\mathcal{P}) \leq 16\pi. \tag{1.1.3}$$

The aim of this paper is to show, using a mixture of analytic and numerical tools, that (1.1.3) is sharp. Main results of this paper were announced (without proofs) in [JLNP, section 4].

1.2 The Bolza surface

Let $\Pi : \mathcal{P} \to \mathbb{S}^2$ be a branched covering of degree $d = 2$. The Riemann-Hurwitz formula (see [GH]) implies that this cover is ramified at 6 points. We choose these points to be the intersections of the round sphere $\mathbb{S}^2$ centered at the origin with the coordinate axes in $\mathbb{R}^3$. The surface $\mathcal{P}$ can be realized as

$$\left\{ (z,w) \in \mathbf{C}^2 : w^2 = F(z) := z\frac{(z-1)(z-i)}{(z+1)(z+i)} \right\} .$$

This surface has the conformal structure of the Bolza surface. It has an octahedral group of holomorphic automorphisms and its symmetry group is the largest among surfaces of genus two [I, KW]. Interestingly enough, the Bolza surface appears in some other extremal problems, in particular for systoles (see [KS]).

To simplify calculations it is convenient to rotate the equatorial plane by $\pi/4$. The equation of $\mathcal{P}$ becomes

$$\mathcal{P} := \left\{ (z,w) \in \mathbf{C}^2 : w^2 = F(z) := z \frac{(z - e^{\pi i/4})(z - e^{3\pi i/4})}{(z + e^{\pi i/4})(z + e^{3\pi i/4})} \right\} . \qquad (1.2.1)$$

The projection Π is defined by $\Pi : (z, w) \to z$. The set of ramification points in the complex z plane is $R := \{0, \infty, \pm e^{\pi i/4}, \pm e^{3\pi i/4}\}$. The spherical and complex models are related by the stereographic projection; the induced metric in the complex plane (which we assume coincides with the equatorial plane of $\mathbb{S}^2$) is

$$4dzd\bar{z}/(1 + |z|^2)^2. \qquad (1.2.2)$$

Let g_0 be the metric on $\mathcal{P}$ which is the pullback of the round metric (1.2.2) on $\mathbb{S}^2$. One can see that the metric g_0 has conical singularities at the points of ramification. It has curvature $+1$ everywhere except the branching points. Because of the presence of singularities we have to specify what we mean by the first positive eigenvalue of the Laplacian on $(\mathcal{P}, g_0)$. We set

$$\lambda_1(\mathcal{P}, g_0) := \inf_{u \in H^1_0(\mathcal{P},g_0),\ u \neq 0,\ \langle u,1 \rangle = 0} \frac{\| \nabla u\|^2}{\|u\|^2} ,$$

where the scalar product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ are taken in the space $L_2(\mathcal{P}, g_0)$. The Sobolev space $H^1_0(\mathcal{P}, g_0)$ of functions supported away from the singularities is obtained by the closure of $C^\infty_0(\mathcal{P}, g_0) := \{v \in C^\infty(\mathcal{P}, g_0) \mid \overline{\Pi \operatorname{supp} v} \cap R = \emptyset\}$ with respect to the norm $\| \nabla v\|^2 + \|v\|^2$.

1.3 Main results

We start with the following

Conjecture 1.3.1. *The equality in* (1.1.3) *is attained for the metric* g_0 *on* $\mathcal{P}$*, i.e.*

$$\lambda_1(\mathcal{P}, g_0)\mathrm{Area}(\mathcal{P}, g_0) = 16\pi.$$

Since $(\mathcal{P}, g_0)$ is a double cover of the standard $\mathbb{S}^2$, we have

$$\mathrm{Area}(\mathcal{P}, g_0) = 2\mathrm{Area}(\mathbb{S}^2) = 8\pi.$$

Therefore, in order to prove Conjecture 1.3.1 it suffices to show that

$$\lambda_1(\mathcal{P}, g_0) = \lambda_1(\mathbb{S}^2) = 2. \tag{1.3.2}$$

Unfortunately, we are unable to prove (1.3.2), and therefore establish Conjecture 1.3.1. We can however reduce the conjecture to the following spectral problem on a quarter-sphere $Q \subset \mathbb{S}^2$ that can be treated using numerical methods. Namely, let, in usual spherical coordinates (ϕ, θ),

$$Q = \{(\phi, \theta) : \ 0 < \phi < \pi/2, 0 < \theta < \pi\}.$$

We split the boundary ∂Q into two parts: $\partial Q = \overline{\partial_1 Q \sqcup \partial_2 Q}$, where

$$\partial_1 Q = \{(0, \theta) : |\theta - \pi/2| < \pi/4\} \cup \{(\pi/2, \theta) : \ 0 < \theta < \pi/2\},$$

$$\partial_2 Q = \{(0, \theta) : \ |\theta - \pi/2| > \pi/4\} \cup \{(\pi/2, \theta) : \pi/2 < \theta < \pi\},$$

and consider the spectral boundary value problem for the Laplace-Beltrami operator on Q:

$$-\Delta u = \Lambda u \quad \text{on } Q, \quad u|_{\partial_1 Q} = 0, \quad (\partial u/\partial n)|_{\partial_2 Q} = 0. \tag{1.3.3}$$

Let Λ_1 denote the first eigenvalue of the problem (1.3.3) (which we understand as usual in the variational sense).

Conjecture 1.3.4.

$$\Lambda_1 \geq 2.$$

Our main result is

Theorem 1.3.5. *Conjecture 1.3.4 implies Conjecture 1.3.1.*

Theorem 1.3.5 is proved in sections 2 and 3.

Extensive numerical calculations (see Section 4) show that $\Lambda_1 \gtrsim 2.27$ which implies Conjecture 1.3.1. The best lower bound we are able to prove is just $\Lambda_1 > 0.75$, which follows from Dirichlet-Neumann bracketing (replace the Dirichlet condition by the Neumann one on the arc $(0, \theta)$, $\pi/4 < \theta < 3\pi/4$), see [Ke]).

Finally, we note that the spectral problem (1.3.3) easily reduces via the stereographic projection to the following mixed Dirichlet-Neumann problem on a half-disk $D := \{(r, \psi) \in \mathbb{R}^2 : r < 1, 0 < \psi < \pi\}$ (here (r, ψ) are usual planar polar coordinates):

$$-\Delta v = \frac{4\Lambda}{(1+r^2)^2} v \quad \text{on } D\,, \quad v|_{\partial_1 D} = 0\,, \quad (\partial v/\partial n)|_{\partial_2 D} = 0\,. \tag{1.3.6}$$

Here $\partial_1 D := \{(r, 0) : r \in (0,1)\} \cup \{(1, \psi) : |\psi - \pi/2| < \pi/4\}$ and $\partial_2 D := \{(r, \pi) : r \in (0,1)\} \cup \{(1, \psi) : |\psi - \pi/2| > \pi/4\}$.

Problems (1.3.3) and (1.3.6) are quite remarkable in their own right — each of them is an example of a mixed Dirichlet-Neumann boundary value problem whose spectrum is invariant under a swap of Dirichlet and Neumann boundary conditions. Namely, the spectrum of (1.3.6) coincides with the spectrum of

$$-\Delta v = \frac{4\Lambda}{(1+r^2)^2} v \quad \text{on } D\,, \quad v|_{\partial_2 D} = 0\,, \quad (\partial v/\partial n)|_{\partial_1 D} = 0\,. \tag{1.3.7}$$

We refer to [JLNP] for a further discussion on Dirichlet-Neumann swap isospectrality.

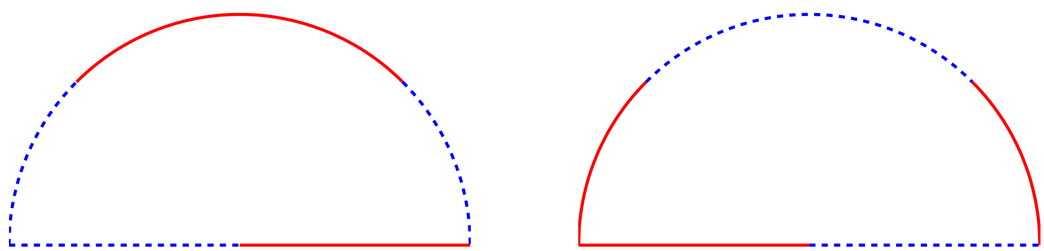

Figure 1: Geometry of boundary value problems (1.3.6) (left) and (1.3.7) (right). Here and further on, the solid red line denotes Dirichlet boundary condition and the dashed blue line — the Neumann one.

Remark 1.3.8. One can check that a surface with a finite number of conical singularities can be approximated by a sequence of smooth surfaces of the same genus and area in such a way that the corresponding sequence of the first non-zero eigenvalues converges to λ_1 on the original surface. Thus, Conjecture 1.3.1 means that (1.1.3) is sharp in the class of smooth metrics, although the equality is not necessarily attained. For a general result about the convergence of the whole spectrum see [Ro].

2 Symmetries

2.1 Hyperelliptic involution

Let $T : \mathcal{P} \to \mathcal{P}$, $T^2 = \mathrm{Id}$ be a map that intertwines the preimages of points of $\mathbb{S}^2$ under a two-sheeted covering $\Pi : \mathcal{P} \to \mathbb{S}^2$. Clearly, the Laplace operator Δ commutes with T.

By the spectral theorem, we can consider separately the restrictions of the Laplacian onto the spaces of functions which are either even or odd with respect to T. The even functions on $\mathcal{P}$ can be identified with the functions on $\mathbb{S}^2$. Therefore, as $\lambda_1(\mathbb{S}^2) = 2$, we have $\lambda_1(\mathcal{P}) \leq 2$, and the equality in (1.3.2) will be achieved if and only if the first eigenvalue λ_1^{odd} of the Laplacian acting on the odd subspace satisfies $\lambda_1^{\text{odd}} \geq 2$.

2.2 Isometries of $\mathcal{P}$

Consider the following isometries of $\mathbb{S}^2$ (as usual, we identify $\mathbb{S}^2$ and $\mathbb{C}$ by stereographic projection):

$$\begin{aligned} &\sigma_1 : z \mapsto \bar{z} \quad \text{or} \quad (\chi, \eta, \xi) \mapsto (\chi, -\eta, \xi), \\ &\sigma_2 : z \mapsto -\bar{z} \quad \text{or} \quad (\chi, \eta, \xi) \mapsto (-\chi, \eta, \xi), \\ &\sigma_3 : z \mapsto 1/\bar{z} \quad \text{or} \quad (\chi, \eta, \xi) \mapsto (\chi, \eta, -\xi). \end{aligned} \tag{2.2.1}$$

Here $z = x+iy$ is a point in the equatorial plane upon which a point $(\chi, \eta, \xi) \in \mathbb{S}^2$ is stereographically projected.

The hyperelliptic involution T is given by $T : (z, w) \to (z, -w)$. For $1 \leq j \leq 3$, a symmetry σ_j of $\mathbb{S}^2$ has two corresponding symmetries s_j and $T \circ s_j$ satisfying

$$\Pi \circ s_j = \Pi \circ T \circ s_j = \sigma_j \circ \Pi\,. \tag{2.2.2}$$

Those symmetries, with account of (1.2.1) are given by the explicit formulae

$$\begin{aligned} &s_1 : (z, w) \mapsto (\bar{z}, \bar{z}/\bar{w}), \\ &s_2 : (z, w) \mapsto (-\bar{z}, i\bar{w}), \\ &s_3 : (z, w) \mapsto (1/\bar{z}, \bar{w}/\bar{z}). \end{aligned} \tag{2.2.3}$$

As an illustration, we demonstrate how the last of these formulae is obtained: if $w^2 = F(z)$, then by (1.2.1),

$$F\left(\frac{1}{\bar{z}}\right) = \frac{1}{\bar{z}} \frac{(1/\bar{z} - e^{\pi i/4})(1/\bar{z} - e^{3\pi i/4})}{(1/\bar{z} + e^{\pi i/4})(1/\bar{z} + e^{3\pi i/4})} = \frac{\overline{F(z)}}{\bar{z}^2}\,,$$

thus giving the expression for s_3.

It easily seen that all s_j commute with T and satisfy

$$\begin{aligned} s_j^2 &= \mathrm{Id},\ j=1,2,3; \\ s_1 s_3 &= s_3 s_1,\ s_2 s_3 = s_3 s_2; \\ s_2 s_1 &= T s_1 s_2. \end{aligned} \tag{2.2.4}$$

Remark 2.2.5. In the proof of Theorem 1.3.5 we will use only the symmetries s_1 and s_3. Calculations for s_2 are presented for the sake of completeness (see Remark 3.4.1).

2.3 Fixed point sets of isometries

Let $\mathrm{Fix}(S)$ denote a fixed point set of a mapping S. As easily seen from (2.2.1), the sets $\mathrm{Fix}(\sigma_j)$, for $j = 1, 2, 3$, lie in the union of the coordinate lines and a unit circle of $\mathbb{C}$, and we introduce the following notation for future reference. The coordinate lines are divided into two rays each by the ramification point $r_0 := 0$, and we denote

$$a_1 := \{z = t, t > 0\}, \quad a_2 := \{z = it, t > 0\},$$

$$a_3 := \{z = t, t < 0\}, \quad a_4 := \{z = it, t < 0\}.$$

The circle is divided into four arcs by the ramification points $r_1 := e^{-\pi i/4}$, $r_2 := e^{\pi i/4}$, $r_3 := e^{3\pi i/4}$, and $r_4 := e^{-3\pi i/4}$, and we denote the arcs by

$$a_{k+4} := \{z = e^{t\pi i/4}, t \in (2k-3, 2k-1)\}, \qquad k = 1, 2, 3, 4,$$

so that the arc a_5 goes from r_1 to r_2, the arc a_6 goes from r_2 to r_3, the arc a_7 goes from r_3 to r_4, and finally a_8 goes from r_4 to r_1.

In this notation, the fixed point sets $\mathrm{Fix}(\sigma_j)$ are written as

$$\mathrm{Fix}(\sigma_1) = a_1 \cup a_3, \quad \mathrm{Fix}(\sigma_2) = a_2 \cup a_4, \quad \mathrm{Fix}(\sigma_3) = a_5 \cup a_6 \cup a_7 \cup a_8. \tag{2.3.1}$$

Note that each of the rays a_j ($j = 1, 2, 3, 4$) intersects an arc a_{j+4} at a single point which we denote z_j:

$$z_1 = 1, \quad z_2 = i, \quad z_3 = -1, \quad z_4 = -i,$$

see Figure 2.

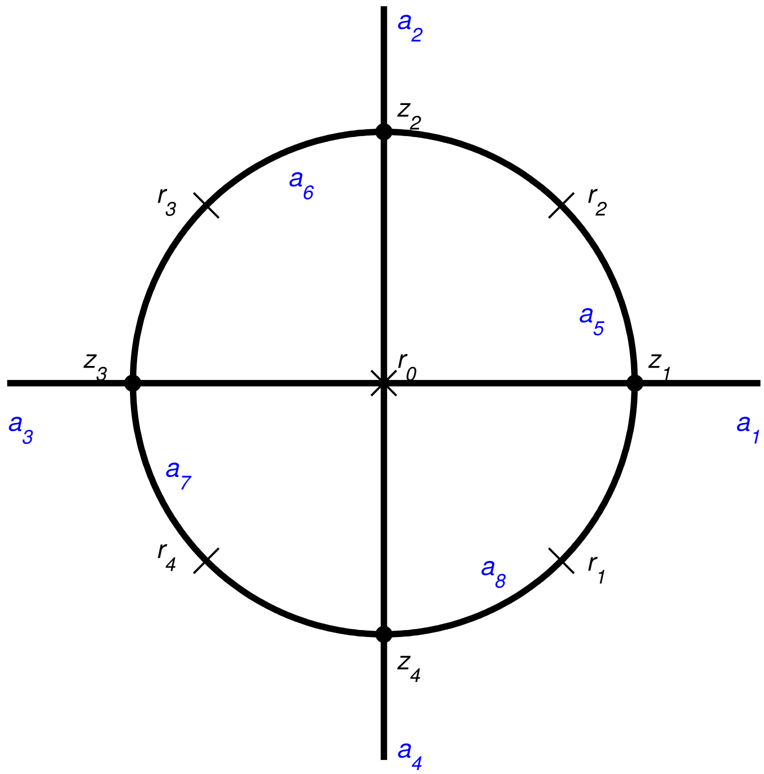

Figure 2: Ramification points, rays, arcs and intersections

2.4 Fixed point sets of s_1, s_2, s_3

Each of the points z_j has exactly two pre-images $p_j^{(m)} := (w_j^{(m)}, z_j) \in \Pi^{-1} z_j$, $m = 1, 2$, where $w_j^{(1,2)}$ are the solutions of the equation $(w_j)^2 = F(z_j)$, with F given in (1.2.1). These solutions are easily found from (1.2.1); we are of course at liberty to choose which of the two solutions is denoted $w_j^{(1)}$ and which is denoted $w_j^{(2)}$. For definiteness we set

$$\begin{aligned} w_1^{(1)} &= i\,, & w_1^{(2)} &= -i\,; \\ w_2^{(1)} &= \frac{1+i}{2+\sqrt{2}}\,, & w_2^{(2)} &= -\frac{1+i}{2+\sqrt{2}}\,; \\ w_3^{(1)} &= 1\,, & w_3^{(2)} &= -1\,; \\ w_4^{(1)} &= \frac{1-i}{2-\sqrt{2}}\,, & w_4^{(2)} &= -\frac{1-i}{2-\sqrt{2}}\,. \end{aligned} \tag{2.4.1}$$

For future use, we need to know the images of points $p_j^{(m)}$ under the symmetries s_l, $l = 1, 2, 3$. These are easily calculated from (2.2.3); it turns out that $s_l p_j^{(m)} = p_k^{(n)}$ with some indices $k \in \{1, 2, 3, 4\}$, $n \in \{1, 2\}$. The results of the calculations are summarized in the following Table 1.

(j, m)	(k, n)		
	$l = 1$	$l = 2$	$l = 3$
(1,1)	(1,1)	(3,1)	(1,2)
(1,2)	(1,2)	(3,2)	(1,1)
(2,1)	(4,1)	(2,1)	(2,1)
(2,2)	(4,2)	(2,2)	(2,2)
(3,1)	(3,2)	(1,2)	(3,2)
(3,2)	(3,1)	(1,1)	(3,1)
(4,1)	(2,1)	(4,2)	(4,1)
(4,2)	(2,2)	(4,1)	(4,2)

Table 1: The points $p_j^{(m)}$ and their images $p_k^{(n)} = s_l p_j^{(m)}$ under symmetries s_l. The table lists the pairs of indices (j, m) and the resulting pairs (k, n) for $l = 1, 2, 3$. Note also that T acts by interchanging the second indices $1 \leftrightarrow 2$.

For each of the rays or arcs a_k, $k = 1, \ldots, 8$ defined in the previous section, its pre-image $\Pi^{-1} a_k$ has two connected components which we denote $b_k^{(1)}, b_k^{(2)}$ related by $b_k^{(m)} = T b_k^{(m')}$, $m, m' \in \{1, 2\}$, $m \neq m'$. Again, the choice of which component we denote by an upper index (1) is up to us and in order to fix the notation we postulate that $b_k^{(m)} \ni w_{k'}^{(m)}$, $k' = ((k-1) \mod 4) + 1$, e.g. $w_1^{(1)} \in b_1^{(1)} \cap b_5^{(1)}$, $w_3^{(2)} \in b_3^{(2)} \cap b_7^{(2)}$, etc.

We now have at our disposal all the information we need in order to obtain the fixed point sets of s_1, s_2, s_3. We start with the following two simple Lemmas.

Lemma 2.4.2. $\Pi \operatorname{Fix}(s_j) \subseteq \operatorname{Fix}(\sigma_j)$.

Proof. Let $z \in \Pi \operatorname{Fix}(s_j)$. Then there exists $p \in \mathcal{P}$ such that $\Pi p = z$ and $s_j p = p$. Thus $\Pi s_j p = z$ and by (2.2.2) $\sigma_j \Pi p = \sigma_j z = z$, so that $z \in \operatorname{Fix}(\sigma_j)$. □

Lemma 2.4.3. *Let $a_k \subseteq \operatorname{Fix}(\sigma_j)$. Then, for $m = 1, 2$, either $b_k^{(m)} \subseteq \operatorname{Fix}(s_j)$ or $b_k^{(m)} \subseteq \operatorname{Fix}(T \circ s_j)$.*

Proof. We have $\Pi b_k^{(m)} = a_k$, so that $\sigma_j \Pi b_k^{(m)} = \sigma_j a_k = a_k$, and so by (2.2.2) $\Pi s_j b_k^{(m)} = a_k = \Pi b_k^{(m)} = \Pi T b_k^{(m)}$. The result follows from the obvious observation: if $\Pi \alpha = \Pi \beta$, then either $\alpha = \beta$ or $\alpha = T\beta$. □

The lemmas lead to the following

Proposition 2.4.4.

$$\begin{aligned}
\operatorname{Fix}(s_1) &= b_1^{(1)} \cup b_1^{(2)}\,, \\
\operatorname{Fix}(Ts_1) &= b_3^{(1)} \cup b_3^{(2)}\,, \\
\operatorname{Fix}(s_2) &= b_2^{(1)} \cup b_2^{(2)}\,, \\
\operatorname{Fix}(Ts_2) &= b_4^{(1)} \cup b_4^{(2)}\,, \\
\operatorname{Fix}(s_3) &= b_6^{(1)} \cup b_6^{(2)} \cup b_8^{(1)} \cup b_8^{(2)}\,, \\
\operatorname{Fix}(Ts_3) &= b_5^{(1)} \cup b_5^{(2)} \cup b_7^{(1)} \cup b_7^{(2)}\,.
\end{aligned}$$

Proof. By Lemmas 2.4.2 and 2.4.3, for any given j the fixed sets $\operatorname{Fix}(s_j)$ and $\operatorname{Fix}(Ts_j)$ consist only of the pre-images of the components a_k of the corresponding fixed sets $\operatorname{Fix}(\sigma_j)$ (given by (2.3.1)). However we still need to describe which component $b_k^{(m)}$, $m = 1, 2$, lies in $\operatorname{Fix}(s_j)$ and which in $\operatorname{Fix}(Ts_j)$. As each component $b_k(m)$ is uniquely determined by the point $w_k^{(m)}$ given by (2.4.1), it is sufficient just to check in Table 1 whether $s_j w_k^{(m)} = w_k^{(m)}$ or $T s_j w_k^{(m)} = w_k^{(m)}$.

For example, to find $\operatorname{Fix}(s_2)$ we need only to inspect $b_2^{(m)}$ and $b_4^{(m)}$. As, by Table 1, $s_2 w_2^{(m)} = w_2^{(m)}$ and $T s_2 w_4^{(m)} = w_4^{(m)}$, we have $\operatorname{Fix}(s_2) = b_2^{(1)} \cup b_2^{(2)}$ and $\operatorname{Fix}(Ts_2) = b_4^{(1)} \cup b_4^{(2)}$. The rest of Proposition 2.4.4 is obtained in the same manner. □

3 Proof of Theorem 1.3.5

We divide the proof of Theorem 1.3.5 into several steps.

3.1 Even eigenfunctions with respect to T

Consider the subspace $V_+ \subset L^2(\mathcal{P})$ consisting of all even eigenfunctions with respect to T. Any such eigenfunction has a well-defined projection on $\mathbb{S}^2$. Therefore, if there exists a first eigenfunction of $\mathcal{P}$ that belongs to V_+, its projection is an eigenfunction on $\mathbb{S}^2$ and hence the corresponding eigenvalue is greater or equal than two (recall that $\lambda_1(\mathbb{S}^2) = 2$). Hence, in this case the Conjecture 1.3.1 is verified.

3.2 Use of symmetries s_1, s_3.

Denote by G_{13} the subgroup of the automorphism group of $\mathcal{P}$ generated by the symmetries

$$\{T, s_1, s_3\}.$$

It follows from (2.2.4) that G_{13} is commutative. Note also that all the elements of G_{13} commute with the Laplacian on $\mathcal{P}$. Therefore, we can choose a basis of $L^2(\mathcal{P})$ consisting of joint eigenfunctions of all $s \in G_{13}$ and Δ. Given a joint eigenfunction f of all $s \in G_{13}$, we denote by $\mu(f, s)$ the corresponding eigenvalue of s, i.e.

$$f(sx) = \mu(f, s) f(x)\,.$$

Since $s_j^2 = T^2 = \mathrm{Id}$ for $j = 1, 3$, we see that $\mu(f, s) = \pm 1$ for all $s \in G_{13}$.

3.3 Odd eigenfunctions with respect to T

Consider now the space $V_- \subset L^2(\mathcal{P})$ consisting of all eigenfunctions of the Laplacian which are *odd* with respect to T. Let ϕ_1 be a joint eigenfunction of $\{T, s_1, s_3, \Delta\}$, corresponding to the smallest eigenvalue of $\boldsymbol{\Delta}\big|_{V_-}$

Now, since $\mu(\phi_1, T) = -1$ and $s_3^2 T = T$, we have $\mu(\phi_1, s_1)\mu(\phi_1, s_1 T) = \mu(\phi_1, T) = -1$, and similarly $\mu(\phi_1, s_3)\mu(\phi_1, s_3 T) = -1$.

Without loss of generality we may assume that $\mu(\phi_1, s_1) = -1$. We recall from section 2.3 that the fixed point set $\mathrm{Fix}\, s_1$ consists of the arcs $b_1^{(1)}, b_1^{(2)}$. Thus ϕ_1 must *vanish* on these arcs.

Consider now the symmetries s_3, s_3T. We must have one of the following two cases:

i) $\mu(\phi_1, s_3T) = -1$, $\mu(\phi_1, s_3) = 1$;

ii) $\mu(\phi_1, s_3) = -1$, $\mu(\phi_1, s_3T) = 1$.

Consider first Case i).

Proposition 3.3.1. *In Case* i) *the function* ϕ_1 *vanishes on the arcs*

$$b_1^{(1)}, b_1^{(2)}, b_5^{(1)}, b_5^{(2)}, b_7^{(1)}, b_7^{(2)},$$

and its normal derivative $\partial_n \phi_1$ *vanishes on the arcs*

$$b_3^{(1)}, b_3^{(2)}, b_6^{(1)}, b_6^{(2)}, b_8^{(1)}, b_8^{(2)}.$$

Proof. By Proposition 2.4.4, the fixed-point set of s_3T consists of the arcs $b_5^1, b_5^2, b_7^1, b_7^2$. Accordingly, ϕ_1 vanishes on all those arcs, as well as on b_1^1, b_1^2. Moreover, ϕ_1 has $\mu(\phi_1, s_3) = \mu(\phi_1, s_1T) = 1$. It follows that the normal derivative of $\partial_n \phi_1$ vanishes on the fixed-point sets of those symmetries. It remains to apply once more Proposition 2.4.4 in order to complete the proof. □

Consider next Case ii).

Proposition 3.3.2. *In Case* ii) *the function* ϕ_1 *vanishes on the arcs*

$$b_3^{(1)}, b_3^{(2)}, b_6^{(1)}, b_6^{(2)}, b_8^{(1)}, b_8^{(2)},$$

and its normal derivative $\partial_n \phi_1$ *vanishes on the arcs*

$$b_1^{(1)}, b_1^{(2)}, b_5^{(1)}, b_5^{(2)}, b_7^{(1)}, b_7^{(2)}.$$

Proposition 3.3.2 is proved in the same way as Proposition 3.3.1.

3.4 Final step of the proof

Since ϕ_1 is an odd function with respect to the hyperelliptic involution T, its projection upon $\mathbb{S}^2$ is not well-defined. However, the projection of $|\phi_1|$ to $\mathbb{S}^2$ is well-defined. Denote it by ψ_1.

In Case i), the function ψ_1 can be chosen as a test function for the mixed Dirichlet-Neumann boundary value problem (1.3.3). Assume now Conjecture

1.3.4 is true and the first eigenvalue of (1.3.3) satisfies $\Lambda_1 \geq 2$. Then the Rayleigh quotient of ψ_1 and hence of ϕ_1 satisfies the same inequality. But this means that ψ_1 cannot be the first eigenfunction on $\mathcal{P}$ since we get a contradiction with (1.1.3). Therefore, the first eigenfunction of $\mathcal{P}$ is even with respect to T, and as was shown in section 3.1 this implies Conjecture 1.3.1.

Similarly, in Case ii), the function ψ_1 can be chosen as a test function for the mixed Dirichlet-Neumann boundary volume problem which is obtained from (1.3.3) by swapping the Dirichlet and the Neumann conditions. However, it was shown in [JLNP] that this problem is isospectral to (1.3.3). Therefore, repeating the same arguments as above we prove that Conjecture 1.3.1 holds. This completes the proof of Theorem 1.3.5. □

Remark 3.4.1. In the proof of Theorem 1.3.5 we have used only the symmetries s_1 and s_3. Alternatively, we could have used s_2 and s_3. One can check directly using Proposition 2.4.4 that applying s_2 one obtains a mixed Dirichlet-Neumann boundary value problem which is equivalent to (1.3.3) and hence no additional information about the first eigenfunction is obtained.

3.5 A family of extremal surfaces of genus two

The purpose of this section is to prove the following

Corollary 3.5.1. *Conjecture 1.3.4 implies that there exists a continuous family $\mathcal{P}_t$ of surfaces of genus 2 such that $\lambda_1 \mathrm{Area}(\mathcal{P}_t) = 16\pi$.*

Proof. Consider the Riemann surface $\mathcal{P}_t$ defined by the equation

$$\left\{ (z,w) : w^2 = \frac{z\left(z - e^{i(\pi/2-t)}\right)\left(z - e^{i(\pi/2+t)}\right)}{\left(z - e^{-i(\pi/2-t)}\right)\left(z - e^{-i(\pi/2+t)}\right)} \right\}$$

where $t \in (0, \pi/2)$. Note that $\mathcal{P}_{\pi/4} = \mathcal{P}$. It is easy to see that for any t, $\mathcal{P}_t$ is symmetric with respect to s_1 and s_3. Arguing in the same way as in the proof of Theorem 1.3.5 and using a stereographic projection, we reduce the problem on $\mathcal{P}_t$ to the following two mixed Dirichlet-Neumann boundary value problems on the half-disk D:

$$-\Delta v = \frac{4\Lambda}{(1+r^2)^2} v \quad \text{on } D\,, \quad v|_{\partial_1(t)} = 0\,, \quad (\partial v/\partial n)|_{\partial_2(t)} = 0\,. \qquad (3.5.2)$$

and

$$-\Delta v = \frac{4\Lambda}{(1+r^2)^2} v \quad \text{on } D\,, \quad v|_{\partial_2(t)} = 0\,, \quad (\partial v/\partial n)|_{\partial_1(t)} = 0\,. \qquad (3.5.3)$$

Here $\partial_1(t) := \{(r,0) : r \in (0,1)\} \cup \{(1,\psi) : |\psi - \pi/2| < t\}$ and $\partial(t)D := \{(r,\pi) : r \in (0,1)\} \cup \{(1,\psi) : \pi/2 > |\psi - \pi/2| > t\}$.

We remark that for $t \neq \pi/4$ these two problems are not isospectral. Using Dirichlet-Neumann bracketing it is easy to see that (3.5.2) has a smaller first eigenvalue than (3.5.3) if $t < \pi/4$ and a larger one if $t > \pi/4$. Denote the minimal first eigenvalue of the two problems by $\Lambda_1(t)$. According to Conjecture 2 and numerical calculations, $\Lambda_1(\pi/4) > 2$. Since the first eigenvalues of both problems depend continuously and monotonically on parameter t, and since $\Lambda_1(0) = \Lambda_1(\pi/2) = 0.75$ (see section 1.3), there exist numbers $t_1^* \in (0, \pi/4)$ and $t_2^* \in (\pi/4, \pi/2)$ such that $\Lambda_1(t_1^*) = \Lambda_1(t_2^*) = 2$ and so $\Lambda_1(t) \geq 2$ for $t \in [t_1^*, t_2^*]$. Arguing is above, we deduce that for all surfaces $\mathcal{P}_t$ corresponding to these values of t, estimate (1.1.3) is sharp. This completes the proof of the theorem. □

Corollary 3.5.1 implies that 16π is a *degenerate* maximum for $\lambda_1 \mathrm{Area}(M)$ for surfaces of genus two. This is not the case for surfaces of lower genus on which the metric maximizing the first eigenvalue is unique. Note also that the extremal metrics in genera zero and one are analytic, while the surfaces $\mathcal{P}_t$ have singular points.

4 Numerical investigations

4.1 Basics of the Finite Element Method

In this section we describe the numerical experiments used to estimate the first eigenvalue of (1.3.7).

We define the space $\mathcal{H}$ as the closure of $\{v \in C^\infty(D) | \overline{\operatorname{supp} v} \cap \partial_2 D = \emptyset\}$ with respect to the $H^1(D)$ norm.

The variational setting for the eigenvalue problem is to find the smallest eigenvalue $\lambda \in \mathbb{R}$, and the associated eigenvector $v \in \mathcal{H}$ such that for all $u \in \mathcal{H}$,

$$\int_D \nabla v \cdot \nabla u \, dD = 4\lambda \int_D \frac{vu}{(1+r^2)^2} \, dD \,. \tag{4.1.1}$$

We use finite elements to approximate the eigenvalues and eigenfunctions of (4.1.1). The general procedure we follow is:

1. Discretize the region D using a triangular mesh $\mathcal{T}_h = \bigcup_{i=1}^{N_h} \tau_i$, with a size of an individual triangle $\tau \in \mathcal{T}_h$ parameterized by $h > 0$.

2. Introduce a finite-dimensional subspace V_h of $\mathcal{H}$, consisting of finite element basis functions $\{\phi_i\}_{i=1}^{N_h}$ on $\mathcal{T}_h$;

3. Denote $(v_h,\ \lambda_h) \in V_h \times \mathbb{R}$, with $v_h = (v_1, v_2, \ldots, v_{N_h})^t$, the solution of the finite-dimensional generalized eigenvalue problem

$$A_h v_h = \lambda_h B_h v_h \,, \tag{4.1.2}$$

where

$$(A_h)_{ij} := \int_D \nabla\phi_i \cdot \nabla\phi_j \, dD \,, \quad (B_h)_{ij} := \int_D \frac{4\phi_i\phi_j}{(1+r^2)^2} \, dD \,. \tag{4.1.3}$$

Clearly, problem (4.1.2) is the discrete analog of (4.1.1). The eigenpair (v_h, λ_h) is computed using some iterative algorithm, until a prescribed tolerance `tol` is reached.

4. Steps 1-3 are repeated with smaller and smaller h until some other stopping criterion is attained.

We now present the results of some numerical experiments based on this strategy.

4.2 Conforming finite elements

In the first set of experiments, the choice of approximating subspaces V_h was a sequence of *P1-conforming finite element spaces (Courant triangles)*, see [Br]. This means that for a given $h > 0$, and a triangulation $\mathcal{T}_h$ of the domain,

$$V_h := \{v \in \mathcal{H};\ v|_\tau = \text{ polynomial of degree } \leq 1 \text{ for every } \tau \in \mathcal{T}_h\} \,.$$

The discrete generalized eigenvalue problem for each h was solved using an Arnoldi iteration with shift 2.6. For details on the Arnoldi iteration, see, e.g., [GovL, TrBa].

Experiment 1: The initial triangulation is based on a graded mesh, with more triangles located near the singularities of the eigenfunction (see Figure 3).

The refinement strategy was based on simply subdividing each triangle in $\mathcal{T}_h$ into 4 while preserving the quality of the mesh, yielding a new triangulation $\mathcal{T}_{h/2}$. The eigenvalue solver was run until a tolerance of $\texttt{tol} = 10^{-16}$ was achieved. The computation was performed using `FreeFem++` for generating the finite elements

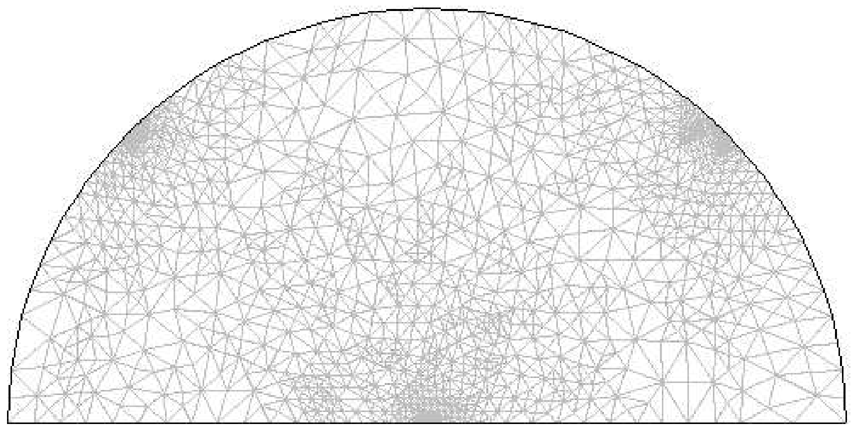

Figure 3: A finite element mesh

λ_h	Err	N_h	No. of Triangles	No. of Arnoldi iterates
2.45590105457	2.00434573363e-05	169	288	11
2.36301118569	1.32592470511e-06	625	1152	11
2.32089716556	8.16135748742e-08	2401	4608	12
2.30111238184	4.80693483786e-09	9409	18432	11
2.29161462311	2.79565739833e-10	37249	73728	11

Table 2: Using P1-conforming finite elements

and the meshes, and `ARPACK` for the eigenvalue solve. The meshes were refined until the measure of error,

$$\texttt{Err} := \int_D |\nabla u_h|^2 - 4\lambda_h \frac{u_h^2}{(1+r^2)^2}\, dD\,,$$

satisfied $|\texttt{Err}| < 5 \times 10^{-10}$. The results are tabulated in Table 2.

Experiment 2: This experiment was conducted using `MATLAB`'s finite element package `PDEToolbox`, and the eigenvalue solve was performed using `ARPACK` routines. A sequence of triangular meshes was created, starting from the coarsest mesh, and refining 5 times. The major difference between this and the previous experiment is in the manner in which the zero Dirichlet data is enforced.

λ_h	N_h	No.of Triangles
2.55310562723060	77	126
2.40400118356918	279	504
2.33742285062686	1061	2016
2.30582934149898	4137	8064
2.29039772121374	16337	32256
2.28276090970583	64929	129024
2.27895954902635	258881	516096

Table 3: Using P1-conforming finite elements in MATLAB

4.3 Nonconforming finite elements

In the second set of experiments, we used *P1-nonconforming finite elements (Crouzeix elements)*, see [Br]. These are defined as

$$V_h := \{v \in L^2(D); v|_\tau \text{ is linear for each } \tau \in \mathcal{T}_h, \\ v \text{ is continuous at the midpoints of triangle edges}\}$$

for a given $h > 0$ and a triangulation $\mathcal{T}_h$. Note that V_h is not a subspace of $\mathcal{H}$; for more information on the use of nonconforming elements in eigenvalue problems, see [ArDu]. As before, the discrete generalized eigenvalue problem is solved using an Arnoldi iteration with a shift of 2.2 until a tolerance of $\texttt{tol} = 10^{-16}$ is achieved. The refinement strategy was to subdivide each triangle in $\mathcal{T}_h$ into 4 subtriangles, yielding a new mesh $\mathcal{T}_{h/2}$. The meshes were refined until a measure of error

$$\texttt{Err} := \int_D |\nabla v_h|^2 - 4\lambda_h \frac{v_h^2}{(1+r^2)^2}\, dD\,,$$

satisfied $|\texttt{Err}| < 5 \times 10^{-10}$. The results are presented in Table 4.

In each of the experiments above, we found that the computed eigenvalues appeared to converge to a value greater than 2.27. The associated eigenfunctions also appear to converge to a function whose contour lines are shown in Figure 4.

λ_h	Err	N_h	No. of Triangles	No. of Arnoldi iterates
2.13042989031	-1.5494060025e-05	169	288	11
2.20743747322	-7.89999667122e-07	625	1152	11
2.24561396752	-3.8455570927e-08	2401	4608	11
2.26440630518	-1.87263030138e-09	9409	18432	11
2.27364314423	-8.78059287464e-11	37249	73728	11

Table 4: Using P1-nonconforming finite elements

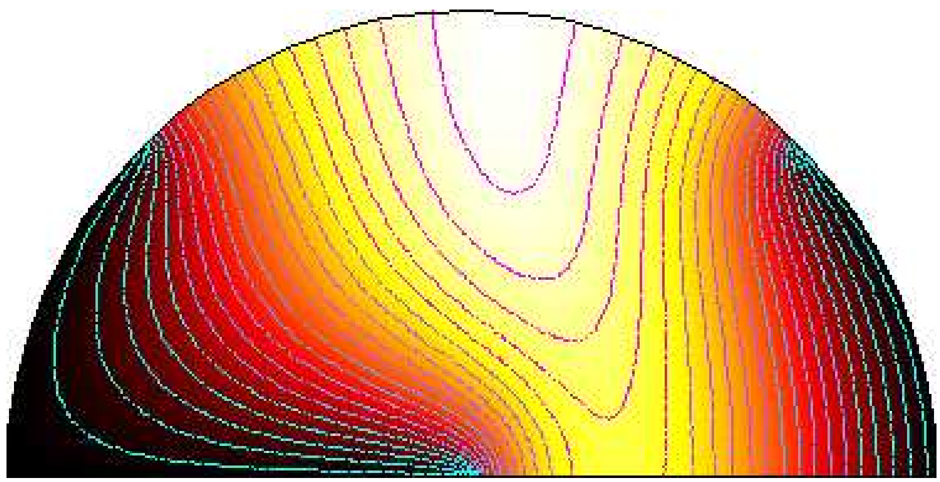

Figure 4: Contour lines of the first eigenfunction

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