ADAPTIVE WAVELET ALGORITHMS FOR SOLVING OPERATOR EQUATIONS: THESIS SUMMARY

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This thesis treats various aspects of adaptive wavelet algorithms for solving operator equations. For a separable Hilbert space H, a linear functional $f \in H'$, and a boundedly invertible linear operator $A : H \to H'$, we consider the problem of finding $u \in H$ satisfying

$$Au = f.$$

Typically A is given by a variational formulation of a boundary value problem or integral equation, and H is a Sobolev space formulated on some domain or manifold, possibly incorporating essential boundary conditions. Often we will assume that A is self-adjoint and H-elliptic. General operators can be treated, e.g., by forming normal equations, although in particular situations quantitatively more attractive alternatives exist.

In their pioneering works [*Math. Comp.*, 70:27–75, 2001] and [*Found. Comput. Math.*, 2(3):203–245, 2002], Cohen, Dahmen and DeVore introduced adaptive wavelet paradigms for solving the problem numerically. Utilizing a Riesz basis $\Psi = \{\psi_i \in H : i \in \mathbb{N}\}$ for H, the idea is to transform the original problem into a problem involving coefficient vector $\mathbf{u} \in \ell_2$ of u with respect to the basis Ψ. This $\mathbf{u}$ is the unique solution of

$$\mathbf{Au} = \mathbf{f},$$

where $\mathbf{A} : \ell_2 \to \ell_2$ is an infinitely sized stiffness matrix with elements $\mathbf{A}_{ik} = [A\psi_k](\psi_i) \in \mathbb{R}$, and $\mathbf{f} \in \ell_2$ is an infinitely sized load vector with elements $\mathbf{f}_i = f(\psi_i) \in \mathbb{R}$. Under certain assumptions concerning the cost of evaluating the entries of the stiffness matrix, the methods from the aforementioned works of Cohen, Dahmen, and DeVore for solving this infinite matrix-vector problem were shown to be of optimal computational complexity. In this thesis, we will verify those assumptions, extend the scope of problems for which the adaptive wavelet algorithms can be applied directly, and most importantly, develop and analyze modified adaptive algorithms with improved quantitative properties.

Chapter 1 (*Introduction*) contains a general introduction to the thesis.

Chapter 2 (*Basic principles*) contains a short introduction to the theory of adaptive wavelet algorithms. We start with recalling essential properties of wavelet bases, and briefly present basic results on best N-term approximation. Then we describe how an optimally convergent algorithm can be constructed using any linearly convergent iteration in the energy space. We include proofs of the most fundamental results, along with references to relevant literature.

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In Chapter 3 (*Adaptive Galerkin methods*), an adaptive wavelet method for solving linear operator equations is constructed that is a modification of the method from [*Math. Comp.*, 70:27–75, 2001], in the sense that there is no recurrent coarsening of the iterands. In spite of this, it will be shown that the method has optimal computational complexity. Numerical results for a simple model problem indicate that the new method is more efficient than the existing method.

In Chapter 4 (*Using polynomial preconditioners*), we investigate the possibility of using polynomial preconditioners in the context of adaptive wavelet methods. We propose a version of a preconditioned adaptive wavelet algorithm and show that it has optimal computational complexity.

In Chapter 5 (*Adaptive algorithm for nonsymmetric and indefinite elliptic problems*), we modify the adaptive wavelet algorithm from Chapter 3 so that it applies directly, i.e., without forming the normal equation, not only to self-adjoint elliptic operators but also to operators of the form $L = A + B$, where A is self-adjoint elliptic and B is compact, assuming that the resulting operator equation is well-posed. We show that the algorithm has optimal computational complexity.

Aiming at a further improvement of quantitative properties, in Chapter 6 (*Adaptive algorithm with truncated residuals*), a new class of adaptive wavelet algorithms for solving elliptic operator equations is introduced, which are proven to have optimal complexity assuming a certain property of the stiffness matrix. This assumption is confirmed for elliptic differential operators.

In Chapter 7 (*Computability of differential operators*), restricting us to differential operators, we develop a numerical integration scheme that computes the entries of the stiffness matrix at the expense of an error that is consistent with the approximation error, whereas in each column the average computational cost per entry is $\mathcal{O}(1)$. As a consequence, we can conclude that the "fully discrete" adaptive wavelet algorithm has optimal computational complexity.

In Chapter 8 (*Computability of singular integral operators*), we prove an analogous result for singular integral operators, by carefully distributing computational costs over the matrix entries in combination with choosing efficient quadrature schemes.

Chapter 9 (*Conclusion*) finishes with a summary and discussion of the presented research topics, as well as with some suggestions for future research.