556: MATHEMATICAL STATISTICS I

MULTIVARIATE JENSEN'S INEQUALITY

Jensen's Inequality - if $g(x)$ is **convex**, so that for $0 < \lambda < 1$,

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$$

for all x and y, then if X is a random variable with expectation μ,

$$E_{f_X}[g(X)] \geq g(E_{f_X}[X])$$

- extends to the multivariable case in a number of ways.

- If $\underset{\sim}{X}$ is a k-dimensional vector random variable, but g

  $$g : \mathbb{R}^k \longrightarrow \mathbb{R}$$

  is a convex **scalar** function, then

  $$E_{f_{\underset{\sim}{X}}}\left[g(\underset{\sim}{X})\right] \geq g(E_{f_{\underset{\sim}{X}}}\left[\underset{\sim}{X}\right])$$

  for which the proof is similar to the original version.

- If $\underset{\sim}{g}$

  $$g : \mathbb{R}^k \longrightarrow \mathbb{R}^d$$

  is a vector function, then the above result can be applied componentwise to the elements

  $$(g_1(\underset{\sim}{x}), g_2(\underset{\sim}{x}), \ldots, g_d(\underset{\sim}{x}))^{\mathsf{T}}.$$

- If g is a matrix-valued function, for example

  $$g(\underset{\sim}{x}) = \underset{\sim}{x}\underset{\sim}{x}^{\mathsf{T}}$$

  then we can also consider matrix-type inequalities; for example, for two $k \times k$ matrices, Σ_1 and Σ_2, we might write

  $$\Sigma_1 \geq \Sigma_2 \qquad \text{if} \qquad \underset{\sim}{x}^{\mathsf{T}}(\Sigma_1 - \Sigma_2)\underset{\sim}{x} \geq 0 \quad \forall \underset{\sim}{x} \in \mathbb{R}^k$$

  and legitimately write, say,

  $$E_{f_{\underset{\sim}{X}}}\left[\underset{\sim}{X}\underset{\sim}{X}^{\mathsf{T}}\right] \geq \underset{\sim}{\mu}\underset{\sim}{\mu}^{\mathsf{T}}$$

  where $\underset{\sim}{\mu} = E_{f_{\underset{\sim}{X}}}[\underset{\sim}{X}]$. However, general results relating to convex multivariable functions go beyond the scope of the course.