556: MATHEMATICAL STATISTICS I

THE BOREL-CANTELLI LEMMA

DEFINITION Limsup and liminf events
Let $\{E_n\}$ be a sequence of events in sample space Ω. Then

$$E^{(S)} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$$

is the **limsup** event of the infinite sequence; event $E^{(S)}$ occurs if and only if

- **for all** $n \geq 1$, **there exists** an $m \geq n$ such that E_m occurs.
- **infinitely many** of the E_n occur.

Similarly, let

$$E^{(I)} = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m$$

is the **liminf** event of the infinite sequence; event $E^{(I)}$ occurs if and only if

- **there exists** $n \geq 1$, such that **for all** $m \geq n$, E_m occurs.
- **only finitely many** of the E_n do not occur.

THEOREM The Borel-Cantelli Lemma

Let $\{E_n\}$ be a sequence of events in sample space Ω. Then

(a) If

$$\sum_{n=1}^{\infty} \mathrm{P}(E_n) < \infty, \qquad \Longrightarrow \qquad \mathrm{P}\left(E^{(S)}\right) = 0,$$

that is,

$$P[E_n \text{ occurs infinitely often }] = 0.$$

(b) If the events $\{E_n\}$ are **independent**

$$\sum_{n=1}^{\infty} \mathrm{P}(E_n) = \infty \qquad \Longrightarrow \qquad \mathrm{P}\left(E^{(S)}\right) = 1.$$

that is,

$$P[E_n \text{ occurs infinitely often }] = 1.$$

Note: This result is useful for assessing almost sure convergence. For a sequence of random variables $\{X_n\}$ and limit random variable X, suppose, for $\epsilon > 0$, that $A_n(\epsilon)$ is the event

$$A_n(\epsilon) \equiv \{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}$$

The BC Lemma says

(a) **if** $\displaystyle\sum_{n=1}^{\infty} P(A_n(\epsilon)) = \sum_{n=1}^{\infty} P[\,|X_n - X| > \epsilon\,] < \infty$ **then** $X_n \xrightarrow{a.s.} X$

(b) **if** $\displaystyle\sum_{n=1}^{\infty} P(A_n(\epsilon)) = \sum_{n=1}^{\infty} P[\,|X_n - X| > \epsilon\,] = \infty$ with the X_n **independent** **then** $X_n \overset{a.s.}{\nrightarrow} X$

Proof. **NOT EXAMINABLE**
(a) Note first that

$$\sum_{n=1}^{\infty} \mathrm{P}(E_n) < \infty \Longrightarrow \lim_{n\to\infty} \sum_{m=n}^{\infty} \mathrm{P}(E_m) = 0.$$

because if the sum on the left-hand side is finite, then the tail-sums on the right-hand side tend to zero as $n \to \infty$. But for every $n \geq 1$,

$$E^{(S)} \;=\; \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m \subseteq \bigcup_{m=n}^{\infty} E_m \qquad \therefore \qquad \mathrm{P}\left(E^{(S)}\right) \;\leq\; \mathrm{P}\left(\bigcup_{m=n}^{\infty} E_m\right) \;\leq\; \sum_{m=n}^{\infty} \mathrm{P}(E_m). \tag{1}$$

Thus, taking limits as $n \longrightarrow \infty$, we have that

$$\mathrm{P}\left(E^{(S)}\right) \;\leq\; \lim_{n\to\infty} \sum_{m=n}^{\infty} \mathrm{P}(E_m) \;=\; 0.$$

(b) Consider $N \geq n$, and the union of events

$$E_{n,N} = \bigcup_{m=n}^{N} E_m.$$

$E_{n,N}$ corresponds to the collection of sample outcomes that are in *at least one* of the collections corresponding to events $E_n, ..., E_N$. Therefore, $E'_{n,N}$ is the collection of sample outcomes in Ω that are **not in any** of the collections corresponding to events $E_n, ..., E_N$, and hence

$$E'_{n,N} = \bigcap_{m=n}^{N} E'_m \tag{2}$$

Now,

$$E_{n,N} \subseteq \bigcup_{m=n}^{\infty} E_m \qquad \Longrightarrow \qquad \mathrm{P}(E_{n,N}) \;\leq\; \mathrm{P}\left(\bigcup_{m=n}^{\infty} E_m\right)$$

and hence, by assumption and independence,

$$\begin{aligned} 1 - \mathrm{P}\left(\bigcup_{m=n}^{\infty} E_m\right) \;&\leq\; 1 - \mathrm{P}\left(\bigcup_{m=n}^{N} E_m\right) = 1 - \mathrm{P}(E_{n,N}) = \mathrm{P}\left(E'_{n,N}\right) = \mathrm{P}\left(\bigcap_{m=n}^{N} E'_m\right) \;=\; \prod_{m=n}^{N} \mathrm{P}\left(E'_m\right) \\ &=\; \prod_{m=n}^{N} \left(1 - \mathrm{P}\left(E_m\right)\right) \leq\; \exp\left\{-\sum_{m=n}^{N} \mathrm{P}\left(E_m\right)\right\}, \end{aligned}$$

as $1 - x \leq \exp\{-x\}$ for $0 < x < 1$. Now, taking the limit of both sides as $N \to \infty$, for fixed n,

$$1 - \mathrm{P}\left(\bigcup_{m=n}^{\infty} E_m\right) \;\leq\; \lim_{N\to\infty} \exp\left\{-\sum_{m=n}^{N} \mathrm{P}\left(E_m\right)\right\} = 0$$

as, by assumption $\sum_{n=1}^{\infty} \mathrm{P}(E_n) = \infty$. Thus, for each n, we have that

$$\mathrm{P}\left(\bigcup_{m=n}^{\infty} E_m\right) = 1 \qquad \therefore \qquad \lim_{n\to\infty} \mathrm{P}\left(\bigcup_{m=n}^{\infty} E_m\right) = 1.$$

But the sequence of events $\{A_n\}$ defined for $n \geq 1$ by

$$A_n = \bigcup_{m=n}^{\infty} E_m$$

is monotone non-increasing, and hence, by continuity,

$$\mathrm{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathrm{P}(A_n). \tag{3}$$

From (4), we have that the right hand side of equation (5) is equal to 1, and, by definition,

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m. \tag{4}$$

Hence, combining (4), (5) and (6) we have finally that

$$\mathrm{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\right) = 1 \qquad \Longrightarrow \qquad \mathrm{P}\left(E^{(S)}\right) = 1.$$

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Interpretation and Implications

The Borel-Cantelli result is concerned with the calculation of the probability of the limsup event $E^{(S)}$ occurring for general infinite sequences of events $\{E_n\}$. From previous discussion, we have seen that $E^{(S)}$ corresponds to the collection of sample outcomes in Ω that are in **infinitely many** of the E_n collections. Alternately, $E^{(S)}$ occurs if and only if **infinitely many** $\{E_n\}$ occur. The Borel-Cantelli result tells us conditions under which $\mathrm{P}\left(E^{(S)}\right) = 0$ or 1.

EXAMPLE : Consider the event E defined by

"E occurs" = "run of 100^{100} Heads occurs in an infinite sequence of independent coin tosses"

We wish to calculate P(E), and proceed as follows; consider the infinite sequence of events $\{E_n\}$ defined by

"E_n occurs" = "run of 100^{100} Heads occurs in the nth block of 100^{100} coin tosses"

Then $\{E_n\}$ are independent events, and

$$\mathrm{P}(E_n) = \frac{1}{2^{100^{100}}} > 0 \implies \sum_{n=1}^{\infty}\mathrm{P}(E_n) = \infty,$$

and hence by part (b) of the Borel-Cantelli result,

$$\mathrm{P}\left(E^{(S)}\right) = \mathrm{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\right) = 1$$

so that the probability that infinitely many of the $\{E_n\}$ occur is 1. But, crucially,

$$E^{(S)} \subseteq E \implies \mathrm{P}(E) = 1.$$

Therefore the probability that E occurs, that is that a run of 100^{100} Heads occurs in an infinite sequence of independent coin tosses, is 1.