556: MATHEMATICAL STATISTICS I

DISTRIBUTIONS DERIVED FROM NORMAL RANDOM SAMPLES

Suppose that $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ form a random sample.

(a) By the univariate transformation result

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

(b) By the multivariate transformation result

$$T = \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim Student(n-1)$$

as T can be written $T = Z/\sqrt{V/\nu}$, where $Z \sim N(0,1)$ and $V \sim \chi^2_\nu$ are independent random variables defined by

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \qquad V = \frac{(n-1)s^2}{\sigma^2}$$

(c) **Fisher-F distribution**: By the multivariate transformation result, if $V_1 \sim \chi^2_{k_1}$ and $V_2 \sim \chi^2_{k_2}$ are independent random variables, then

$$Q = \frac{V_1/k_1}{V_2/k_2}$$

has pdf

$$f_Q(x) = \frac{\Gamma\left(\dfrac{k_1+k_2}{2}\right)}{\Gamma\left(\dfrac{k_1}{2}\right)\Gamma\left(\dfrac{k_2}{2}\right)} \left(\frac{k_1}{k_2}\right)^{k_1/2} x^{k_1/2-1} \left(1 + \frac{k_1}{k_2}x\right)^{-(k_1+k_2)/2} \qquad x > 0$$

and zero otherwise. We say that $Q \sim Fisher(k_1, k_2)$.

Now, if $X_1, \ldots, X_{n_X} \sim N(\mu_X, \sigma_X^2)$ and $Y_1, \ldots, Y_{n_Y} \sim N(\mu_Y, \sigma_Y^2)$ are independent random samples from different distributions, then as

$$\frac{(n_X-1)s_X^2}{\sigma_X^2} \sim \chi^2_{n_X-1} \qquad \frac{(n_Y-1)s_Y^2}{\sigma_Y^2} \sim \chi^2_{n_Y-1}$$

it follows that

$$\frac{s_X^2/\sigma_X^2}{s_Y^2/\sigma_Y^2} \sim Fisher(n_X-1, n_Y-1)$$

Note that if $X \sim Fisher(k_1, k_2)$, then

$$E_{f_X}[X] = \frac{k_2}{k_2-2} \ \ (\text{if } k_2 > 2) \qquad Var_{f_X}[X] = 2\left(\frac{k_2}{k_2-2}\right)^2 \frac{(k_1+k_2-2)}{k_1(k_2-4)} \ \ (\text{if } k_2 > 4)$$

Also

$$X \sim Fisher(k_1, k_2) \implies \frac{1}{X} \sim Fisher(k_2, k_1)$$

$$X \sim Student(k_1) \implies X^2 \sim Fisher(1, k_1)$$

$$X \sim Fisher(k_1, k_2) \implies \frac{(k_1/k_2)X}{1+(k_1/k_2)X} \sim Beta\left(\frac{k_1}{2}, \frac{k_2}{2}\right)$$