MATH 556: RANDOM VARIABLES & PROBABILITY DISTRIBUTIONS

1.8 RANDOM VARIABLES & PROBABILITY MODELS

Definition 1.11 A **random variable** **(r.v.)** X is a function defined on a sample space Ω that associates a *real number* $X(\omega) = x$ with each possible outcome $\omega \in \Omega$.

Formally, we regard X as a (possibly many-to-one) mapping from Ω to $\mathbb{R}$

$$X: \quad \begin{aligned} \Omega &\longrightarrow \mathbb{R} \\ \omega &\longmapsto x \end{aligned}$$

In fact, X operates on events in Ω, so formally it is a function from the σ-algebra of subsets of Ω, to be denoted $\mathscr{C}$. Using suitably defined random variables, we can associate **any** sample space Ω (for **any** experiment) with a sample space that is a set of real numbers, in which the events are subsets of reals.

NOTE : Strictly, when referring to random variables, we should make explicit the connection to original sample space Ω, and write $X(\omega)$ for individual sample outcomes, and

$$P[X \in B] = P[\{\omega : X(\omega) \in B\}]$$

for events. However, generally, we will suppress this and merely refer to X.

For an event $E \subseteq \Omega$, $E \in \mathscr{C}$, we consider how X acts on E. If $E \equiv \{\omega_1, \omega_2, \ldots\}$, then

$$X(E) \equiv \{X(\omega_1), X(\omega_2), \ldots\} \equiv \{x \; : \; X(\omega) = x, \text{ for some } \omega \in E\}$$

Probability Considerations

We note the equivalence of events in Ω and their images under X, that is we must have

$$P(E) = \Pr[X \in X(E)].$$

Let P_X denote the probability function associated with X that operates on sets of real values. Consider a set of real values, B, and let

$$P_X(B) = \Pr[X \in B] = P(A)$$

where $A \equiv \{w \in \Omega \; : \; X(\omega) \in B\}$.

We will assign probability to subsets B of $\mathbb{R}$ that are equivalent to events (subsets) in Ω, and that form the basis of a σ-algebra of subsets of Ω. It is sufficient to consider sets B in the σ-algebra $\mathcal{B}$ that contains all open, half-open or closed subsets of the reals, that is, sets of the form

$$(a, b) \qquad (a, b] \qquad [a, b) \qquad [a, b]$$

and countable unions and intersections of these sets.

Probability Functions

Consider the real function of a real argument, F_X, defined by

$$F_X(x) = P_X((-\infty, x])$$

for real values of x. Note that

$$X \in (-\infty, x] \text{ is equivalent to } X \leq x$$

F_X defines the **probability distribution** of X. The nature of F_X determines how we can manipulate this function.

Three cases to consider:

1. F_X is a *step-function*, that is, F_X changes only at a certain (countable) set of x values.
2. F_X is a *continuous* function.
3. F_X is a mixture of 1. and 2.

1.8.1 DISCRETE RANDOM VARIABLES

Definition 1.12 A random variable X is **discrete** if F_X is a step-function.

If X is discrete then the set of all values at which F_X changes, to be denoted $\mathbb{X}$, is **countable**, that is

- $\mathbb{X} \equiv \{x_1, x_2, \ldots, x_n\}$ (that is, a **finite** list)
- $\mathbb{X} \equiv \{x_1, x_2, \ldots\}$ (that is, a countably **infinite** list).

If X is discrete, then it follows the event $X \in B$ can be decomposed

$$X \in B \qquad \Longleftrightarrow \qquad X \in \{x_i : x_i \in B\}$$

so that F_X can be represented as a sum of probabilities

$$F_X(x) = \sum_{x_i \leq x} P_X(X \in \{x_i\}) = \sum_{x_i \leq x} \Pr[X = x_i].$$

Definition 1.13 PROBABILITY MASS FUNCTION

The function f_X, defined on $\mathbb{X}$ by

$$f_X(x) = \Pr[X = x] \qquad x \in \mathbb{X}$$

that assigns probability to each $x \in \mathbb{X}$ is the (discrete) **probability mass function**, or **pmf**.

NOTE: For completeness, we define

$$f_X(x) = 0 \qquad x \notin \mathbb{X}$$

so that f_X is defined for all $x \in \mathbb{R}$. Thus $\mathbb{X}$ is the *support* of random variable X, that is, the set of $x \in \mathbb{R}$ such that $f_X(x) > 0$

PROPERTIES OF MASS FUNCTION f_X

Elementary properties of the mass function are straightforward to establish using the probability axioms.

A function f_X is a probability mass function for discrete random variable X with support $\mathbb{X}$ of the form $\{x_1, x_2, \ldots\}$ if and only if

$$\text{(i) } f_X(x_i) \geq 0 \qquad\qquad \text{(ii) } \sum_i f_X(x_i) = 1$$

Clearly as $f_X(x) = \Pr[X = x]$, we must have that

$$0 \leq f_X(x) \leq 1$$

for all $x \in \mathbb{R}$.

Definition 1.14 DISCRETE CUMULATIVE DISTRIBUTION FUNCTION

The **cumulative distribution function**, or **cdf**, F_X of a discrete r.v. X is defined by

$$F_X(x) = \Pr[X \leq x] \qquad x \in \mathbb{R}.$$

Connection between F_X and f_X: Let X be a discrete random variable with support $\mathbb{X} \equiv \{x_1, x_2, \ldots\}$, where

$$x_1 < x_2 < \ldots$$

and pmf f_X and cdf F_X. For any real value x, if $x < x_1$, then $F_X(x) = 0$, and for $x \geq x_1$,

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i)$$

so that, for $i = 2, 3, \ldots$,

$$f_X(x_i) = F_X(x_i) - F_X(x_{i-1})$$

with, for completeness, $f_X(x_1) = F_X(x_1)$.

1.8.2 PROPERTIES OF DISCRETE CDF F_X

(i) In the limiting cases,

$$\lim_{x \to -\infty} F_X(x) = 0 \qquad\qquad \lim_{x \to \infty} F_X(x) = 1.$$

(ii) F_X is **continuous from the right** (but not continuous) on $\mathbb{R}$ that is, for $x \in \mathbb{R}$,

$$\lim_{h \to 0^+} F_X(x+h) = F_X(x)$$

but, if $x \in \mathbb{X}$,

$$\lim_{h \to 0^+} F_X(x-h) \neq F_X(x)$$

that is, the "left limit" is not equal to the "right limit" at x values in $\mathbb{X}$.

(iii) F_X is **non-decreasing**, that is

$$a < b \Longrightarrow F_X(a) \leq F_X(b)$$

(iv) For $a < b$,

$$P[a < X \leq b] = F_X(b) - F_X(a)$$

Notes:

- The functions f_X and/or F_X can both be used to describe the probability distribution of random variable X.
- The function f_X is non-zero only at the elements of $\mathbb{X}$.
- The function F_X is a **step-function**, which takes the value zero at minus infinity, the value one at infinity, and is non-decreasing with points of discontinuity at the elements of $\mathbb{X}$.
- The right-continuity of F_X is denoted in plots by the use of a filled circle, $\bullet$, as in the example below.
- In the discrete case, F_X is **not differentiable** for all $x \in \mathbb{R}$; at points of continuity (that is, for $x \notin \mathbb{X}$), it is differentiable, and the derivative is zero.

1.8.3 CONTINUOUS RANDOM VARIABLES

Definition 1.15 A random variable X is **continuous** if the function F_X defined on $\mathbb{R}$ by

$$F_X(x) = P[X \leq x]$$

for $x \in \mathbb{R}$ is a **continuous** function on $\mathbb{R}$, that is, for $x \in \mathbb{R}$,

$$\lim_{h \to 0} F_X(x+h) = F_X(x).$$

Definition 1.16 CONTINUOUS CUMULATIVE DISTRIBUTION FUNCTION
The **cumulative distribution function**, or **cdf**, F_X of a continuous r.v. X is defined by

$$F_X(x) = P[X \leq x] \qquad x \in \mathbb{R}.$$

Definition 1.17 PROBABILITY DENSITY FUNCTION
A random variable is **absolutely continuous** if the cumulative distribution function F_X can be written

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

for some function f_X, termed the **probability density function**, or **pdf**, of X. For any suitable set B,

$$\Pr[X \in B] = \int_B f_X(x)\, dx$$

Directly from the definition, at values of x where F_X is differentiable x,

$$f_X(x) = \frac{d}{dt}\{F_X(t)\}_{t=x}$$

PROPERTIES OF CONTINUOUS F_X AND f_X

(i) The pdf f_X need not exist, but continuous r.v.s where f_X cannot be defined in this way will be ignored. The function f_X can be defined piecewise on intervals of $\mathbb{R}$.

(ii) For the cdf of a continuous r.v.,

$$\lim_{x \to -\infty} F_X(x) = 0 \qquad \lim_{x \to \infty} F_X(x) = 1$$

(iii) If X is continuous,

$$f_X(x) \neq P[X = x] = \lim_{h \to 0}[F_X(x+h) - F_X(x)] = 0$$

(iv) For $a < b$,

$$P[a < X \leq b] = P[a \leq X < b] = P[a \leq X \leq b] = P[a < X < b] = F_X(b) - F_X(a)$$

It follows that a function f_X is a pdf for a continuous random variable X **if and only if**

(i) $f_X(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f_X(x) dx = 1$

This result follows direct from definitions and properties of F_X.

Note that in the continuous case, there is no requirement that f_X is bounded above.

Example 1.5 Consider a coin tossing experiment where a fair coin is tossed repeatedly under identical experimental conditions, with the sequence of tosses independent, until a Head is obtained. For this experiment, the sample space, Ω is then the set of sequences

$$(\{H\}, \{TH\}, \{TTH\}, \{TTTH\} \ldots)$$

with associated probabilities 1/2, 1/4, 1/8, 1/16, ...

Define discrete random variable $X : \Omega \longrightarrow \mathbb{R}$, by $X(\omega) = x \Longleftrightarrow$ first H on toss x. Then

$$f_X(x) = P[X = x] = \left(\frac{1}{2}\right)^x \qquad x = 1, 2, 3, \ldots$$

and zero otherwise. For $x \geq 1$, let $k(x)$ be the largest integer not greater than x. Then

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i) = \sum_{i=1}^{k(x)} f_X(i) = 1 - \left(\frac{1}{2}\right)^{k(x)}$$

and $F_X(x) = 0$ for $x < 1$.

Graphs of the probability mass function (top) and cumulative distribution function (bottom) are shown in Figure 3. Note that the mass function is only non-zero at points that are elements of X, and that the cdf is defined for all real values of x, but is only continuous from the right. F_X is therefore a step-function.

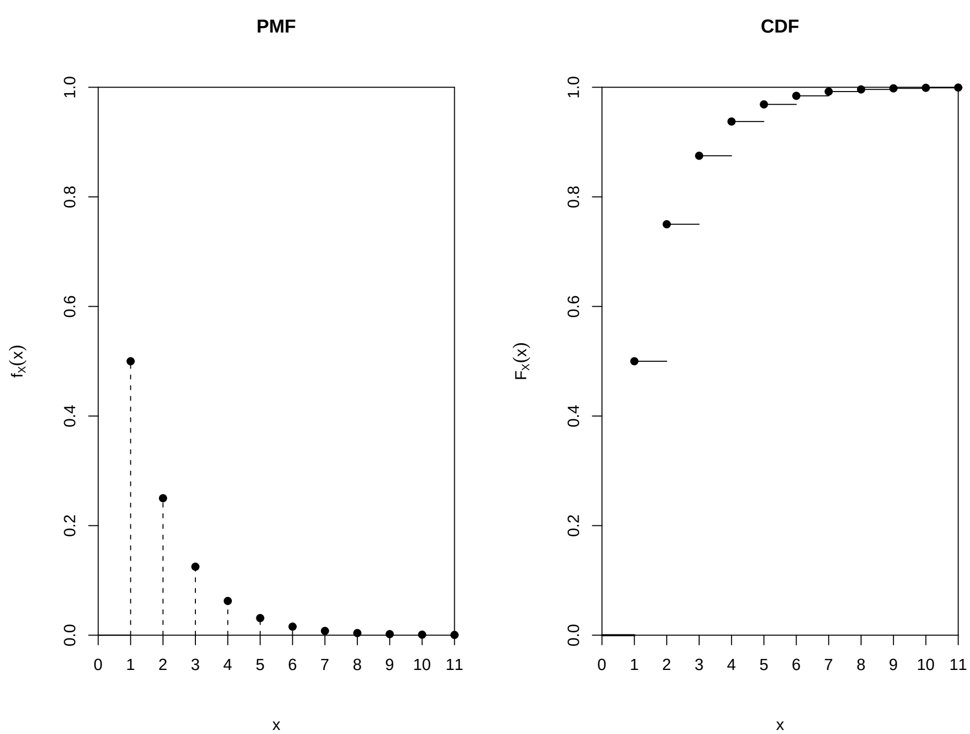

Figure 3: PMF $f_X(x) = \left(\frac{1}{2}\right)^x, x = 1, 2, 3, \ldots$ and CDF $F_X(x) = 1 - \left(\frac{1}{2}\right)^{k(x)}$

Example 1.6 Consider an experiment to measure the length of time that an electrical component functions before failure. The sample space of outcomes of the experiment, Ω is $^+$, and if A_x is the event that the component functions for longer than $x > 0$ time units, suppose that $\mathrm{P}(A_x) = \exp\left\{-x^2\right\}$.

Define continuous random variable $X : \Omega \longrightarrow \mathbb{R}^+$, by $X(\omega) = x \iff$ component fails at time x. Then, if $x > 0$,

$$F_X(x) = P[X \leq x] = 1 - P(A_x) = 1 - \exp\left\{-x^2\right\}$$

and $F_X(x) = 0$ if $x \leq 0$. Hence if $x > 0$,

$$f_X(x) = \frac{d}{dt}\left\{F_X(t)\right\}_{t=x} = 2x\exp\left\{-x^2\right\}$$

and zero otherwise.

Graphs of the probability density function (top) and cumulative distribution function (bottom) are shown in Figure 4. Note that both the pdf and cdf are defined for all real values of x, and that both are continuous functions.

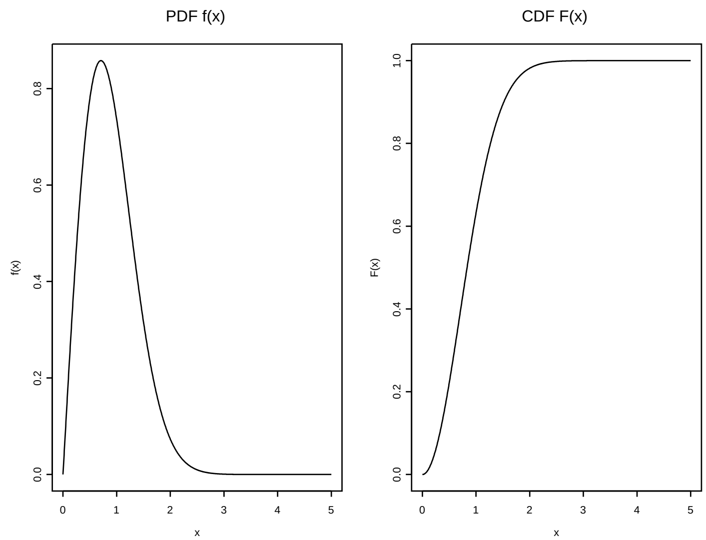

Figure 4: PDF $f_X(x) = 2x\exp\left\{-x^2\right\},\ x > 0$, and CDF $F_X(x) = 1 - \exp\left\{-x^2\right\},\ x > 0$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_0^x f_X(t)dt$$

as $f_X(x) = 0$ for $x \leq 0$, and also that

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^{\infty} f_X(x)dx = 1.$$