MATH 556 - EXERCISES 3 : SOLUTIONS

1 (a) By direct calculation, using a derivation as in lecture notes,

$$C_X(t) = E_{f_X}[e^{itX}] = \int_{-\infty}^{\infty} e^{itx}\frac{1}{2}|x|\exp\{-|x|\}\,dx = \int_0^{\infty}\cos(tx)xe^{-x}\,dx$$

as the pdf is an even function around zero. Integrating by parts,

$$\begin{aligned}
C_X(t) &= \left[\frac{1}{t}\sin(tx)xe^{-x}\right]_0^{\infty} - \frac{1}{t}\int_0^{\infty}\sin(tx)[-xe^{-x}+e^{-x}]\,dx\\
&= \frac{1}{t}\int_0^{\infty}\sin(tx)xe^{-x}\,dx - \frac{1}{t}\int_0^{\infty}\sin(tx)e^{-x}\,dx\\
&= \frac{1}{t}\left[-\frac{1}{t}\cos(tx)xe^{-x}\right]_0^{\infty} + \frac{1}{t^2}\int_0^{\infty}\cos(tx)[-xe^{-x}+e^{-x}]\,dx\\
&\qquad -\frac{1}{t}\left[-\frac{1}{t}\cos(tx)e^{-x}\right]_0^{\infty} + \frac{1}{t^2}\int_0^{\infty}\cos(tx)e^{-x}\,dx\\
&= -\frac{1}{t^2}\int_0^{\infty}\cos(tx)xe^{-x}\,dx - \frac{1}{t^2} + \frac{2}{t^2}\int_0^{\infty}\cos(tx)e^{-x}\,dx
\end{aligned}$$

where the first term is equal to $-C_X(t)/t^2$. Now

$$\begin{aligned}
\int_0^{\infty}\cos(tx)e^{-x}\,dx &= \left[-\frac{1}{t}\sin(tx)e^{-x}\right]_0^{\infty} - \frac{1}{t}\int_0^{\infty}\sin(tx)e^{-x}\,dx = 0 - \frac{1}{t}\int_0^{\infty}\sin(tx)e^{-x}\,dx\\
&= -\frac{1}{t}\left[-\frac{1}{t}\cos(tx)e^{-x}\right]_0^{\infty} - \frac{1}{t^2}\int_0^{\infty}\cos(tx)e^{-x}\,dx = 0 - \frac{1}{t^2}\int_0^{\infty}\cos(tx)e^{-x}\,dx
\end{aligned}$$

Thus, on rearrangement, verifying the result from lectures, we have

$$\int_0^{\infty}\cos(tx)e^{-x}\,dx = \frac{1}{1+t^2}.$$

Returning to the previous expression, we have that

$$C_X(t) = -\frac{1}{t^2}C_X(t) - \frac{1}{t^2} + \frac{2}{t^2}\frac{1}{1+t^2} \qquad \therefore \qquad C_X(t) = \frac{1-t^2}{(1+t^2)^2} \qquad t\in\mathbb{R}.$$

Note that an alternative proof can be obtained using Gamma integrals: we have that

$$\begin{aligned}
C_X(t) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{2}|x|e^{-|x|}\,dx &= \frac{1}{2}\int_0^{\infty}e^{itx}xe^{-x}\,dx + \frac{1}{2}\int_0^{\infty}e^{-itx}xe^{-x}\,dx\\
&= \frac{1}{2}\int_0^{\infty}xe^{-x(1-it)}\,dx + \frac{1}{2}\int_0^{\infty}xe^{-x(1+it)}\,dx
\end{aligned}$$

Now, by appealing to the Gamma pdf integral, we can deduce that

$$\int_0^{\infty}xe^{-x(1-it)}\,dx = \frac{\Gamma(2)}{(1-it)^2} \qquad \text{and} \qquad \int_0^{\infty}xe^{-x(1+it)}\,dx = \frac{\Gamma(2)}{(1+it)^2}$$

so therefore, as $\Gamma(2)=1$,

$$C_X(t) = \frac{1}{2}\left[\frac{\Gamma(2)}{(1-it)^2} + \frac{\Gamma(2)}{(1+it)^2}\right] = \frac{1-t^2}{(1+t^2)^2}$$

Note that the Gamma function notation usage here is legitimate; the Gamma function is defined for complex arguments. In fact, the complex integral is technically more complicated than it appears. Fortunately the mgf exists in a neighbourhood of zero, so we can mimic the entire computation by looking at the mgf, M_X, and then substituting in it for t at the last line.

(b) By direct calculation

$$C_X(t) = \int_{-\infty}^{\infty} e^{itx} \exp\{-x - e^{-x}\}\, dx = \int_{-\infty}^{\infty} \exp\{-(1-it)x - e^{-x}\}\, dx = \int_{0}^{\infty} y^{-it} e^{-y}\, dx = \Gamma(1-it)$$

after setting $y = e^{-x}$, using the Gamma integral technique from above.

(c) By direct calculation, using the series expansion and integrating term by term;

$$\begin{aligned} C_X(t) = E_{f_X}[e^{itX}] &= \int_{-\infty}^{\infty} e^{itx} \sum_{k=0}^{\infty} (-1)^k \exp\{-(2k+1)\pi|x|\}\, dx \\ &= \sum_{k=0}^{\infty} (-1)^k \int_{-\infty}^{\infty} e^{itx} \exp\{-(2k+1)\pi|x|\}\, dx \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \int_{-\infty}^{\infty} e^{i(t/(2k+1)\pi)y} \exp\{-|y|\}\, dy \end{aligned}$$

setting $y = (2k+1)\pi x$, after exchanging the order of integration and differentiation, which is legitimate in this context as the sum and integral are convergent. Using the result from lectures computing the cf for the Double Exponential distribution,

$$\int_{-\infty}^{\infty} e^{ity} \frac{1}{2} \exp\{-|y|\}\, dy = \frac{1}{1+t^2} \quad \therefore \quad \int_{-\infty}^{\infty} e^{i\frac{t}{(2k+1)\pi}y} \exp\{-|y|\}\, dy = \frac{2}{1 + \left\{\dfrac{t}{(2k+1)\pi}\right\}^2}$$

so that

$$C_X(t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \frac{2}{1 + \left\{\dfrac{t}{(2k+1)\pi}\right\}^2} = \sum_{k=0}^{\infty} (-1)^k \frac{2(2k+1)\pi}{(2k+1)^2\pi^2 + t^2} \tag{1}$$

It can be shown using advanced methods (see the course website) that

$$C_X(t) = \frac{2e^{t/2}}{e^{t/2} + e^{-t/2}} = \frac{1}{\cosh(t/2)} \qquad t \in \mathbb{R}$$

and this result can be verified using the inversion formula. It is then clear that the pdf and its cf are identical in form.

There is a comprehensive list of cfs and integrals in the books

- W. Feller *An Introduction to Probability Theory and Its Applications, Volume 2*, (1971).
- M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables* (1965), Dover Publications.
- I. S. Gradshteyn, I. M. Ryzhik et al., *Table of Integrals, Series, and Products, Sixth Edition* (2000), Academic Press.

2 The cf is integrable, and $|C_X(t)| \longrightarrow 0$ as $|t| \longrightarrow \infty$, so the inversion formula for continuous pdfs can be used:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} C_X(t)\, dt \quad = \quad \frac{1}{2\pi}\int_{-1}^{1} e^{-itx}(1-|t|)\, dt = \frac{1}{\pi}\int_{0}^{1}(1-t)\cos(tx)\, dt$$

after writing $e^{-itx} = \cos(tx) - i\sin(tx)$, and splitting the integral into two halves over $(-1 < t < 0)$ and $(0 < t < 1)$ in the usual way. Integrating by parts yields

$$f_X(x) = \frac{1}{\pi x^2}(1-\cos x) \qquad x \in \mathbb{R}$$

It is straightforward to verify that this is a valid pdf.

3 (a) Here, $|C_X(t)| \longrightarrow 0$ as $|t| \longrightarrow \infty$, so we are dealing with a continuous distribution. Thus, by the inversion formula,

$$f_X(x) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} C_X(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\frac{2(1-\cos t)}{t^2}\, dt$$

But note that here $C_X(t)$ is of the same functional form (up to proportionality) as the pdf found for question 2. Thus, using identical arguments to those given in lectures, we must have that

$$f_X(x) \propto (1-|x|) \qquad -1 < x < 1$$

and zero otherwise. In fact

$$f_X(x) = (1-|x|) \qquad -1 < x < 1$$

and zero otherwise.

(b) Here, C_X is in the form of a pure $\cos$ function, and therefore can be written as the sum of complex exponentials, that is

$$C_X(t) = \cos(\theta t) = \frac{1}{2}\left[(\cos(\theta t) + i\sin(\theta t)) + (\cos(\theta t) - i\sin(\theta t))\right] = \frac{1}{2}e^{it\theta} + \frac{1}{2}e^{-it\theta}$$

and hence it follows that X has a **discrete distribution** with pmf given by

$$f_X(x) = \begin{cases} 1/2 & x = -\theta \\ 1/2 & x = \theta \\ 0 & \text{otherwise} \end{cases}$$

4 Suppose that $C(t)$ is a valid cf for random variable X. Then, on repeated differentiation, we see that

$$C^{(1)}(0) = C^{(2)}(0) = C^{(3)}(0) = 0.$$

Hence, we can deduce from the results given in lectures about moments that

$$E_{f_X}[X] = E_{f_X}[X^2] = Var_{f_X}[X] = 0$$

so that f_X must be a **degenerate** distribution, with $\Pr[X = x_0] = 1$ for some x_0. But this conflicts with the properties of C, namely that $C(t) \longrightarrow 0$ as $t \longrightarrow \pm\infty$, so C cannot be a valid cf.

5 Let $C_\epsilon(t)$ denote the characteristic-type function associated with g_ϵ, that is

$$C_\epsilon(t) = \int_{-\infty}^{\infty} e^{itx} g_\epsilon(t)\ dt.$$

Then

$$\begin{aligned} |C_X(t) - C_\epsilon(t)| &= \left| \int_{-\infty}^{\infty} e^{itx}(f_X(x) - g_\epsilon(x))\ dx \right| \leq \int_{-\infty}^{\infty} \left| e^{itx}(f_X(x) - g_\epsilon(x)) \right|\ dx \\ &\leq \int_{-\infty}^{\infty} |(f_X(x) - g_\epsilon(x))|\ dx < \epsilon \end{aligned}$$

by assumption, and hence $C_X(t)$ and $C_\epsilon(t)$ are (uniformly) arbitrarily close. But

$$C_\epsilon(t) = \int_{-\infty}^{\infty} e^{itx} g_\epsilon(x)\ dx = \int_{-\infty}^{\infty} e^{itx} \left\{ \sum_{k=1}^{K} c_k I_{A_k}(x) \right\}\ dx = \sum_{k=1}^{K} c_k \frac{e^{iu_k t} - e^{il_k t}}{it}$$

where $A_k = (l_k, u_k]$, say, with $l_1 = -\infty$ and $u_K = \infty$. Thus

$$|C_\epsilon(t)| \leq \frac{2}{t} \sum_{k=1}^{K} c_k \longrightarrow 0 \qquad \text{as} \qquad |t| \longrightarrow \infty.$$

as $|(e^{iu_k t} - e^{il_k t})/i| \leq |e^{iu_k t}/i| + |e^{il_k t}/i| \leq 2$. Hence, as $|C_\epsilon(t)| \longrightarrow 0$, $|C_X(t)| \longrightarrow 0$ also.

6 If $X \sim Cauchy$, then $Y = 1/X$ has pdf given by the general transformation theorem

$$f_Y(y) = f_X(1/y) \times |J(y)| = \frac{1}{\pi} \frac{1}{1 + (1/y)^2}\ \times\ |-1/y^2| = \frac{1}{\pi} \frac{1}{1 + y^2} \qquad y \in \mathbb{R}$$

so in fact $Y \sim Cauchy$ also. Now, the sample mean rv $\overline{X}$ has characteristic function given by

$$\{C_X(t/n)\}^n = \{\exp\{-|t/n|\}\}^n = \exp\{-|t|\}$$

so $\overline{X}$ also has a $Cauchy$ distribution. Combining these results gives that $Z_n = 1/\overline{X} \sim Cauchy$. The cdf of the Cauchy distribution takes the form

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1 + y^2} dy = \frac{1}{\pi} \arctan(x) + \frac{1}{2} \qquad x \in \mathbb{R}$$

and hence

$$P[|Z_n| \leq c] = F_X(c) - F_X(-c) = \frac{1}{\pi} \arctan(c) - \frac{1}{\pi} \arctan(-c) = \frac{2}{\pi} \arctan(c)$$

7 From the course formula sheet, if $X \sim Gamma(\alpha, \beta)$, then the mgf for X is given by

$$M_X(t) = \left(\frac{\beta}{\beta - t} \right)^{\alpha} \qquad -\beta < t < \beta,$$

say. If we take

$$Z_{nj} \sim Gamma(\alpha/n, \beta) \qquad j = 1, \ldots, n$$

as a collection of iid variables, and define Z_n as their sum, then the mgf of Z_n is

$$\left\{ \left(\frac{\beta}{\beta - t} \right)^{\alpha/n} \right\}^n = \left(\frac{\beta}{\beta - t} \right)^{\alpha}$$

and Z_n and X have the same distribution. We compute the corresponding cfs by substitution. Hence X has an infinitely divisible distribution.