MATH 556 - EXERCISES 2: SOLUTIONS

1 We have $f_R(r) = 6r(1-r)$, for $0 < r < 1$, and hence

$$F_R(r) = r^2(3-2r) \qquad 0 < r < 1$$

with the usual cdf behaviour outside of this range.

- Circumference: $Y = 2\pi R$, so $\mathbb{Y} = (0, 2\pi)$, and from first principles, for $y \in \mathbb{Y}$,

$$\begin{aligned} F_Y(y) &= \mathrm{P}[\, Y \le y \,] = \mathrm{P}[\, 2\pi R \le y \,] = \mathrm{P}[\, R \le y/2\pi \,] = F_R(y/2\pi) = \frac{3y^2}{4\pi^2} - \frac{2y^3}{8\pi^3} \\ \implies f_Y(y) &= \frac{6y}{8\pi^3}(2\pi - y) \qquad 0 < y < 2\pi \end{aligned}$$

- Area: $Z = \pi R^2$, so $\mathbb{Z} = (0, \pi)$, and from first principles, for $z \in \mathbb{Z}$, recalling that f_R is only positive when $0 < z < \pi$,

$$\begin{aligned} F_Z(z) &= \mathrm{P}[\, Z \le z \,] = \mathrm{P}[\, \pi R^2 \le z \,] = \mathrm{P}[\, R \le \sqrt{z/\pi} \,] = F_R(z/2\pi) = \frac{3z}{\pi} - 2\left\{\frac{z}{\pi}\right\}^{3/2} \\ \implies f_Z(z) &= 3\pi^{-3/2}(\sqrt{\pi} - \sqrt{z}) \qquad 0 < z < \pi. \end{aligned}$$

2 If $\mathbb{X}^{(2)} = (0,1) \times (0,1)$ is the (joint) range of vector random variable (X, Y). We have

$$f_{X,Y}(x,y) = cx(1-y) \qquad 0 < x < 1,\ 0 < y < 1$$

so that

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \qquad \text{and} \qquad \mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are the ranges of X and Y respectively, and

$$f_X(x) = c_1 x \qquad \text{and} \qquad f_Y(y) = c_2(1-y) \tag{1}$$

for some constants satisfying $c_1 c_2 = c$. Hence, the two conditions for independence are satisfied in (1), and X and Y are independent.

Secondly, we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dxdy = 1 \qquad\qquad c^{-1} = \int_0^1\int_0^1 x(1-y)\, dxdy = 1$$

and as

$$\int_0^1\int_0^1 x(1-y)\, dxdy = \left\{\int_0^1 x\, dx\right\}\left\{\int_0^1 (1-y)\, dy\right\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

we have $c = 4$.

Finally, we have $A = \{(x,y) \,:\, 0 < x < y < 1\}$, and hence, recalling that the joint density is only non-zero when $x < y$, we first fix a y and integrate dx on the range $(0, y)$, and then integrate dy on the range $(0,1)$, that is

$$\begin{aligned} P[\, X < Y \,] &= \iint\limits_A f_{X,Y}(x,y)\, dxdy = \int_0^1 \left\{\int_0^y 4x(1-y)\, dx\right\} dy \\ &= \int_0^1 \left\{\int_0^y x\, dx\right\} 4(1-y)\, dy = \int_0^1 2y^2(1-y)\, dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^4\right]_0^1 = \frac{1}{6} \end{aligned}$$

3 First sketch the support of the density; this will make it clear that the boundaries of the support are different for $0 < y \le 1$ and $y > 1$.

(i) The marginal distributions are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{1/x}^{x} \frac{1}{2x^2} y\, dy = \frac{1}{2x^2}(\log x - \log(1/x)) = \frac{\log x}{x^2} \qquad 1 \le x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \begin{cases} \displaystyle\int_{1/y}^{\infty} \frac{1}{2x^2 y}\, dx = \frac{1}{2} & 0 \le y \le 1 \\ \displaystyle\int_{y}^{\infty} \frac{1}{2x^2 y}\, dx = \frac{1}{2y^2} & 1 \le y \end{cases}$$

(ii) Conditionals:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{x^2 y} & 1/y \le x \text{ if } 0 \le y \le 1 \\ \dfrac{y}{x^2} & y \le x \text{ if } 1 \le y \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{2y \log x} \qquad 1/x \le y \le x \text{ if } x \ge 1$$

(iii) Marginal expectation of Y;

$$E_{f_Y}[\,Y\,] = \int -\infty \infty y f_Y(y)\, dy = \int_0^1 \frac{y}{2}\, dy + \int_1^{\infty} \frac{1}{2y}\, dy = \infty$$

as the second integral is divergent.

4 (i) We set

$$\begin{array}{l} U = X/Y \\ V = -\log(XY) \end{array} \iff \begin{array}{l} X = U^{1/2} e^{-V/2} \\ Y = U^{-1/2} e^{-V/2} \end{array}$$

note that, as X and Y lie in $(0,1)$ we have $XY < X/Y$ and $XY < Y/X$, giving constraints $e^{-V} < U$ and $e^{-V} < 1/U$, so that $0 < e^{-V} < \min\{U, 1/U\}$. The Jacobian of the transformation is

$$|J(u,v)| = \left| \begin{array}{cc} \dfrac{u^{-1/2} e^{-v/2}}{2} & -\dfrac{u^{1/2} e^{-v/2}}{2} \\ -\dfrac{u^{-3/2} e^{-v/2}}{2} & -\dfrac{u^{-1/2} e^{-v/2}}{2} \end{array} \right| = u^{-1} e^{-v}/2.$$

Hence

$$f_{U,V}(u,v) = u^{-1} e^{-v}/2 \qquad 0 < e^{-v} < \min\{u, 1/u\},\ u > 0$$

The corresponding marginals are given below: let $g(y) = -\log(\min\{u, 1/u\})$, then

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \int_{g(y)}^{\infty} \frac{e^{-v}}{2u}\, dv = \left[-\frac{e^{-v}}{2u}\right]_{g(y)}^{\infty} = \frac{\min\{u, 1/u\}}{2u} \qquad u > 0$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, du = \int_{e^{-v}}^{e^{v}} \frac{e^{-v}}{2u}\, du = \left[\frac{\log u}{2} e^{-v}\right]_{e^{-v}}^{e^{v}} = v e^{-v} \qquad v > 0$$

(ii) Now let

$$\begin{matrix} V = X + Y \\ \\ Z = X - Y \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} X = \dfrac{V+Z}{2} \\ \\ Y = \dfrac{V-Z}{2} \end{matrix}$$

and the Jacobian of the transformation is $1/2$. The transformed variables take values on the square A in the (V, Z) plane with corners at $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$ bounded by the lines $z = -v$, $z = 2 - v$, $z = v$ and $z = v - 2$. Then

$$f_{V,Z}(v,z) = \frac{1}{2} \qquad (v,z) \in A$$

and zero otherwise (sketch the square A). Hence, integrating in horizontal strips in the (V, Z) plane,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{V,Z}(v,z)\, dv = \begin{cases} \displaystyle\int_{-z}^{2+z} \frac{1}{2}\, dv & = 1 + z \qquad -1 < z \leq 0 \\ \\ \displaystyle\int_{z}^{2-z} \frac{1}{2}\, dv & = 1 - z \qquad\quad 0 < z < 1 \end{cases}$$

5 We have $K_X(t) = \log M_X(t)$, hence

$$K_X^{(1)}(t) = \frac{d}{ds}\{K_X(t)\}_{s=t} = \frac{d}{ds}\{\log M_X(t)\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = E_{f_X}[\, X\,]$$

as $M_X(0) = 1$. Similarly

$$K_X^{(2)}(t) = \frac{M_X(t) M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\{M_X(t)\}^2}$$

and hence

$$K_X^{(2)}(0) = \frac{M_X(0) M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\{M_X(0)\}^2} = E_{f_X}[\, X^2\,] - \{E_{f_X}[\, X\,]\}^2$$

and hence $K_X^{(2)}(0) = Var_{f_X}[\, X\,]$

6 (i) Put $U = X/Y$ and $V = Y$; the inverse transformations are therefore $X = UV$ and $Y = V$. In terms of the multivariate transformation theorem, we have transformation functions defined by

$$\begin{aligned} g_1(t_1, t_2) &= t_1/t_2 \qquad & g_1^{-1}(t_1, t_2) &= t_1 t_2 \\ g_2(t_1, t_2) &= t_2 \qquad & g_2^{-1}(t_1, t_2) &= t_2 \end{aligned}$$

and the Jacobian of the transformation is given by

$$|J(u,v)| = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = |v|$$

and hence

$$f_{U,V}(u,v) = f_{X,Y}(uv, v)\ |v| = \left(\frac{1}{2\pi}\right) \exp\left\{-\frac{1}{2}(u^2v^2 + v^2)\right\} |v| \qquad (u,v) \in \mathbb{R}^2$$

and zero otherwise, and so, for any real u,

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv \quad = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right) \exp\left\{-\frac{1}{2}(u^2v^2 + v^2\right\} |v| \; dv$$

$$= \left(\frac{1}{\pi}\right) \int_0^{\infty} v \exp\left\{-\frac{v^2}{2}(1+u^2)\right\} \; dv$$

$$= \left(\frac{1}{\pi}\right) \left[-\frac{1}{(1+u^2)} \exp\left\{-\frac{v^2}{2}(1+u^2)\right\}\right]_0^{\infty} = \frac{1}{\pi(1+u^2)}$$

with the final step following by direct integration. Thus U has a *Cauchy* distribution.

(ii) Now put $T = X/\sqrt{S/\nu}$ and $R = S$; the inverse transformations are therefore $X = T\sqrt{R/\nu}$ and $S = R$. In terms of the multivariate transformation theorem, we have transformation functions from $(X, S) \to (T, R)$ defined by

$$g_1(t_1, t_2) = t_1/\sqrt{t_2/\nu} \qquad\qquad g_1^{-1}(t_1, t_2) = t_1\sqrt{t_2/\nu}$$

$$g_2(t_1, t_2) = t_2 \qquad\qquad g_2^{-1}(t_1, t_2) = t_2$$

and the Jacobian of the transformation is given by

$$|J(t,r)| = \begin{vmatrix} \sqrt{\dfrac{r}{\nu}} & \dfrac{t}{2\sqrt{r\nu}} \\ 0 & 1 \end{vmatrix} = \left|\sqrt{\frac{r}{\nu}}\right| = \sqrt{\frac{r}{\nu}}$$

and hence

$$f_{T,R}(t,r) = f_{X,S}\left(t\sqrt{\frac{r}{\nu}}, r\right)\sqrt{\frac{r}{\nu}} \quad = \quad f_X\left(t\sqrt{\frac{r}{\nu}}\right) f_S(r) \sqrt{\frac{r}{\nu}} \qquad t \in \mathbb{R}, s \in \mathbb{R}^+$$

and zero otherwise, and so, for any real t,

$$f_T(t) \quad = \int_{-\infty}^{\infty} f_{T,R}(t,r)\, dr$$

$$= \int_0^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{rt^2}{2\nu}\right\} \frac{(1/2)^{(\nu/2)}}{\Gamma(\nu/2)} r^{\nu/2-1} e^{-r/2} \sqrt{\frac{r}{\nu}}\, dr$$

$$= \left(\frac{1}{2\pi}\right)^{1/2} \frac{(1/2)^{(\nu/2)}}{\Gamma(\nu/2)} \frac{1}{\sqrt{\nu}} \int_0^{\infty} r^{(\nu+1)/2-1} \exp\left\{-\frac{r}{2}\left(1+\frac{t^2}{\nu}\right)\right\}\, dr$$

$$= \left(\frac{1}{2\pi}\right)^{1/2} \frac{(1/2)^{(\nu/2)}}{\sqrt{\nu}\,\Gamma(\nu/2)} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \int_0^{\infty} z^{(\nu+1)/2-1} \exp\left\{-\frac{z}{2}\right\}\, dz$$

after setting

$$z = r\left(1+\frac{t^2}{\nu}\right).$$

Hence

$$f_T(t) \quad = \quad = \left(\frac{1}{2\pi}\right)^{1/2} \frac{(1/2)^{(\nu/2)}}{\sqrt{\nu}\,\Gamma(\nu/2)} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \frac{\Gamma((\nu+1)/2+1)}{(1/2)^{(\nu+1)/2}}$$

as the integrand is proportional to a Gamma pdf. Thus

$$f_T(t) = \frac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)} \left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{(1+t^2/\nu)^{(\nu+1)/2}}$$

which is the $Student(\nu)$ density.

(iii) We have that $X|Y = y \sim N(0, y^{-1})$ and $Y \sim Gamma(\nu/2, \nu/2)$. Now, we have

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y) \qquad x \in \mathbb{R}, y \in \mathbb{R}^+$$

and zero otherwise, and so, for any real x,

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \\
&= \int_0^{\infty} \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \frac{\left(\dfrac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\dfrac{\nu}{2}\right)} y^{\nu/2-1} e^{-\nu y/2}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \frac{\left(\dfrac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\dfrac{\nu}{2}\right)} \int_0^{\infty} y^{(\nu+1)/2-1} \exp\left\{-\frac{y}{2}\left(\nu + x^2\right)\right\} dy \\
&= \frac{1}{\sqrt{2\pi}} \frac{\left(\dfrac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\dfrac{\nu}{2}\right)} \frac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\left(\dfrac{1}{2}\left(\nu+x^2\right)\right)^{(\nu+1)/2}}
\end{aligned}$$

as the integrand is proportional to a Gamma pdf. Therefore f_X is given by

$$f_X(x) = \frac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)} \left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{(1+x^2/\nu)^{(\nu+1)/2}}$$

which is again the $Student(\nu)$ density.

Exercise 6 give the two alternative ways of specifying the Student-t distribution, either as a function of independent Normal and Gamma/Chi-squared variables, or as the marginal obtained by "*scale-mixing*" a Normal distribution by a Gamma distribution (that is, rather than having a fixed variance $\sigma^2 = 1/Y$; we regard Y as a *random variable* having a Gamma distribution, so that (X,Y) have a joint distribution

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y)$$

from which we calculate $f_X(x)$ by integration.