MATH 556 - PRACTICE EXAM QUESTIONS: SOLUTIONS

1. Due to the symmetry of form, this joint pdf factorizes simply as

$$f_{X,Y}(x,y) = \left\{ \sqrt{c_1} \exp\left\{-\frac{x}{2}\right\} \right\} \left\{ \sqrt{c_1} \exp\left\{-\frac{y}{2}\right\} \right\} = f_X(x) f_Y(y) \qquad x,y > 0$$

and hence the variables are independent. Now

$$\int_0^\infty \exp\left\{-\frac{x}{2}\right\} dx = 2$$

so therefore $\sqrt{c_1} = \frac{1}{2}$, and hence $c_1 = \frac{1}{4}$.
Now random variable U, defined by

$$U = \frac{1}{2}(X - Y).$$

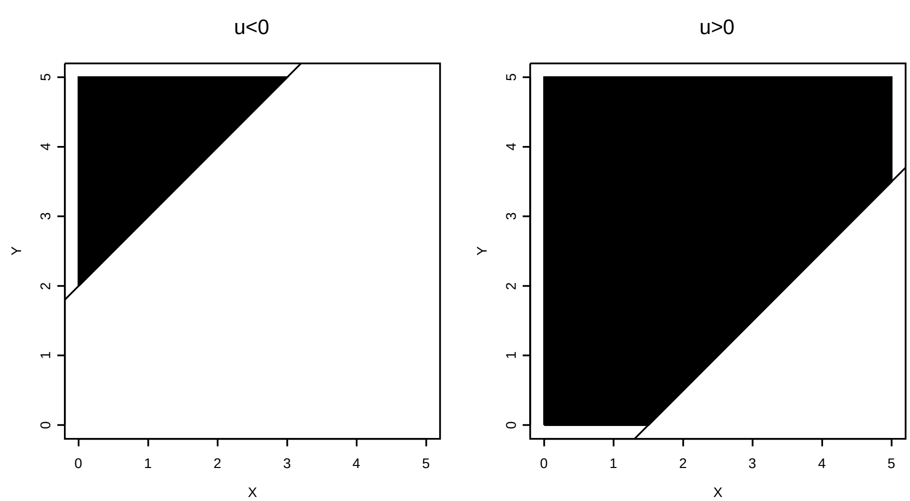
has range $\mathbb{U} \equiv R$ (X and Y are positive but unbounded random variables. Hence, for $u \in \mathbb{R}$, the cdf of U, F_U, is given by

$$F_U(u) = P[U \leq u] = P\left[\frac{1}{2}(X - Y) \leq u\right] = \iint_{A_u} f_{X,Y}(x,y)\, dx dy$$

where $A_u \equiv \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : (x-y)/2 \leq u\}$. The boundary of the region A_u is determined by the three lines

$$x = 0,\ y = 0 \text{ and } y = x - 2u$$

This region is shaded in black in the figures below in the two cases $u < 0$ and $u \geq 0$ respectively; in these pictures the shaded region extends over all x and y above and to the left of the line $y = x - 2u$.



Integrating first dx for a fixed y, we see that the integral is always $x = 0$ to $x = y+2u$, irrespective of whether $u < 0$ or $u \geq 0$. However, the lower limit of the outer dy integral is $y = -2u$ if $u < 0$, and is zero if $u \geq 0$. Combining these together we have the lower limit of

$$l(u) = \max\{0, -2u\}$$

and hence

$$\begin{aligned}
F_U(u) &= \int_{l(u)}^{\infty} \left\{ \int_0^{y+2u} \frac{1}{4} \exp\left\{ -\frac{1}{2}(x+y) \right\} dx \right\} dy \\
&= \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{ -\frac{y}{2} \right\} \left\{ \int_0^{y+2u} \frac{1}{2} \exp\left\{ -\frac{x}{2} \right\} dx \right\} dy \\
&= \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{ -\frac{y}{2} \right\} \left[-\exp\left\{ -\frac{x}{2} \right\} \right]_0^{y+2u} dy \\
&= \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{ -\frac{y}{2} \right\} \left(1 - \exp\left\{ -\frac{(y+2u)}{2} \right\} \right) dy \\
&= \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{ -\frac{y}{2} \right\} dy - \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{ -\frac{2(y+u)}{2} \right\} dy \\
&= \exp\left\{ -\frac{l(u)}{2} \right\} - \frac{1}{2} \exp\{-u\} \int_{l(u)}^{\infty} \exp\{-y\}\, dy \\
&= \exp\left\{ -\frac{l(u)}{2} \right\} - \frac{1}{2} \exp\{-(u + l(u))\}
\end{aligned}$$

If $u < 0$, $l(u) = -2u$, and hence

$$F_U(u) = e^u - \frac{1}{2}e^u = \frac{1}{2}e^u$$

and if $u \geq 0$, $l(u) = 0$, and hence

$$F_U(u) = 1 - \frac{1}{2}e^{-u}$$

Thus

$$f_U(u) = \begin{cases} \frac{1}{2}e^u & u < 0 \\ \frac{1}{2}e^{-u} & u \geq 0 \end{cases} = \frac{1}{2}\exp\{-|u|\} \qquad u \in \mathbb{R}$$

2. Joint pdf is constant on the ellipse $\mathcal{E}$, thus the normalizing constant is the reciprocal of the area of the ellipse, that is $1/(\pi ab)$. The range of the random variables can be re-written

$$\mathbb{X}^{(2)} \equiv \left\{(x,y) : -a < x < a, -b\left(1-x^2/a^2\right)^{1/2} < y < b\left(1-x^2/a^2\right)^{1/2}\right\}$$

and hence, by double integration,

$$\begin{aligned}
\iint_{\mathcal{E}} f_{X,Y}(x,y)\ dxdy &= \int_{-a}^{a} \left\{ \int_{-b\left(1-x^2/a^2\right)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} c_2 dy \right\} dx \\
&= \int_{-a}^{a} 2c_2 b\left(1-x^2/a^2\right)^{1/2} dx \\
&= abc_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2 t\ dt \qquad (\text{setting } x = a\sin t) \\
&= abc_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2t + 1)\ dt \\
&= abc_2 \left[\frac{1}{2}\sin 2t + t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi abc_2
\end{aligned}$$

and hence $c_2 = 1/(\pi ab)$.

(a) For the marginal pdf of X, f_X, for fixed x,

$$f_X(x) = \int_{-b\left(1-x^2/a^2\right)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} \frac{1}{\pi ab} dy = \frac{2}{\pi a}\left(1-x^2/a^2\right)^{1/2} \qquad -a < x < a,$$

(b) By symmetry of form, we must have for the marginal for Y

$$f_Y(y) = \frac{2}{\pi b}\left(1-y^2/b^2\right)^{1/2} \qquad -b < y < b,$$

and because the two functions

$$\left(1-x^2/a^2\right)^{1/2} \qquad\qquad \left(1-y^2/b^2\right)^{1/2}$$

are symmetric about zero, we must have that

$$E_{f_X}[X] = E_{f_Y}[Y] = 0.$$

Finally for the covariance, we have that

$$
\begin{aligned}
Cov_{f_{X,Y}}[X,Y] &= E_{f_{X,Y}}[XY] - E_{f_X}[X]\,E_{f_Y}[Y] = E_{f_{X,Y}}[XY] \\
&= \int_{-a}^{a} \left\{ \int_{-b(1-x^2/a^2)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} xy f_{X,Y}(x,y)\,dy \right\} dx \\
&= \int_{-a}^{a} \left\{ \int_{-b(1-x^2/a^2)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} y dy \right\} \frac{x}{\pi ab} dx \\
&= \int_{-a}^{a} \left[\frac{y^2}{2} \right]_{-b(1-x^2/a^2)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} \frac{x}{\pi ab} dx \\
&= 0
\end{aligned}
$$

Hence X and Y are uncorrelated.

(c) X and Y **not** independent as there exists at least one pair $(x,y) \in \mathbb{R}^2$ such that

$$f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$$

(for example, any point within the rectangle $(-a,a) \times (-b,b)$ that is outside the ellipse has joint probability density zero, but $f_X(x) > 0$ and $f_Y(y) > 0$).

3. (a) Need expectations, variances and covariance. We have for X

$$E_{f_X}[X] = 0 \qquad E_{f_X}\left[X^2\right] = 1 \qquad Var_{f_X}[X] = 1$$

and for Y

$$E_{f_Y}[Y] = E_{f_X}\left[X^2\right] = 1 \qquad E_{f_Y}\left[Y^2\right] = E_{f_X}\left[X^4\right] = 3 \qquad Var_{f_Y}[Y] = 2$$

and for the covariance

$$E_{f_{X,Y}}[XY] = E_{f_X}\left[X^3\right] = 0 \therefore Cov_{f_{X,Y}}[X,Y] = 0 - 0 \times 1 = 0$$

and hence the correlation is also zero.

X and Y are not independent (merely uncorrelated); we have the joint distribution non-zero only on the line $y = x^2$, whereas f_X and f_Y are positive on the whole of $\mathbb{R} \times \mathbb{R}^+$.

(b) (i) By elementary properties of independent standard normal random variables (using mgfs for example)

$$X_1 - X_2 \sim Normal\,(0, 2)$$

and thus

$$Y_1 = X_1 - X_2 + 1 \sim Normal\,(1, 2)$$

(ii) By properties of the multivariate normal distribution, using multivariate transformation results

$$Y \sim N\,(b, \Sigma)$$

where

$$\Sigma = AA^T = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

and hence the covariance is

$$\Sigma_{12} = -2$$

and the correlation is

$$\frac{\Sigma_{12}}{\sqrt{\Sigma_{11} \times \Sigma_{22}}} = \frac{-2}{\sqrt{2 \times 4}} = -\frac{1}{\sqrt{2}}$$

4. (a) Note first that

$$f_X(x) = \frac{1}{(1+x)^2} \qquad x > 0$$

and zero otherwise. Then

$$\begin{aligned}
P[X_1X_2 < 1] &= \int_0^\infty \int_0^{1/x_1} f_{X_1,X_2}(x_1,x_2)\, dx_2 dx_1 \\
&= \int_0^\infty \int_0^{1/x_1} \frac{1}{(1+x_1)^2}\frac{1}{(1+x_2)^2}\, dx_2 dx_1 = \int_0^\infty \left[\frac{x_2}{1+x_2}\right]_0^{1/x_1} \frac{1}{(1+x_1)^2}\, dx_1 \\
&= \int_0^\infty \frac{1/x_1}{1+1/x_1}\frac{1}{(1+x_1)^2}\, dx_1 = \int_0^\infty \frac{x_1}{(1+x_1)^3}\, dx_1 \\
&= \left[-\frac{1}{2}\frac{x_1}{(1+x_1)^2}\right]_0^\infty + \int_0^\infty \frac{1}{2}\frac{1}{(1+x_1)^2}\, dx_1 = 0 + \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

(b) Using the multivariate transformation theorem

(a) We have that $\mathbb{Y}^{(2)} \equiv \mathbb{R} \times \mathbb{R}^+$, and

$$g_1(t_1,t_2) = \frac{t_1}{\sqrt{t_1^2+t_2^2}} \qquad g_2(t_1,t_2) = \sqrt{t_1^2+t_2^2}$$

(b) Inverse transformations:

$$\left.\begin{aligned} Y_1 &= \frac{Z_1}{\sqrt{Z_1^2+Z_2^2}} \\ Y_2 &= \sqrt{Z_1^2+Z_2^2} \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} Z_1 &= Y_1Y_2 \\ Z_2 &= \sqrt{1-Y_1^2}Y_2 \end{aligned}\right.$$

and thus

$$g_1^{-1}(t_1,t_2) = t_1t_2 \qquad g_2^{-1}(t_1,t_2) = \sqrt{1-t_1^2}t_2$$

(c) Range: we have that $-1 < Y_1 < 1$ and $Y_2 > 0$, so $\mathbb{Y}^{(2)} = (-1,1) \times \mathbb{R}^+$

(d) The Jacobian for points $(y_1,y_2) \in \mathbb{Y}^{(2)}$ is

$$D_{y_1,y_2} = \begin{bmatrix} \dfrac{\partial z_1}{\partial y_1} & \dfrac{\partial z_1}{\partial y_2} \\ \dfrac{\partial z_2}{\partial y_1} & \dfrac{\partial z_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ \dfrac{-y_1y_2}{\sqrt{1-y_1^2}} & \sqrt{1-y_1^2} \end{bmatrix} \Rightarrow |J(y_1,y_2)| = \frac{y_2}{\sqrt{1-y_1^2}}$$

(e) For the joint pdf we have for $(y_1,y_2) \in \mathbb{Y}^{(2)}$, by independence of Z_1 and Z_2

$$\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{Z_1,Z_2}\left(y_1y_2, \sqrt{1-y_1^2}y_2\right) \times \frac{y_2}{\sqrt{1-y_1^2}} \\
&= \frac{1}{\pi}\frac{y_2 \exp\left\{-y_2^2/2\right\}}{\sqrt{1-y_1^2}}
\end{aligned}$$

and zero otherwise, where, by inspection,

$$f_{Y_1}(y_1) = \frac{1}{\pi\sqrt{1-y_1^2}} \qquad -1 < y_1 < 1 \qquad\qquad f_{Y_2}(y_2) = y_2 \exp\left\{-y_2^2/2\right\} \qquad y_2 > 0$$

Note that Y_1 and Y_2 are independent, as their joint pdf factorizes into the respective marginal pdfs at all points of $\mathbb{R}^2$.

5. (a) Given a f_X, we construct a tilted version with tilt given parameter θ as follows; we consider

$$f_{X|\theta}(x|\theta) \propto f_X(x) \exp\{\theta X\}$$

in such a way so that the resulting function $f_{X|\theta}(x|\theta)$ is a valid pdf. Clearly this function is non-negative, and integrable if

$$\int_{-\infty}^{\infty} f_X(x) \exp\{\theta x\}\, dx < \infty$$

If this holds, then

$$f_{X|\theta}(x|\theta) = \frac{f_X(x) \exp\{\theta X\}}{\displaystyle\int_{-\infty}^{\infty} f_X(x) \exp\{\theta x\}\, dx} = \frac{f_X(x) \exp\{\theta X\}}{M_X(\theta)}$$

where M_X is the mgf corresponding to the original f_X. Finally, if $K_X(t) = \log M_X(t)$ is the corresponding cumulant generating function, then

$$f_{X|\theta}(x|\theta) = f_X(x) \exp\{\theta X - K_X(\theta)\}$$

This is a natural exponential family distribution in its canonical parameterization, that is,

$$f_{X|\theta}(x|\theta) = h(x)c(\theta) \exp\{\theta X\}$$

where $h(x) = f_X(x)$ and $c(\theta) = M_X(\theta)$. This computation can be generalized by considering the derivation with random variable $S = s(X)$ replacing X in the exponent, and M_S replacing M_X.

(b) If $\mathcal{N}$ is given by

$$\mathcal{N} \equiv \left\{\theta \in \mathbb{R} : K_S(\theta) = \log\left[\int e^{s(y)\theta} f_Y(y)\, dy\right] < \infty\right\}$$

(i) $0 \in \mathcal{N}$ as f_Y is a valid pdf and hence integrable. Note that as $Var_{f_S}[s(Y)] > 0$, the distribution of $s(Y)$ is not degenerate, and hence $\mathcal{N}$ contains elements other than zero.

(ii) For $0 \leq \alpha \leq 1$, we consider $\theta = \alpha\theta_1 + (1-\alpha)\theta_2$. Then

$$\begin{aligned}
\int e^{s(y)\theta} f_Y(y)\, dy &= \int \exp\{s(y)(\alpha\theta_1 + (1-\alpha)\theta_2)\} f_Y(y)\, dy \\
&= \int \exp\{s(y)\alpha\theta_1\} \exp\{s(y)(1-\alpha)\theta_2\} f_Y(y)\, dy \\
&= E_{f_Y}\left[g_1(Y;\theta_1)^{\alpha} g_2(Y;\theta_2)^{1-\alpha}\right]
\end{aligned}$$

say, where $g_i(y;\theta) = \exp\{s(y)\theta_i\}$ for $i = 1, 2$. Now using Hölder's Inequality with $p = 1/\alpha$, $q = 1/(1-\alpha)$, we can deduce that

$$E_{f_Y}\left[g_1(Y;\theta_1)^{\alpha} g_2(Y;\theta_2)^{1-\alpha}\right] \leq E_{f_Y}\left[g_1(Y;\theta_1)\right]^{\alpha} E_{f_Y}\left[g_2(Y;\theta_2)\right]^{1-\alpha}$$

$$\int e^{s(y)\theta} f_Y(y)\, dy \leq E_{f_Y}\left[g_1(Y;\theta_1)\right]^{\alpha} E_{f_Y}\left[g_2(Y;\theta_2)\right]^{1-\alpha} < \infty$$

as

$$E_{f_Y}\left[g_i(Y;\theta_i)\right] = \int \exp\{s(y)\theta_i\} f_Y(y)\, dy < \infty \qquad i = 1, 2$$

Hence $\theta \in \mathcal{N}$, and the set $\mathcal{N}$ is convex.

(iii) We need to show that

$$K_S(\alpha\theta_1 + (1-\alpha)\theta_2) \leq \alpha K_S(\theta_1) + (1-\alpha)K_S(\theta_2)$$

Now, let $\theta = \alpha\theta_1 + (1-\alpha)\theta_2$. Then, using the notation from part (ii),

$$\begin{aligned} K_S(\theta) &= \log E_{f_Y}\left[g_1(Y;\theta_1)^\alpha g_2(Y;\theta_2)^{1-\alpha}\right] \\ &\leq \log\left\{E_{f_Y}\left[g_1(Y;\theta_1)\right]^\alpha E_{f_Y}\left[g_2(Y;\theta_2)\right]^{1-\alpha}\right\} \end{aligned}$$

using Hölder's Inequality again. Thus

$$\begin{aligned} K_S(\theta) &\leq \alpha \log E_{f_Y}\left[g_1(Y;\theta_1)\right] + (1-\alpha)\log E_{f_Y}\left[g_2(Y;\theta_2)\right] \\ &= \alpha K_S(\theta_1) + (1-\alpha)K_S(\theta_2) \end{aligned}$$

and the result follows.

6. (a) (i) As $n \to \infty$, for $x \in \mathbb{R}$

$$\left(\frac{1}{1+e^{-x}}\right) < 1 \qquad \therefore \qquad F_{X_n}(x) \to 0$$

and so the limiting function is not a cdf, and no limiting distribution exists.

(ii) If $U_n = X_n - \log n$. Then $\mathbb{U} \equiv (-\infty, \infty)$ and the cdf of U_n is

$$F_{U_n}(u) = P\left[U_n \le u\right] = P\left[X_n - \log n \le u\right] = P\left[X_n \le u + \log n\right] = F_{X_n}(u + \log n)$$

and so

$$F_{Y_n}(y) = \left(\frac{1}{1+e^{-u-\log n}}\right)^n = \left(\frac{1}{1+e^{-u}/n}\right)^n = \left(1 - \frac{e^{-u}}{n+e^{-u}}\right)^n$$

Thus as $n \to \infty$, for all u

$$F_{U_n}(u) \to \exp\left\{-e^{-u}\right\} \qquad \therefore \qquad F_{U_n}(u) \to F_U(u) = \exp\left\{-e^{-u}\right\}$$

and the limiting distribution of U_n does exist, and is continuous on $\mathbb{R}$.

Thus, for large n,

$$P[X_n > k] = P[U_n > k+\log n] = 1-F_{U_n}(k+\log n) \approx 1-F_U(k+\log n) = 1-\exp\left\{-e^{-k-\log n}\right\}$$

(b) Let X_i denote the score on roll i. Then

$$E_{f_{X_i}}[X_i] = \frac{-2+(4\times -1)+6}{6} = 0 \qquad Var_{f_{X_i}}[X_i] = E_{f_{X_i}}[X_i^2] = \frac{4+(4\times 1)+36}{6} = \frac{22}{3}$$

and denote these quantities μ and σ^2 respectively.

(i) The expectation and variance of T_{100} are $100\mu = 0$ and $100\sigma^2 = 2200/3$.

(ii) The Central Limit Theorem gives that for the iid $\{X_i\}$ collection

$$\frac{\sum\limits_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \sim AN(0,1)$$

where AN denotes Asymptotically Normal (as $n \to \infty$). Thus

$$T_n = \sum_{i=1}^{n} X_i \sim AN\left(0, \frac{22n}{3}\right)$$

and

(iii) Using the Weak Law of Large numbers, we can deduce that

$$M_n \xrightarrow{p} \mu = 0$$

as $n \longrightarrow \infty$, that is, the sample mean random quantity converges in probability to zero, that is, the probability distribution of M_n becomes degenerate at zero.

7. (a) (i) The Poisson distribution mgf is

$$M_X(t) = \exp\left\{\lambda(e^t - 1)\right\}.$$

Now, if $Z_\lambda = (X-\lambda)/\sqrt{\lambda}$, we use the mgf result for linear functions, that is if

$$Y = aX + b \Longrightarrow M_Y(t) = e^{bt} M_X(at).$$

Here, $a = 1/\sqrt{\lambda}$ and $b = -\sqrt{\lambda}$, so

$$\begin{aligned} M_{Z_\lambda}(t) &= e^{-\sqrt{\lambda}t} \exp\left\{\lambda(e^{t/\sqrt{\lambda}} - 1)\right\} = \exp\left\{-\lambda^{1/2}t + \lambda\left[\frac{t}{\lambda^{1/2}} + \frac{t^2}{2\lambda} + \frac{t^3}{6\lambda^{3/2}} + \ldots\right]\right\} \\ &= \exp\left\{\frac{t^2}{2} + \frac{t^3}{6\sqrt{\lambda}} + \ldots\right\} \to \exp\left\{\frac{t^2}{2}\right\} \qquad \text{as } \lambda \to \infty \end{aligned}$$

so therefore

$$Z_\lambda \xrightarrow{d} Z \sim Normal(0,1)$$

as $\lambda \to \infty$.

(ii) Let $T_i = X_i + Y_i$. Then, by properties of Poisson random variables, we have that $T_i \sim Poisson\,(\lambda_X + \lambda_Y)$. Hence

$$T = \sum_{i=1}^{n} (X_i + Y_i) \sim Poisson\,(n\,(\lambda_X + \lambda_Y)).$$

so that

$$E_{f_T}[T] = n\,(\lambda_X + \lambda_Y) \qquad Var_{f_T}[T] = n\,(\lambda_X + \lambda_Y)$$

But $M = \dfrac{T}{n}$, so

$$E_{f_M}[M] = \frac{n\,(\lambda_X + \lambda_Y)}{n} = \lambda_X + \lambda_Y \qquad Var_{f_M}[M] = \frac{n\,(\lambda_X + \lambda_Y)}{n^2}$$

which are both finite. Hence, by the *Weak Law of Large Numbers*

$$M \xrightarrow{p} E_{f_M}[M] = \lambda_X + \lambda_Y = \mu$$

(b) (i) $T_n = \max\{X_1, \ldots, X_n\}$ so

$$F_{T_n}(t) = \{F_X(t)\}^n = \left(1 - e^{-\lambda t}\right)^n \qquad t \in \mathbb{R}^+$$

(ii) In the limit as $n \to \infty$ we have the limit for *fixed* t as

$$F_{T_n}(t) \to 0 \qquad \text{for all } t$$

Hence there is *no limiting distribution*.

(iii) If $U_n = \lambda T_n - \log n$, we have from first principles that for $u > -\log n$

$$\begin{aligned}
F_{U_n}(u) &= P\left[U_n \leq u\right] = P\left[\lambda T_n - \log n \leq u\right] \\
&= P\left[T_n \leq \frac{1}{\lambda}\left(u + \log n\right)\right] \\
&= F_{T_n}\left(\frac{1}{\lambda}\left(u + \log n\right)\right) \\
&= \left(1 - e^{-(u+\log n)}\right)^n \\
&= \left(1 - \frac{e^{-u}}{n}\right)^n
\end{aligned}$$

so that

$$F_{U_n}(u) \to \exp\left\{-e^{-u}\right\} \qquad \text{as } n \to \infty$$

which is a valid cdf. Hence the limiting distribution is

$$F_U(u) = \exp\left\{-e^{-u}\right\} \qquad u \in \mathbb{R}$$

8. (a) Using properties of cfs, we have

$$C_{T_n}(t) = \left\{e^{-|t|}\right\}^n = e^{-|nt|}$$

Now using the scale transformation result for mgfs/cfs (given on Formula Sheet), we have that if $V = \lambda U$, then

$$C_V(t) = C_U(\lambda t)$$

we deduce that, in distribution, $T_n = nX$, where $X \sim Cauchy$, so that, by the univariate transformation theorem,

$$f_{T_n}(x) = f_X(x/n)|J(x)| = \frac{1}{\pi}\frac{1}{1+(x/n)^2}\frac{1}{n} = \frac{1}{\pi}\frac{n}{n^2+x^2}$$

(b) From (a), we can deduce immediately that $\overline{X}_n \sim Cauchy$ **for all n.** Hence, using the Cauchy cdf,

$$P[|\overline{X}_n| > \epsilon] = 1 - \frac{2}{\pi}\arctan(\epsilon) \nrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty.$$

and hence $\overline{X}_n \overset{p}{\nrightarrow} 0$ as $n \longrightarrow \infty$.

(c) Many possible methods of solution; recall that the scale mixture formulation specifies a three level hierarchy in this case

$$\begin{array}{lll} \text{LEVEL 3}: & \alpha, \beta > 0 & \qquad\qquad \text{Fixed parameters} \\ \text{LEVEL 2}: & V \sim Gamma(\alpha, \beta) & \\ \text{LEVEL 1}: & X|V = v \sim Normal(0, g(v)) & \end{array}$$

for some non-negative function $g(.)$. The marginal for X is thus

$$f_X(x) = \int_0^\infty f_{X|V}(x|v)f_V(v)\,dv = \int_0^\infty \left(\frac{1}{2\pi g(v)}\right)^{1/2} \exp\left\{-\frac{x^2}{2g(v)}\right\}\frac{\beta^\alpha}{\Gamma(\alpha)}v^{\alpha-1}e^{-\beta v}\,dv.$$

We require the result of this calculation to be the Cauchy pdf. In order to integrate out v, it appears that we must make the integrand proportional to a Gamma pdf, and choosing $g(v) = v^{-1}$ makes this possible; ignoring constants, the integrand becomes

$$v^{\alpha+1/2-1}\exp\left\{-\frac{v(2\beta+x^2)}{2}\right\}$$

which, on integration, yields a term proportional to

$$\frac{\Gamma(\alpha+1/2)}{(2\beta+x^2)^{\alpha+1/2}}.$$

Hence choosing $\alpha = 1/2$, $\beta = 1/2$ yields a term proportional to the Cauchy pdf. Thus the Cauchy distribution is a scale mixture of a Normal density by a $Gamma(1/2, 1/2) \equiv \chi^2_1$ distribution, with "link" function $g(v) = v^{-1}$.