MATH 556 - MID-TERM 2007 SOLUTIONS

1. (a) (i) We have

$$\begin{aligned} M_X(t) &= E_{f_X}\left[e^{tX}\right] = \sum_{x=1}^{\infty} e^{tx} f_X(x) = \sum_{x=1}^{\infty} e^{tx} \frac{-1}{\log(1-\phi)} \frac{\phi^x}{x} = \frac{-1}{\log(1-\phi)} \sum_{x=1}^{\infty} \frac{\left(\phi e^t\right)^x}{x} \\ &= \frac{\log\left(1-\phi e^t\right)}{\log(1-\phi)} \end{aligned}$$

provided $\left|\phi e^t\right| < 1$, or equivalently $t < -\log\phi$, which ensures the required neighbourhood of zero is available as $\phi > 0$. The final result follows by inspection of the summand, and the fact that the pmf is known to sum to 1.

8 MARKS

(ii) For the expectation, by direct calculation

$$E_{f_X}[X] = \sum_{x=1}^{\infty} x f_X(x) = \sum_{x=1}^{\infty} x \frac{-1}{\log(1-\phi)} \frac{\phi^x}{x} = \frac{-1}{\log(1-\phi)} \sum_{x=1}^{\infty} \phi^x = \frac{-1}{\log(1-\phi)} \frac{\phi}{1-\phi}$$

(could also use the mgf)

4 MARKS

(b) By direct integration

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \theta x^2/2 & 0 \le x < 1 \\ \dfrac{1}{2} + \dfrac{\theta}{2}(1 - e^{-2(x-1)}) & x \ge 1 \end{cases}$$

and as we need $F_X(x) \longrightarrow 1$ as $x \longrightarrow \infty$, this implies that $\theta = 1$.

8 MARKS

2. (a) (i) For $0 \le x \le n$, using the Beta integral function

$$f_X(x) = \int_0^1 \binom{n}{x} y^x (1-y)^{n-x}\, dy = \binom{n}{x} \frac{\Gamma(x+1)\Gamma(n-x+1)}{\Gamma(n+2)} = \frac{1}{n+1}$$

5 MARKS

(ii) For $x > 0$, using the Gamma integral

$$f_X(x) = \int_0^{\infty} y e^{-xy} \beta e^{-\beta y} dy = \beta \int_0^{\infty} y e^{-(x+\beta)y} dy = \beta \frac{\Gamma(2)}{(x+\beta)^2} = \frac{\beta}{(x+\beta)^2}$$

5 MARKS

(b) (i) For the cdf

$$F_{Y_1}(y_1) = \Pr[Y_1 \le y_1] = \Pr[X_1^2 \le y_1] = \Pr[-\sqrt{y_1} \le X_1 \le \sqrt{y_1}] = F_{X_1}(\sqrt{y_1}) - F_{X_1}(-\sqrt{y_1})$$

and thus on differentiation

$$f_{Y_1}(y_1) = \frac{1}{2\sqrt{y_1}}\left[f_{X_1}(\sqrt{y_1}) + f_{X_1}(-\sqrt{y_1})\right].$$

Substituting in the Normal pdf, we obtain that

$$f_{Y_1}(y_1) = \frac{1}{(2\pi)^{1/2}} y^{-1/2} \exp\{-y_1/2\} = \frac{(1/2)^{1/2}}{\Gamma(1/2)} y^{-1/2} \exp\{-y_1/2\} \qquad y_1 > 0$$

and zero otherwise, so that $Y_1 \sim Gamma(1/2, 1/2)$.

5 MARKS

(ii) Clearly $Z_1^2, \ldots, Z_n^2$ are independent and identically distributed $Gamma(1/2, 1/2)$ random variables, and thus by using mgfs, we have that, from the formula sheet

$$M_{S_n}(t) = \{M_{Y_1}(t)\}^n = \left(\frac{1/2}{1/2 - t}\right)^{n/2}$$

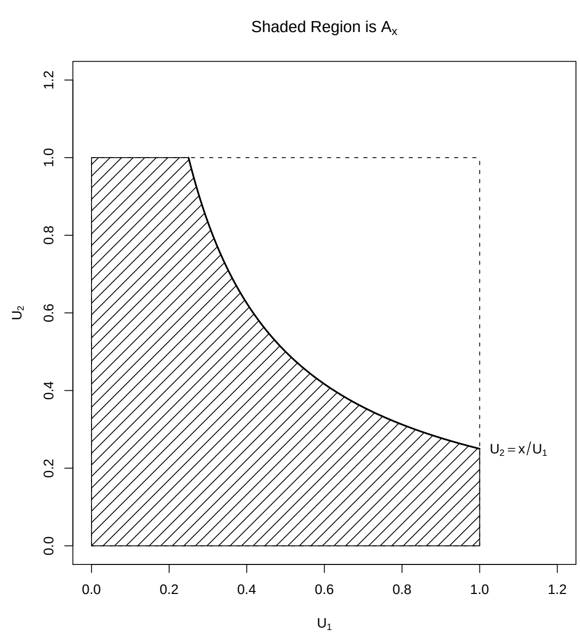
which is the mgf of the $Gamma(n/2, 1/2)$ distribution.

5 MARKS

3. (a) We have that,

$$F_X(x) = \Pr[X \le x] = \Pr[U_1 U_2 \le x] = \iint_{A_x} f_{U_1,U_2}(u_1, u_2)\, du_1\, du_2$$

where $A_x \equiv \{(u_1, u_2) : 0 \le u_1, u_2 \le 1, u_1 u_2 \le x\}$. By inspection, $F_X(x) = 0$ if $x \le 0$ and $F_X(x) = 1$ if $x \ge 1$. For $0 < x < 1$, the figure below indicates form of the region A_x.



Thus, for $0 < x < 1$,

$$F_X(x) = \int_0^1 \int_0^{\min\{1,x/u_1\}} du_2\, du_1 = \int_0^1 \min\{1, x/u_1\}\, du_1 = \int_0^x du_1 + \int_x^1 \frac{x}{u_1}\, du_1$$
$$= x + x\,[\log u_1]_x^1 = x - x\log x$$

so that

$$f_X(x) = -\log x \qquad 0 < x < 1$$

and zero otherwise.

10 MARKS

(b) By linearity of expectations $E_{f_{Y_1}}[Y_1] = \mu_{X1} + \mu_{X2}$ and $E_{f_{Y_2}}[Y_2] = \mu_{X1} - \mu_{X2}$, so

$$\begin{aligned} Cov_{f_{Y_1},f_{Y_2}}[Y_1, Y_2] &= E_{f_{Y_1},f_{Y_2}}[Y_1Y_2] - \mu_1\mu_2 \\ &= E_{f_{X_1},f_{X_2}}[(X_1 + X_2)(X_1 - X_2)] - (\mu_{X1} + \mu_{X2})(\mu_{X1} - \mu_{X2}) \\ &= E_{f_{X_1},f_{X_2}}[(X_1^2 - X_2^2)] - (\mu_{X1}^2 - \mu_{X2}^2) \\ &= \left(E_{f_{X_1}}[X_1^2] - \mu_{X1}^2\right) - \left(E_{f_{X_2}}[X_2^2] - \mu_{X2}^2\right) \\ &= \sigma_{X_1}^2 - \sigma_{X_2}^2 \end{aligned}$$

6 MARKS

We cannot, in general, discern whether Y_1 and Y_2 are independent. If $\sigma_{X_1} \neq \sigma_{X_2}$, then the covariance is non-zero, so Y_1 and Y_2 are not independent. However, even if $\sigma_{X_1} = \sigma_{X_2}$, so that the covariance is zero, we still cannot tell if Y_1 and Y_2 are independent, as uncorrelatedness does not imply independence. It is possible to construct examples where Y_1 and Y_2 are independent or are not independent.

4 MARKS

4. (a) We have by the triangle inequality that

$$|C_X(t)| \leq \alpha|C_1(t)| + (1-\alpha)|C_2(t)| \longrightarrow 0 \qquad \text{as} \qquad |t| \longrightarrow \infty$$

as $C_1(t)$ and $C_2(t)$ are the cfs for continuous rvs. Thus C_X is also the cf for a continuous pdf. Alternatively, let

$$f_X(x) = \alpha f_1(x) + (1-\alpha)f_2(x).$$

This is a valid pdf, and the corresponding cf is

$$\begin{aligned} C_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx &= \int_{-\infty}^{\infty} e^{itx} \left[\alpha f_1(x) + (1-\alpha) f_2(x)\right]\, dx \\ &= \alpha \int_{-\infty}^{\infty} e^{itx} f_1(x)\, dx + (1-\alpha) \int_{-\infty}^{\infty} e^{itx} f_2(x)\, dx \\ &= \alpha C_1(t) + (1-\alpha)C_2(t) \end{aligned}$$

so we have found a pdf with the correct cf.

6 MARKS

(b) From the formula sheet

$$C_Y(t) = e^{2it} C_X(-3t)$$

(c) If we define X_1 and X_2 to be independent and have the same distribution as X, and set

$$Z_1 = X_1 + X_2 \qquad\qquad Z_2 = X_1 - X_2$$

then the cfs of Z_1 and Z_2 have the correct form. Note that

$$C_{-X}(t) = \int_{-\infty}^{\infty} e^{itx} f_{-X}(x)\, dx = \int_{-\infty}^{\infty} e^{-itx} f_X(x)\, dx = C_X(-t) = \overline{C}_X(t)$$

8 MARKS

(d) We have that

$$C_X(t) = \frac{1}{3}(1 + 2\cos(2t)) \qquad t \in \mathbb{R}$$

so that

$$\limsup_{|t| \longrightarrow \infty} |C_X(t)| = 1$$

so this is the cf of a **discrete** distribution. Using the inversion formula,

$$f_X(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itx} C_X(t)\, dt$$

or merely by inspection, we find that

$$f_X(x) = \frac{1}{3} \qquad x \in \{-2, 0, 2\}$$

and zero otherwise. Note that

$$\int_{-\pi}^{\pi} e^{-itx} C_X(t)\, dt = \frac{1}{3} \int_{-\pi}^{\pi} \left[e^{-itx} + e^{it(2-x)} + e^{-it(2+x)} \right]\, dt$$

and from the result in lectures, for integer x,

$$\int_{-\pi}^{\pi} e^{itx}\, dt = \begin{cases} 2\pi & x = 0 \\ 0 & x \neq 0 \end{cases}$$

so that in the inversion formula integral only the cases $x = -2, 0, 2$ give non-zero results.

6 MARKS