# RANDOMIZED BLOCK DESIGNS AND THE ANOVA F-TEST

Consider a **randomized block design** (RBD) with $k$ treatments and $b$ blocks. Assume that each block has $k$ experimental units, and that one unit is assigned to each treatment. Let $x_{ij}$ be the measured response for the experimental unit from block $j$ in treatment $i$ and

- sample mean for **treatment** $i$

$$\overline{x}_i = \frac{1}{b}\sum_{j=1}^{b} x_{ij} \qquad i = 1, \ldots, k$$

- sample mean for block $j$

$$\overline{x_j^{(\mathrm{B})}} = \frac{1}{k}\sum_{i=1}^{k} x_{ij} \qquad j = 1, \ldots, b$$

- overall sample mean

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{b} x_{ij}$$

- Sum of Squares for Treatments (SST)

$$\mathrm{SST} = \sum_{i=1}^{k} b(\overline{x}_i - \overline{x})^2$$

- Sum of Squares for Blocks (SSB)

$$\mathrm{SSB} = \sum_{j=1}^{b} k(\overline{x_j^{(\mathrm{B})}} - \overline{x})^2$$

- Overall Sum of Squares (SS)

$$\mathrm{SS} = \sum_{i=1}^{k}\sum_{j=1}^{b}(x_{ij} - \overline{x})^2$$

The following decomposition holds

$$\mathrm{SS} = \mathrm{SST} + \mathrm{SSB} + \mathrm{SSE} \qquad \therefore \qquad \mathrm{SSE} = \mathrm{SS} - \mathrm{SST} - \mathrm{SSB}$$

For testing

$$\begin{aligned} H_0 &: \mu_1 = \cdots = \mu_k \\ H_a &: \text{At least two treatment means different} \end{aligned}$$

in an RBD, the test statistic is

$$F = \frac{\mathrm{MST}}{\mathrm{MSE}}$$

where

$$\mathrm{MST} = \frac{\mathrm{SST}}{k-1} \qquad \mathrm{MSE} = \frac{\mathrm{SSE}}{n-b-k+1}$$

If $H_0$ is **true**, then $F \sim$ Fisher-F$(k-1, n-b-k+1)$, and the rejection region for the test with significance level $\alpha$ is

$$F > F_\alpha(k-1, n-b-k+1)$$

where $F_\alpha(\nu_1, \nu_2)$ is the $1-\alpha$ percentage point of the Fisher-F distribution with $\nu_1$ and $\nu_2$ degrees of freedom (*see pages 899-905 of McClave and Sincich 10th Ed*)

# EXAMPLE

**Data:** Measurements were made on the amount of sulphur (in parts per million) in soil samples using four different solvents. The soil samples were collected from five different geographical locations in Florida, USA, and represented different soil types.

The **response variable** sulphur level. The single **factor** is the *solvent* and there are $k = 4$ **factor levels**:

1. Calcium Chloride ($CaCl_2$)
2. Ammonium Acetate ($NH_4OAc$)
3. Mono-Calcium Phosphate ($Ca(H_2PO_4)_3$)
4. Water ($H_2O$)

The *soil* types determine the $b = 5$ **blocks**

1. Troup, Jackson Co. (*Paleudults* soil)
2. Lakeland, Walton Co. (*Quartzipsamments* soil)
3. Leon, Duval Co. (*Haplaquads* soil)
4. Chipley, Jackson Co. (*Quartzipsamments* soil)
5. Norfolk, Alachua Co. (*Paleudults* soil)

The data observed in the study were as follows:

| | Block | | | | |
|---|---|---|---|---|---|
| Treatment | Troup | Lakeland | Leon | Chipley | Norfolk |
| $CaCl_2$ | 5.07 | 3.31 | 2.54 | 2.34 | 4.71 |
| $NH_4OAc$ | 4.43 | 2.74 | 2.09 | 2.07 | 5.29 |
| $Ca(H_2PO_4)_3$ | 7.09 | 2.32 | 1.09 | 4.38 | 5.70 |
| $H_2O$ | 4.48 | 2.35 | 2.70 | 3.85 | 4.98 |

Using SPSS, the following ANOVA table was obtained; see the related SPSS screens at

`www.math.mcgill.ca/~dstephens/204/Handouts/Math204-SPSS-RBDANOVA-Screens.pdf`

**Tests of Between-Subjects Effects**

Dependent Variable: Sulphur content (ppm)

| Source | Type III Sum of Squares | df | Mean Square | F | Sig. |
|---|---|---|---|---|---|
| ~~Corrected Model~~ | ~~35.586(a)~~ | ~~7~~ | ~~5.084~~ | ~~6.327~~ | ~~.003~~ |
| ~~Intercept~~ | ~~270.333~~ | ~~1~~ | ~~270.333~~ | ~~336.460~~ | ~~.000~~ |
| solvent | 1.621 | 3 | .540 | .673 | .585 |
| soil | 33.965 | 4 | 8.491 | 10.568 | .001 |
| Error | 9.642 | 12 | .803 | | |
| ~~Total~~ | ~~315.561~~ | ~~20~~ | | | |
| Corrected Total | 45.228 | 19 | | | |

a R Squared = .787 (Adjusted R Squared = .662)

This table contains a much information not needed for the ANOVA F-test; the rows headed

- Corrected Model (row 1)
- Intercept (row 2)
- Total (row 6)

can be ignored. The remaining rows are the standard ANOVA table for the randomized block design. As expected, there is a significant difference between **blocks** (row 4, $F = 10.568$, $p$-value=0.001), but **no significant difference** between **treatments** (row 3, $F = 0.673$, $p$-value=0.585).