MATH 204: PRINCIPLES OF STATISTICS 2
UNDERSTANDING THE ANOVA F-STATISTIC

Suppose that we have $k = 3$ treatment groups in a Completely Randomized Design, with sample sizes $n_1 = n_2 = n_3 = 6$. Suppose first that the treatment means are all equal to zero, that is

$$\mu_1 = \mu_2 = \mu_3 = 0$$

and that the treatment group variance parameter σ^2 is equal to 1. A typical data set is displayed below:

							$\overline{x}_i$	s_i^2
TMT 1	-0.88	0.24	-0.46	0.78	-0.47	-0.38	-0.195	0.358
TMT 2	-0.75	0.11	0.64	1.98	-1.03	1.84	0.465	1.611
TMT 3	1.38	1.20	0.42	0.05	-1.29	-0.04	0.287	0.939

yielding $\overline{x} = 0.186$, and

$$s_P^2 = \frac{1}{n-k}\sum_{i=1}^{k}(n_i - 1)s_i^2 = 0.969.$$

For these data, we have using the definitions from lectures

$$\text{SST} = \sum_{i=1}^{k} n_i(\overline{x}_i - \overline{x})^2 = 1.399 \qquad \text{SSE} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \overline{x}_i)^2 = 14.539$$

and

$$\text{SS} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \overline{x})^2 = 15.938$$

so that the equation SS $=$ SST $+$ SSE holds. For the F-statistic, we have

$$F = \frac{\text{MST}}{\text{MSE}} = \frac{\text{SST}/(k-1)}{\text{SSE}/(n-k)} = \frac{1.399/2}{14.539/15} = 0.722$$

To complete the test, we compare this with the $1 - \alpha$ probability point of the Fisher-F distribution with $(k-1, n-k) = (2, 15)$ degrees of freedom. With $\alpha = 0.05$, from the tables on page 901 in McClave and Sincich, we see that

$$F_\alpha(2, 15) = 3.68$$

and we **do not reject** the ANOVA F-test null hypothesis

$$H_0 \quad : \quad \mu_1 = \mu_2 = \mu_3.$$

This is the **correct** conclusion, as in fact all the true treatment means are zero. Thus a **small** value of the test statistic F supports H_0.

Now suppose that, in fact,

$$\mu_1 = 0 \qquad \mu_2 = 10 \qquad \mu_3 = 20.$$

The equivalent data set to the one above but with the treatment means changed in this way takes the form

							$\overline{x}_i$	s_i^2
TMT 1	-0.88	0.24	-0.46	0.78	-0.47	-0.38	-0.195	0.358
TMT 2	9.25	10.11	10.64	11.98	8.97	11.84	10.465	1.611
TMT 3	21.38	21.20	20.42	20.05	18.71	19.96	20.287	0.939

yielding $\overline{x} = 10.186$, and

$$s_P^2 = \frac{1}{n-k}\sum_{i=1}^{k}(n_i - 1)s_i^2 = 0.969.$$

Note that the sample means have changed accordingly, but that the sample variances **have not changed at all**. On further calculation, we have

$$\text{SST} = 1259.199 \qquad \text{SSE} = 14.539 \qquad \text{SS} = 1273.738$$

so that

$$MST = \frac{1259.199}{2} = 629.600 \qquad MSE = \frac{14.539}{15} = 0.969$$

so that

$$F = \frac{629.600}{0.969} = 649.570.$$

We again compare this with $F_\alpha(2, 15) = 3.68$ (the critical value, C_R), and notice that the F statistic is **much larger** than this critical value. The test statistic thus lies within the rejection region, and hence we **reject $\mathbf{H_0}$**.

This example illustrates that SST measures the variability **between** means across the treatment groups, whereas SSE measures the variability **within** treatment groups, allowing for the possibility that the treatment means may be different. The quantity SS measures the total amount of variability; in the first example SS = SST + SSE gives

$$15.938 = 1.399 + 14.539$$

so most of the variability is contributed by SSE, whereas in the second example, we have

$$1273.738 = 1259.199 + 14.539$$

and most of the variability is contributed by SST.