MATH 556: MATHEMATICAL STATISTICS I

DELTA METHOD: EXAMPLES

Under the conditions of the Central Limit Theorem (CLT), for random variables $X_1, \ldots, X_n$ and their sample mean random variable $\overline{X}_n$

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} X \sim Normal(0, \sigma^2).$$

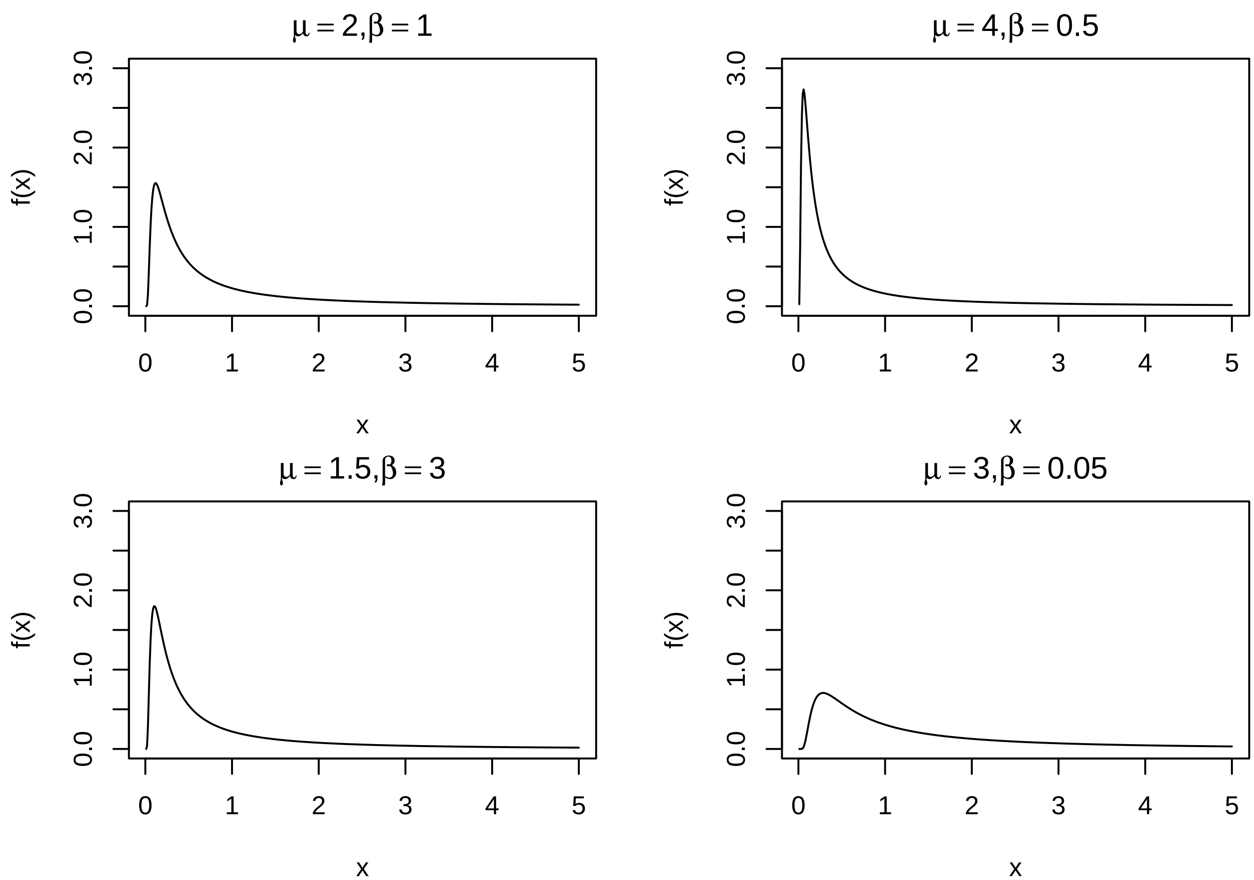
We illustrate results using the *Inverse Gaussian distribution* where

$$f_X(x; \mu, \beta) = \mathbb{1}_{(0,\infty)}(x) \sqrt{\frac{1}{2\pi\beta x^3}} \exp\left\{ -\frac{1}{2\beta} \frac{(x-\mu)^2}{\mu^2 x} \right\}$$

where

$$\mathbb{E}_X[X] = \mu \qquad\qquad \text{Var}_X[X] = \beta\mu^3.$$

so we have that $\sigma^2 = \beta\mu^3$.



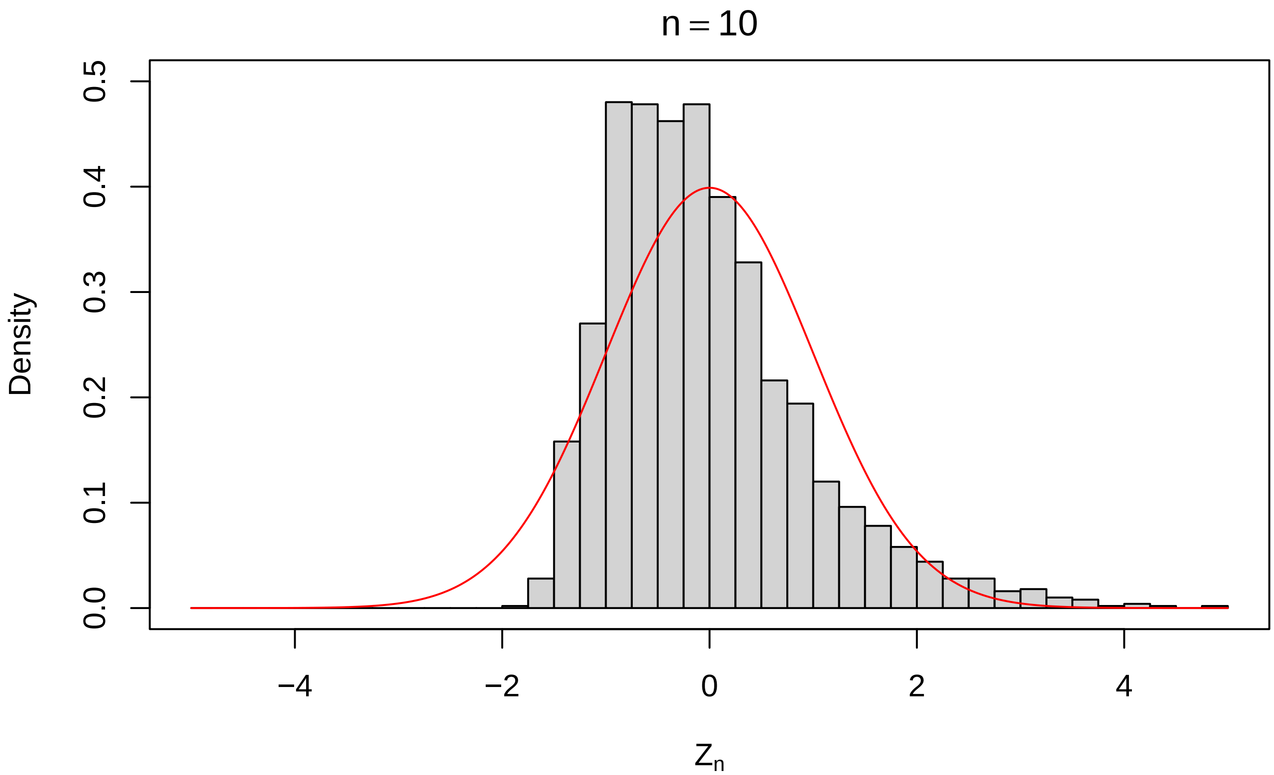
The CLT suggests that as $n \longrightarrow \infty$

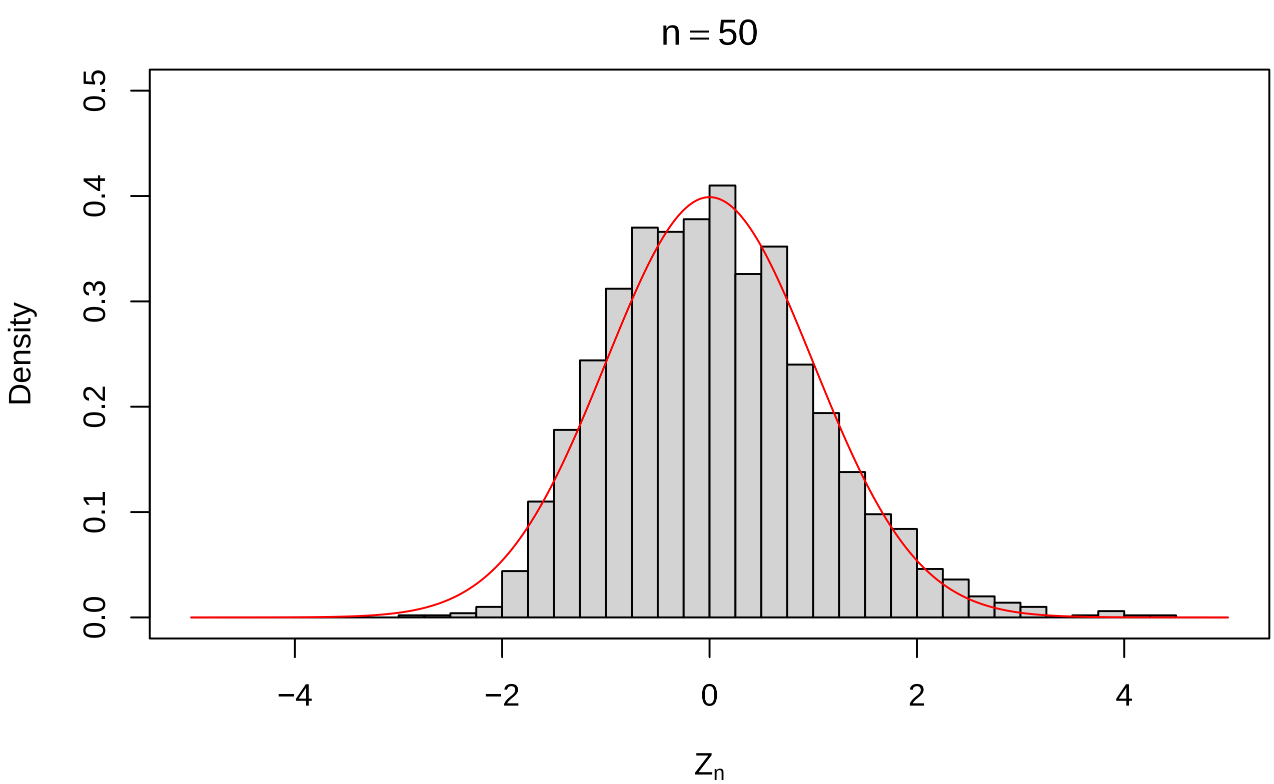
$$Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sqrt{\beta\mu^3}} \xrightarrow{d} Z \sim Normal(0, 1)$$

which we explore below for $n = 10$ and $n = 50$. Overlaying the histograms of $N = 2000$ realizations from the sampling distribution of Z_n are plots of the standard Normal density (red lines).

```
set.seed(194)
n<-10;N<-2000
xv<-seq(-5,5,length=1001);yv<-dnorm(xv)
Zn<-replicate(N,sqrt(n)*(mean(rinvgauss(n,mu1,be1))-mu1)/sig1)
```

```
par(mar=c(4,4,2,1))
ind<-abs(Zn)<5
hist(Zn[ind],breaks=seq(-5,5,by=0.25),freq=FALSE,
        main=substitute(n==nv,list(nv=n)),ylim=range(0,0.5),xlab=expression(Z[n]));box()
lines(xv,yv,col='red')
```





(a) Consider $g(x) = \log x$, so that $\dot{g}(x) = 1/x$, so that

$$L_n = \sqrt{n}(\log \overline{X}_n - \log \mu) \xrightarrow{d} X \sim Normal(0, \sigma^2/\mu^2)$$

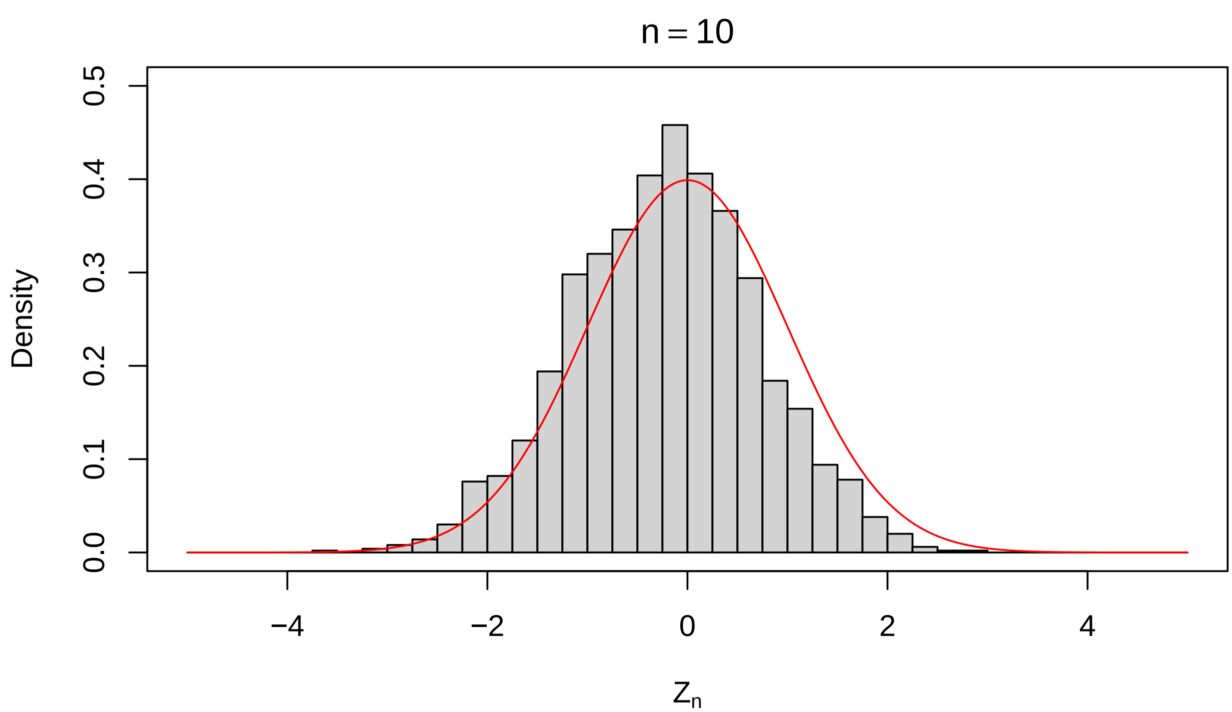
and

$$\log \overline{X}_n \sim \mathcal{AN}(\log \mu, \sigma^2/(n\mu^2))$$

so that

$$Z_n = \frac{\sqrt{n}(\log \overline{X}_n - \log \mu)}{\sqrt{\sigma^2/\mu^2}} \xrightarrow{d} Z \sim Normal(0,1).$$

```
set.seed(194)
n<-10;N<-2000
xv<-seq(-5,5,length=1001);yv<-dnorm(xv)
sval<-sig1/mu1
Zn<-replicate(N,sqrt(n)*(log(mean(rinvgauss(n,mu1,be1)))-log(mu1))/sval)
par(mar=c(4,4,2,1))
hist(Zn,breaks=seq(-5,5,by=0.25),freq=FALSE,
        main=substitute(n==nv,list(nv=n)),ylim=range(0,0.5),xlab=expression(Z[n]));box()
lines(xv,yv,col='red')
```



(b) Consider the function $g(x) = 1/x$, so $\dot{g}(x) = -1/x^2$. Then the Delta method gives

$$R_n = \sqrt{n}\left(\frac{1}{\overline{X}_n} - \frac{1}{\mu}\right) \xrightarrow{d} X \sim Normal(0, \sigma^2/\mu^4)$$

or,

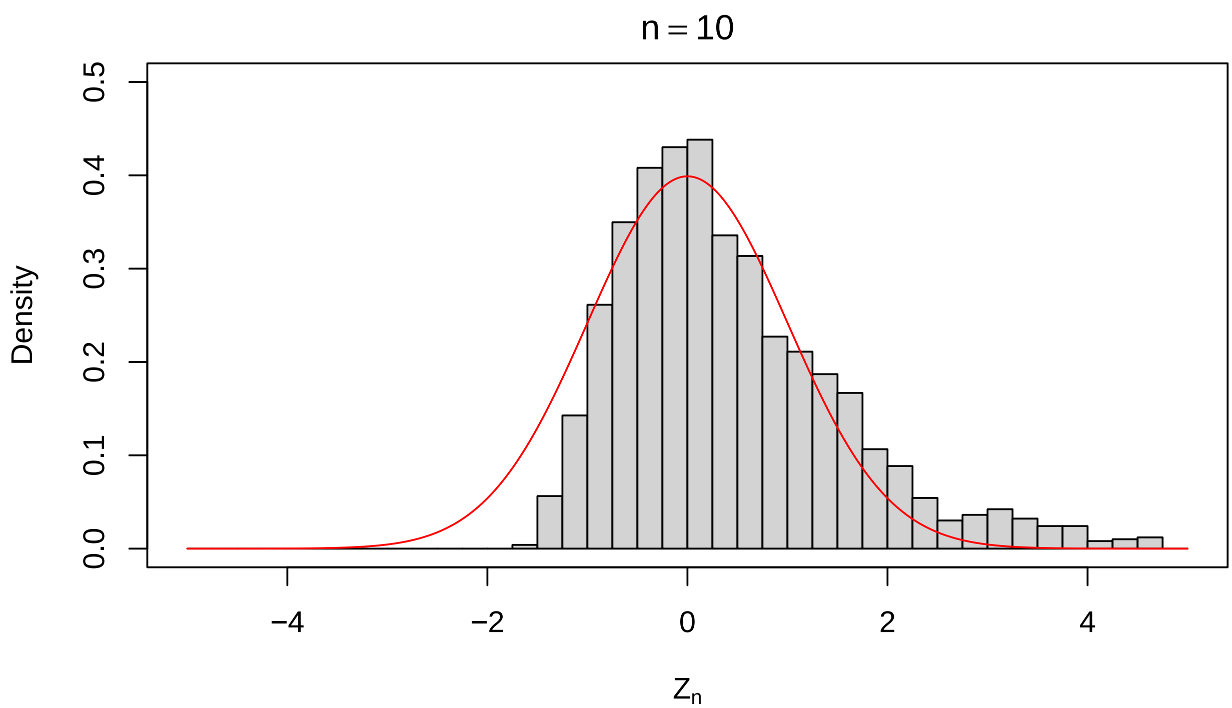
$$\frac{1}{\overline{X}_n} \sim \mathcal{AN}\left(\frac{1}{\mu}, \frac{\sigma^2}{n\mu^4}\right)$$

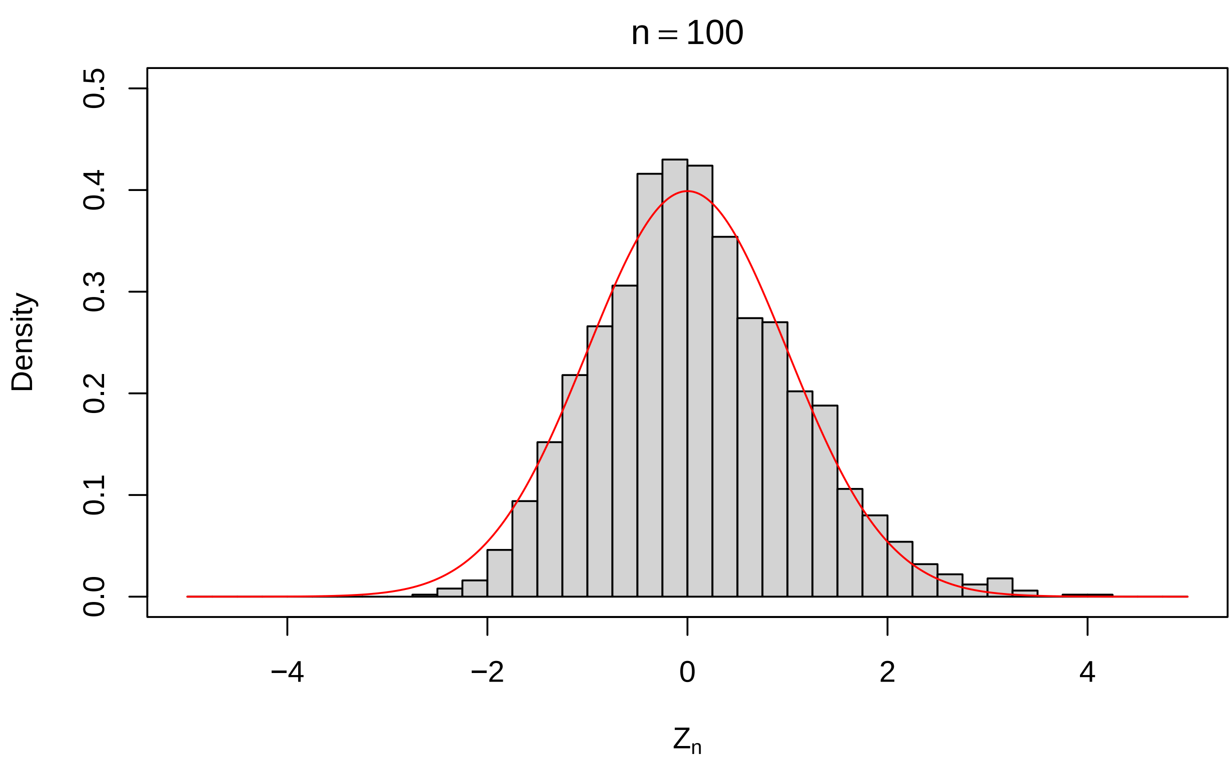
and

$$Z_n = \frac{\sqrt{n}\left(\frac{1}{\overline{X}_n} - \frac{1}{\mu}\right)}{\sqrt{\sigma^2/\mu^4}} \xrightarrow{d} Z \sim Normal(0,1).$$

Here the distribution of R_n is more skewed, and a larger n is required before the Normal approximation is deemed adequate.

```r
set.seed(194);n<-10;N<-2000
xv<-seq(-5,5,length=1001);yv<-dnorm(xv)
Zn<-replicate(N,sqrt(n)*(1/(mean(rinvgauss(n,mu1,be1)))-1/mu1)/(sig1/mu1^2))
par(mar=c(4,4,2,1))
ind<-abs(Zn)<5
hist(Zn[ind],breaks=seq(-5,5,by=0.25),freq=FALSE,
        main=substitute(n==nv,list(nv=n)),ylim=range(0,0.5),xlab=expression(Z[n]));box()
lines(xv,yv,col='red')
```





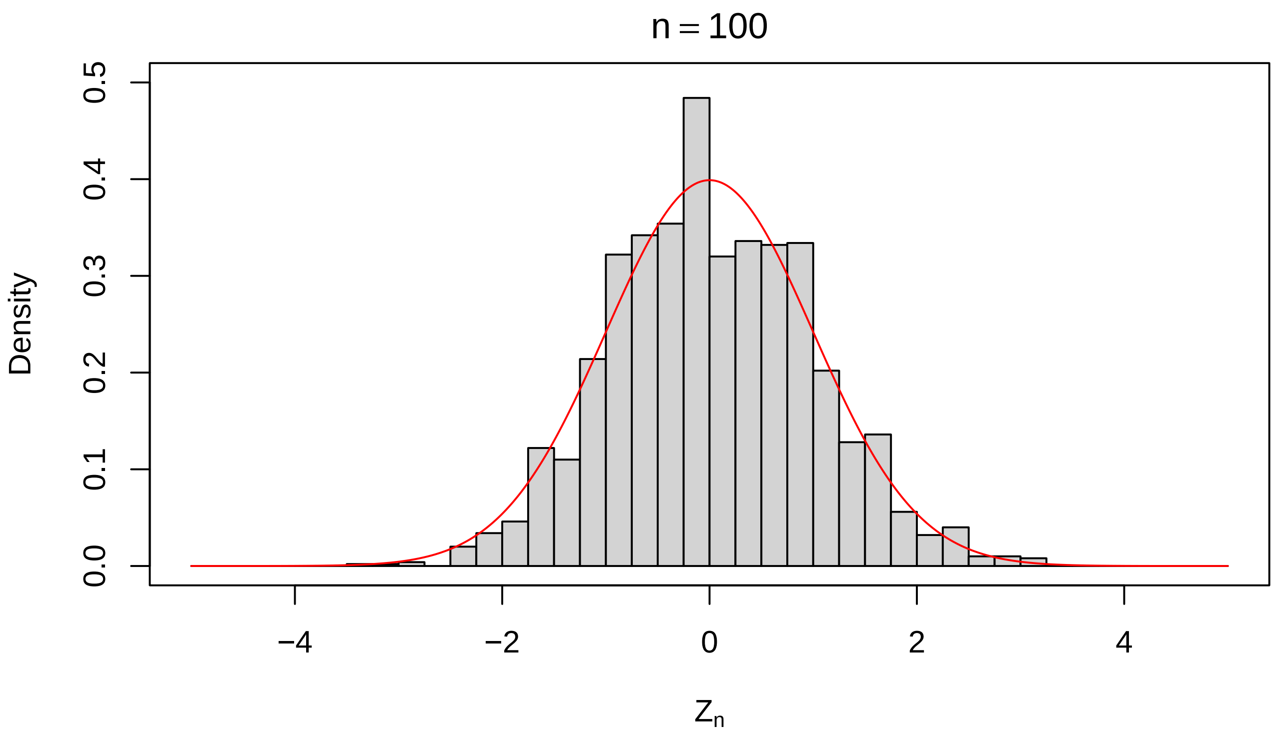
(c) If $X_1, \ldots, X_n$ are iid $Poisson(\lambda)$, then we have

$$M_n = \sqrt{n}(\overline{X}_n - \lambda) \xrightarrow{d} X \sim Normal(0, \lambda)$$

so that

$$Z_n = \frac{\sqrt{n}(\overline{X}_n - \lambda)}{\sqrt{\lambda}} \xrightarrow{d} Z \sim Normal(0, 1).$$

```
n<-100
lam<-3
xv<-seq(-5,5,length=1001);yv<-dnorm(xv)
Zn<-replicate(N,sqrt(n)*(mean(rpois(n,lam))-lam)/sqrt(lam))
par(mar=c(4,4,2,1))
ind<-abs(Zn)<5
hist(Zn[ind],breaks=seq(-5,5,by=0.25),freq=FALSE,
        main=substitute(n==nv,list(nv=n)),ylim=range(0,0.5),xlab=expression(Z[n]));box()
lines(xv,yv,col='red')
```



Consider the case $g(x) = x^2 - 2x$ so that $\dot{g}(x) = 2(x-1)$. If $\lambda \neq 1$, we then have if

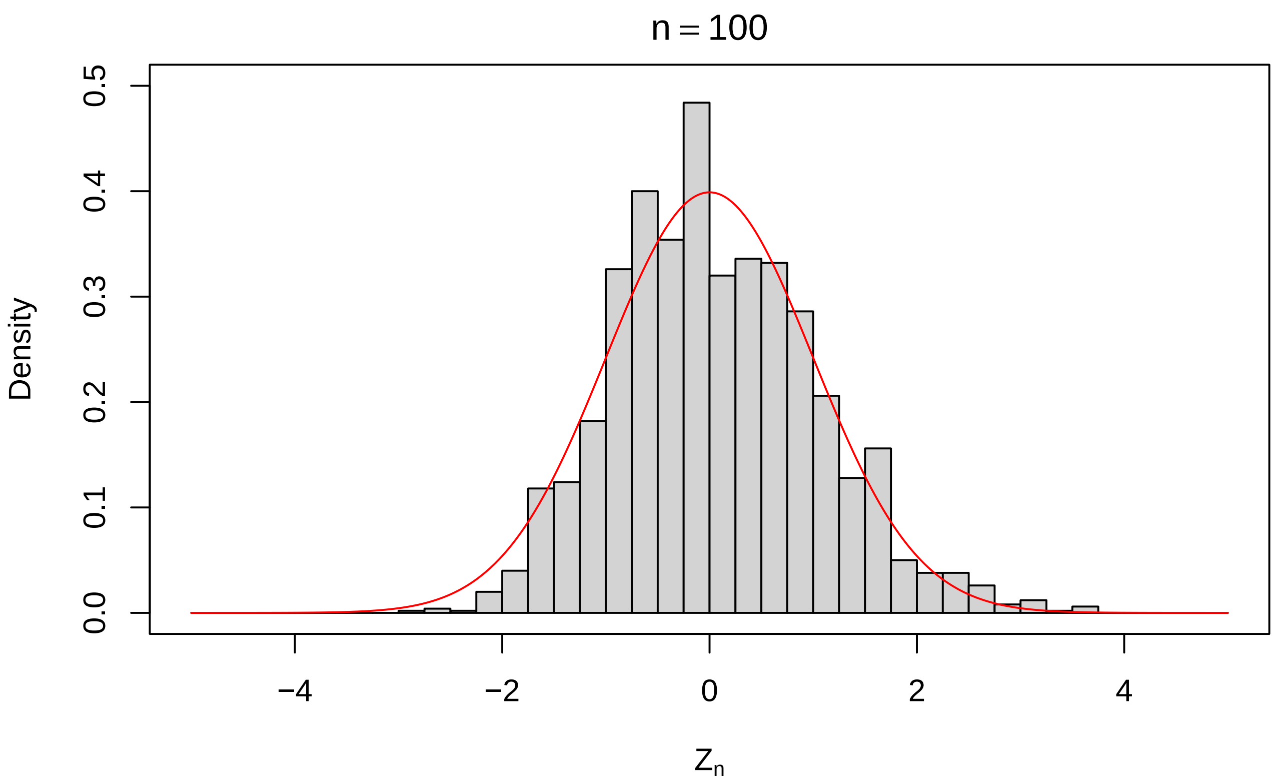
$$g(\overline{X}_n) = \overline{X}_n^2 - 2\overline{X}_n$$

then we have that

$$Z_n = \sqrt{n}\frac{(g(\overline{X}_n) - g(\lambda))}{\sqrt{4\lambda(\lambda-1)^2}} \xrightarrow{d} Z \sim Normal(0, 1)$$

```
Xbarn<-sqrt(lam)*Zn/sqrt(n)+lam
gXn<-Xbarn^2 - 2*Xbarn
mu<-lam^2-2*lam
sigsq<-4*lam*(lam-1)^2
Zn<-sqrt(n)*(gXn-mu)/sqrt(sigsq)
par(mar=c(4,4,2,1))
ind<-abs(Zn)<5
```

```
hist(Zn[ind],breaks=seq(-5,5,by=0.25),freq=FALSE,
        main=substitute(n==nv,list(nv=n)),ylim=range(0,0.5),xlab=expression(Z[n]));box()
lines(xv,yv,col='red')
```



However, if $\lambda = 1$ this approximation does not work as $\dot{g}(\lambda) = 0$. In this case we need to use the second order Delta method that uses the *second order* Taylor expansion at λ

$$g(X_n) = g(\lambda) + \dot{g}(\lambda)(X_n - \lambda) + \frac{\ddot{g}(X^\star)}{2}(X_n - \mu)^2$$

so that as $\dot{g}(\lambda) = 0$ when $\lambda = 1$, we have by the Taylor expansion for $|X_n - X^\star| < |X_n - \lambda|$ that

$$n(g(X_n) - g(\lambda)) = \frac{\ddot{g}(X^\star)}{2}\{\sqrt{n}(X_n - \lambda)\}^2 \xrightarrow{d} \frac{\ddot{g}(\lambda)}{2}\lambda Z^2$$

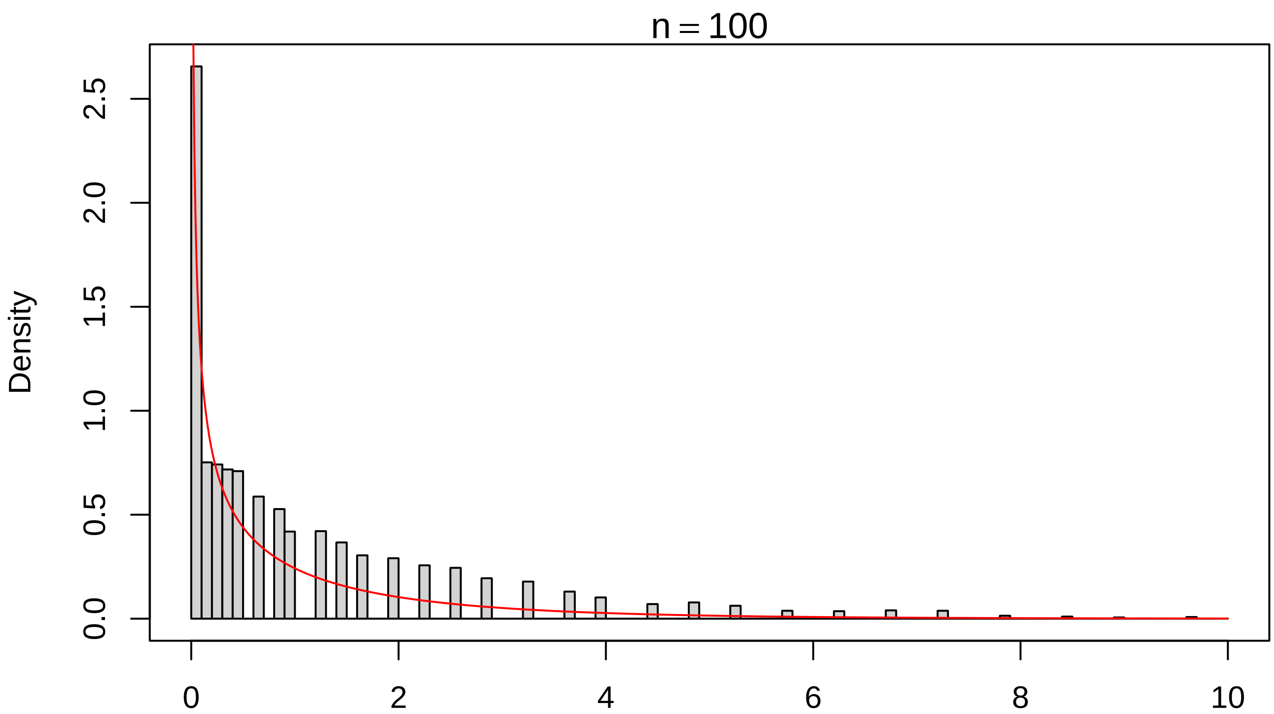
where $Z \sim Normal(0,1)$. Here $\ddot{g}(\lambda) = 2$. Thus, by the usual transformation result

$$V_n = n(g(X_n) - g(\lambda)) \xrightarrow{d} V \sim Gamma\left(\frac{1}{2}, \frac{1}{\lambda}\right)$$

or equivalently

$$W_n = \frac{n(g(X_n) - g(\lambda))}{\lambda} \xrightarrow{d} W \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right) \equiv \chi^2_1.$$

```
n<-100;N<-5000;lam<-1
Xbarn<-replicate(N,mean(rpois(n,lam)))
gXn<-Xbarn^2 - 2*Xbarn
mu<-lam^2-2*lam
Wn<-n*(gXn-mu)/lam;ind<-Wn<10
xv<-seq(0,10,length=1001);yv<-dchisq(xv,1)
par(mar=c(3,4,1,1))
hist(Wn[ind],breaks=seq(0,10,by=0.1),freq=FALSE,
        main=substitute(n==nv,list(nv=n)),xlab=expression(W[n]));box()
lines(xv,yv,col='red')
```




```
n<-1000
Xbarn<-replicate(N,mean(rpois(n,lam)))
par(mar=c(3,4,1,1))
gXn<-Xbarn^2 - 2*Xbarn
Wn<-n*(gXn-mu)/lam;ind<-Wn<10
hist(Wn[ind],breaks=seq(0,10,by=0.1),freq=FALSE,
        main=substitute(n==nv,list(nv=n)),xlab=expression(W[n]));box()
lines(xv,yv,col='red')
```