MATH 556: MATHEMATICAL STATISTICS I
CONVERGENCE IN DISTRIBUTION: WORKED EXAMPLES

EXAMPLE 1: Continuous random variable X_n with support $\mathbb{X} \equiv (0, n]$ for $n > 0$ and cdf

$$F_{X_n}(x) = 1 - \left(1 - \frac{x}{n}\right)^n \qquad 0 < x \leq n$$

and standard cdf behaviour outside of this support. Then as $n \longrightarrow \infty$, $\mathbb{X} \equiv (0, \infty)$, and for all $x > 0$

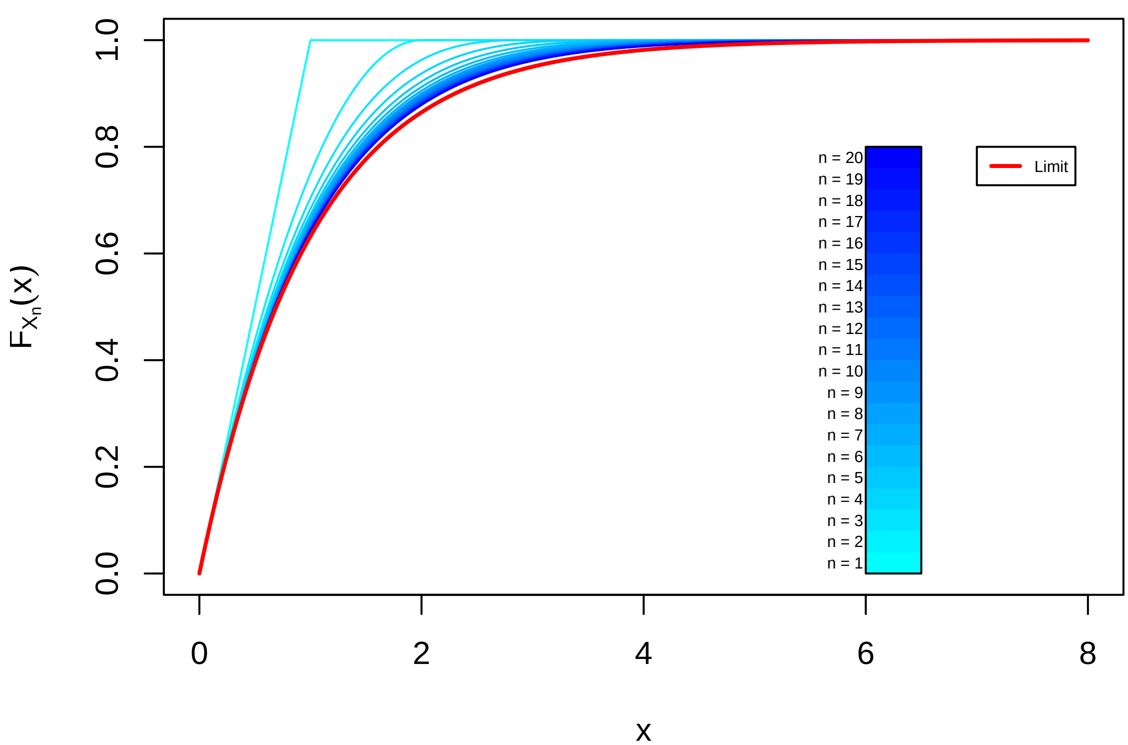
$$F_{X_n}(x) \longrightarrow 1 - e^{-x} \qquad \therefore \qquad F_{X_n}(x) \longrightarrow F_X(x) = 1 - e^{-x}$$

and hence

$$X_n \xrightarrow{d} X$$

with $X \sim \mathit{Exponential}(1)$.

```
library(plotrix)
redblue<-colorRampPalette(c("cyan","blue"))
N<-20
rbc<-redblue(N)
nvec<-c(1:N)
xvec<-seq(0,8,by=0.01)
par(mar=c(4,4,1,1))
plot(xvec,xvec*0,type='n',ylim=range(0,1),xlab='x',ylab=expression(F[X[n]](x)))
for(i in 1:N){
    xv<-pmin(xvec,nvec[i])
    yvec<-1-(1-xv/nvec[i])^nvec[i]
    lines(xvec,yvec,col=rbc[i])
}
lines(xvec,pexp(xvec,1),col='red',lwd=2)
color.legend(6,0.0,6.5,0.8,paste('n =',1:N),rbc,gradient="y",cex=0.5)
legend(7,0.8,c('Limit'),lwd=2,lty=1,col='red',cex=0.5)
```



EXAMPLE 2: Continuous random variable X_n with support $\mathbb{X} \equiv (0, \infty)$ and cdf

$$F_{X_n}(x) = \left(1 - \frac{1}{1+nx}\right)^n \qquad 0 < x < \infty$$

and zero otherwise. Then as $n \longrightarrow \infty$, for all $x > 0$

$$F_{X_n}(x) \longrightarrow e^{-1/x} \qquad \therefore \qquad F_{X_n}(x) \longrightarrow F_X(x) = e^{-1/x}$$

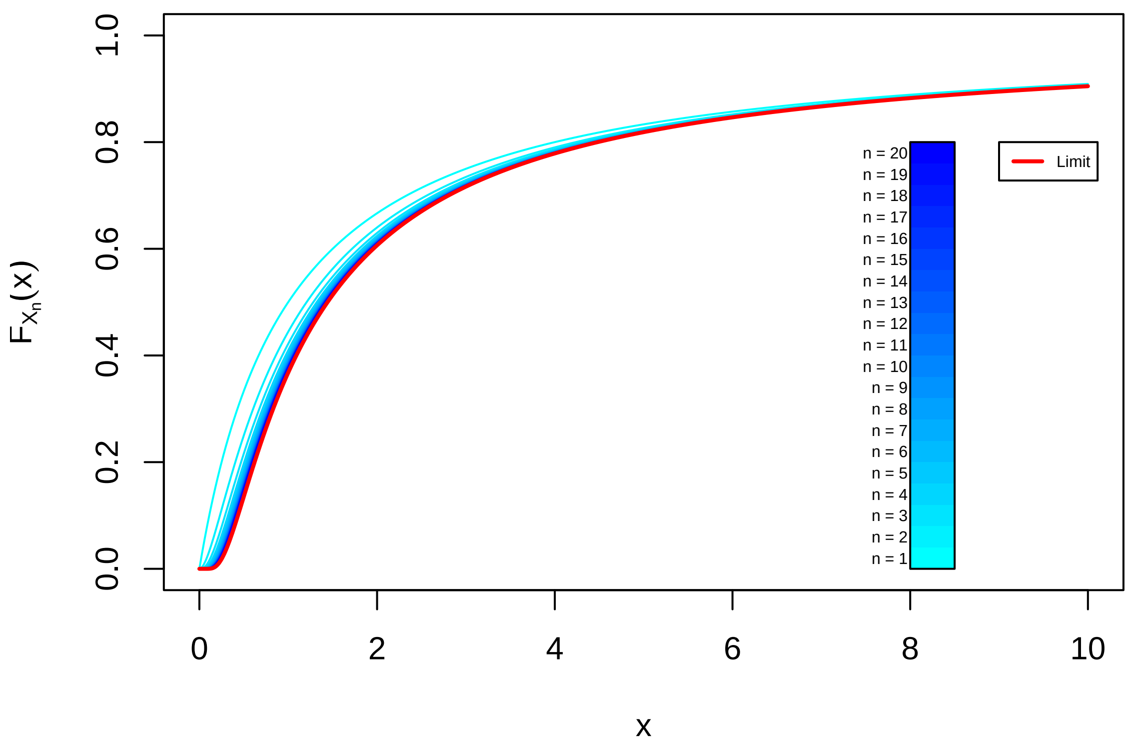
as

$$\lim_{n\longrightarrow\infty} \left(1 - \frac{1}{1+nx}\right)^n = \lim_{n\longrightarrow\infty} \left(1 - \frac{1}{nx}\right)^n = \lim_{n\longrightarrow\infty} \left(1 - \frac{1/x}{n}\right)^n$$

and for any z

$$\lim_{n\longrightarrow\infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

```
N<-20
nvec<-c(1:N)
xvec<-seq(0,10,by=0.01)
par(mar=c(4,4,1,1))
plot(xvec,xvec*0,type='n',ylim=range(0,1),xlab='x',ylab=expression(F[X[n]](x)))
for(i in 1:N){
    yvec<-(1-1/(1+nvec[i]*xvec))^nvec[i]
    lines(xvec,yvec,col=rbc[i])
}
fx<-exp(-1/xvec)
lines(xvec,fx,col='red',lwd=2)
color.legend(8,0.0,8.5,0.8,paste('n =',1:N),rbc,gradient="y",cex=0.5)
legend(9,0.8,c('Limit'),lwd=2,lty=1,col='red',cex=0.5)
```



EXAMPLE 3: Continuous random variable X_n with support $\mathbb{X} \equiv [0,1]$ and cdf

$$F_{X_n}(x) = x - \sin(2n\pi x)/(2n\pi) \qquad 0 \le x \le 1$$

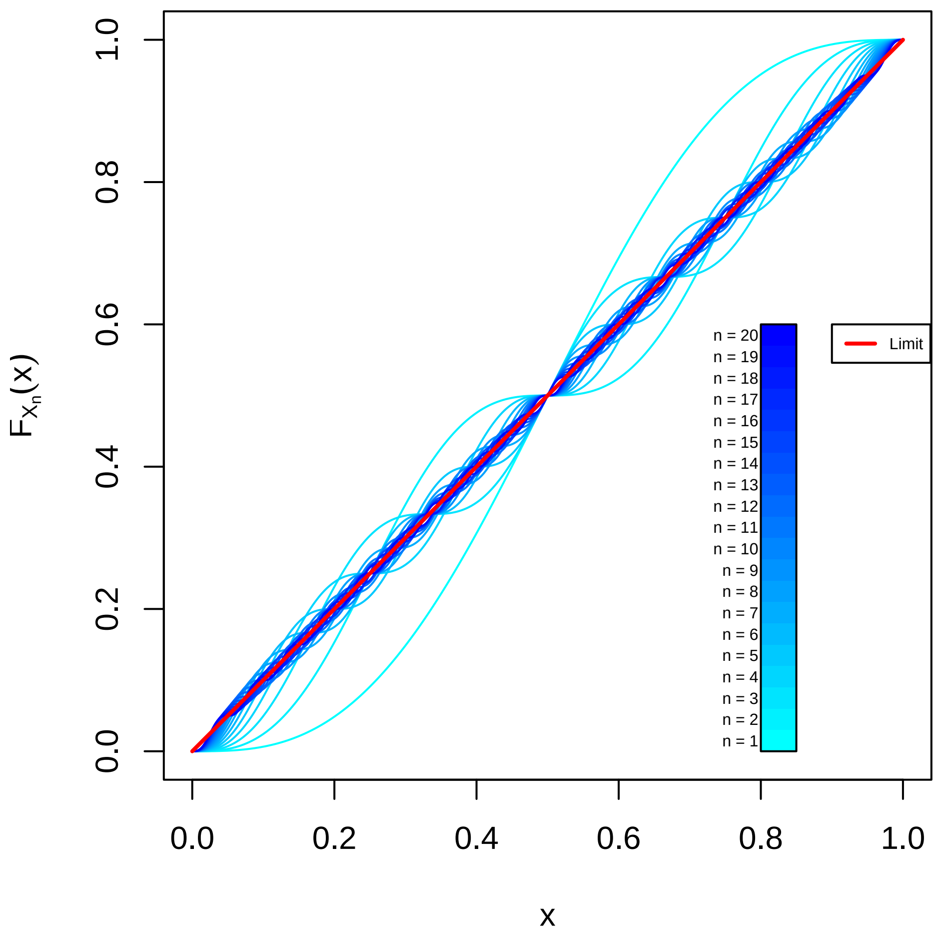
and standard cdf behaviour outside of this support. Then as $n \longrightarrow \infty$, and for all $0 \le x \le 1$

$$F_{X_n}(x) \longrightarrow x \qquad \therefore \qquad F_{X_n}(x) \longrightarrow F_X(x) = x$$

and hence

$$X_n \xrightarrow{d} X \qquad \text{where } X \sim Uniform(0,1)$$

```
N<-20
nvec<-c(1:N)
xvec<-seq(0,1,by=0.001)
par(mar=c(4,4,1,1))
plot(xvec,xvec*0,type='n',ylim=range(0,1),xlab='x',ylab=expression(F[X[n]](x)))
for(i in 1:N){
    yvec<-xvec-sin(2*nvec[i]*pi*xvec)/(2*nvec[i]*pi)
    lines(xvec,yvec,col=rbc[i])
}
lines(xvec,xvec,col='red',lwd=2)
color.legend(0.8,0.0,0.85,0.6,paste('n =',1:N),rbc,gradient="y",cex=0.5)
legend(0.9,0.6,c('Limit'),lwd=2,lty=1,col='red',cex=0.5)
```



NOTE: for the pdf

$$f_{X_n}(x) = 1 - \cos(2n\pi x) \qquad 0 \le x \le 1$$

and for all x there **is no limiting value** $n \longrightarrow \infty$.

EXAMPLE 4: Continuous random variable X_n with support $\mathbb{X} \equiv [0,1]$ and cdf

$$F_{X_n}(x) = 1 - (1-x)^n \qquad 0 \le x \le 1$$

and standard cdf behaviour outside of this support. Then as $n \longrightarrow \infty$, and for $x \in \mathbb{R}$

$$F_{X_n}(x) \longrightarrow \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}.$$

This limiting form is **not** continuous at $x = 0$, as $x = 0$ is not a point of continuity, and the **ordinary definition of convergence in distribution cannot be applied**. However, it is clear that for $\epsilon > 0$,

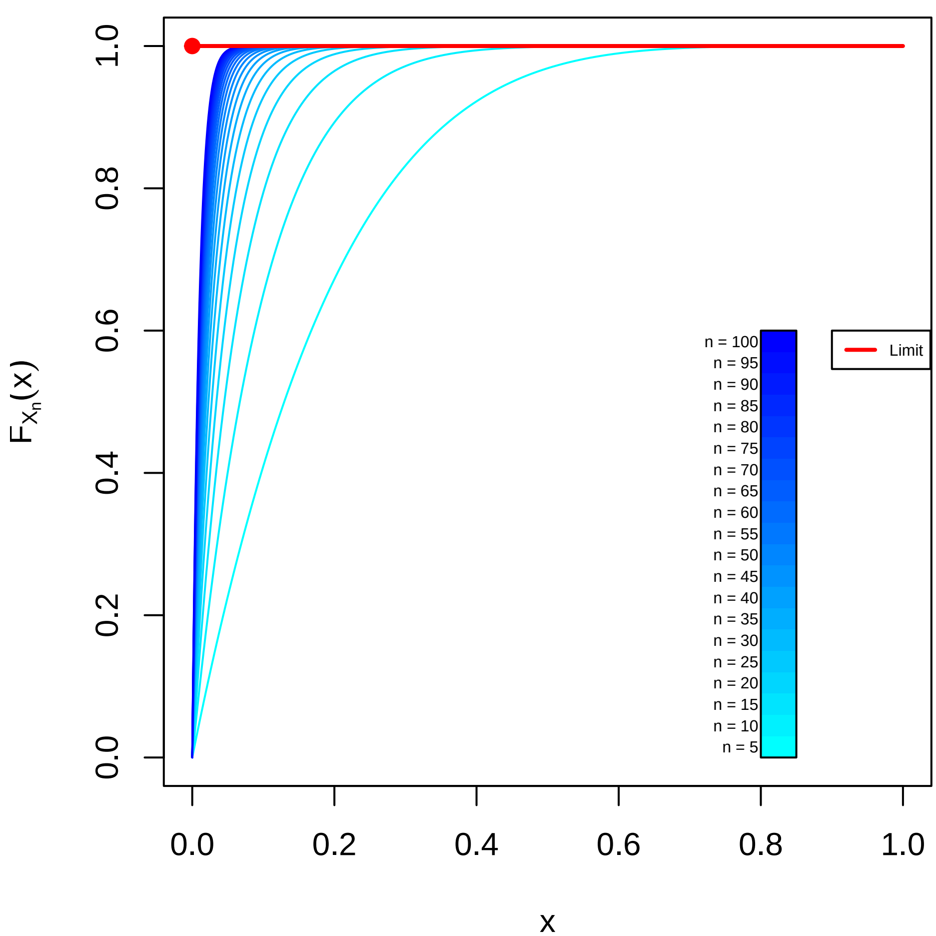
$$P_{X_n}[|X_n| < \epsilon] = 1 - (1-\epsilon)^n \longrightarrow 1 \text{ as } n \longrightarrow \infty$$

so it is still correct to say

$$X_n \xrightarrow{d} X \qquad \text{where } P_X[X = 0] = 1$$

so the limiting distribution is **degenerate at** $x = 0$.

```
N<-20;nvec<-5*c(1:N)
xvec<-seq(0,1,by=0.001)
par(mar=c(4,4,1,1))
plot(xvec,xvec*0,type='n',ylim=range(0,1),xlab='x',ylab=expression(F[X[n]](x)))
for(i in 1:N){
    yvec<-1-(1-xvec)^nvec[i]
    lines(xvec,yvec,col=rbc[i])
}
lines(c(0,1),c(1,1),col='red',lwd=2);points(0,1,pch=19,col='red')
color.legend(0.8,0.0,0.85,0.6,paste('n =',5*c(1:N)),rbc,gradient="y",cex=0.5)
legend(0.9,0.6,c('Limit'),lwd=2,lty=1,col='red',cex=0.5)
```



EXAMPLE 5: Continuous random variable X_n with support $\mathbb{X} \equiv (0, \infty)$ and cdf

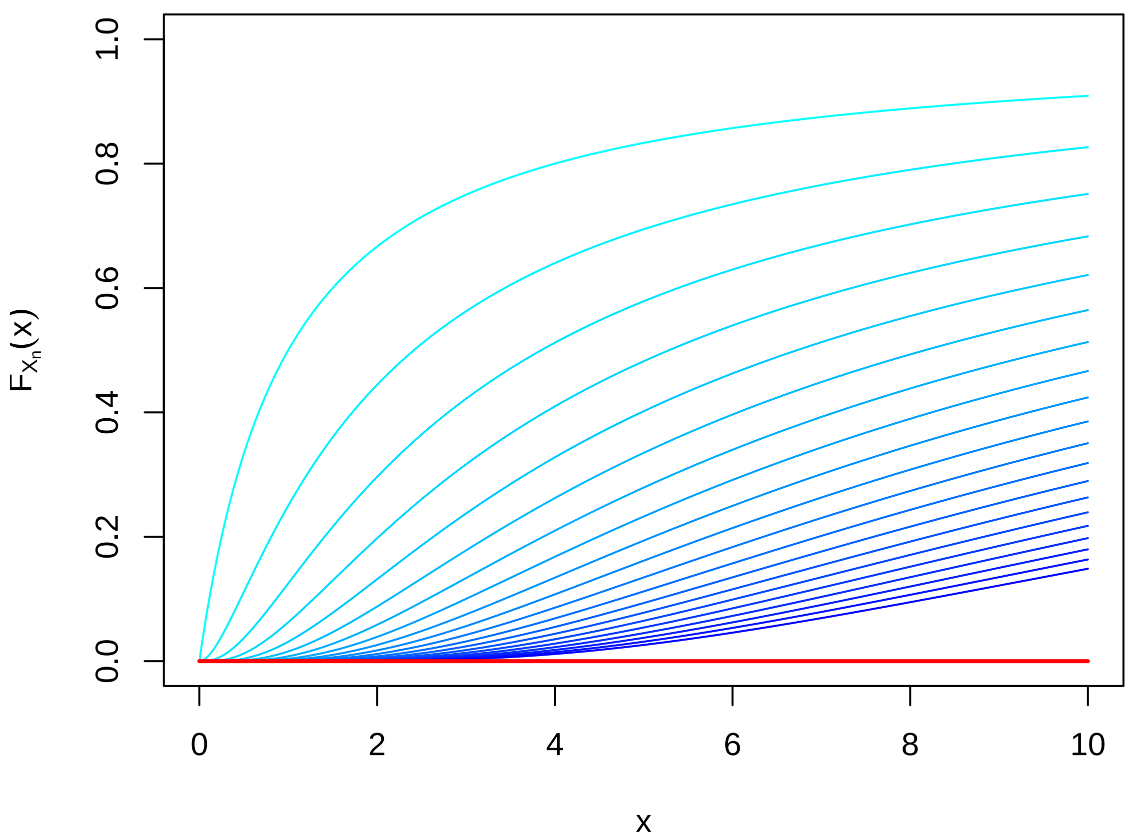
$$F_{X_n}(x) = \left(\frac{x}{1+x}\right)^n \qquad x > 0$$

and zero otherwise. Then as $n \longrightarrow \infty$, and for $x > 0$

$$F_{X_n}(x) \longrightarrow 0$$

Thus there is **no limiting distribution**.

```
N<-20
nvec<-c(1:N)
xvec<-seq(0,10,by=0.01)
par(mar=c(4,4,1,1))
plot(xvec,xvec*0,type='n',ylim=range(0,1),xlab='x',ylab=expression(F[X[n]](x)))
for(i in 1:N){
    yvec<-(xvec/(1+xvec))^nvec[i]
    lines(xvec,yvec,col=rbc[i])
}
lines(c(0,10),c(0,0),col='red',lwd=2)
```



Now let $V_n = X_n/n$. Then $\mathbb{V} \equiv (0, \infty)$ and the cdf of V_n is

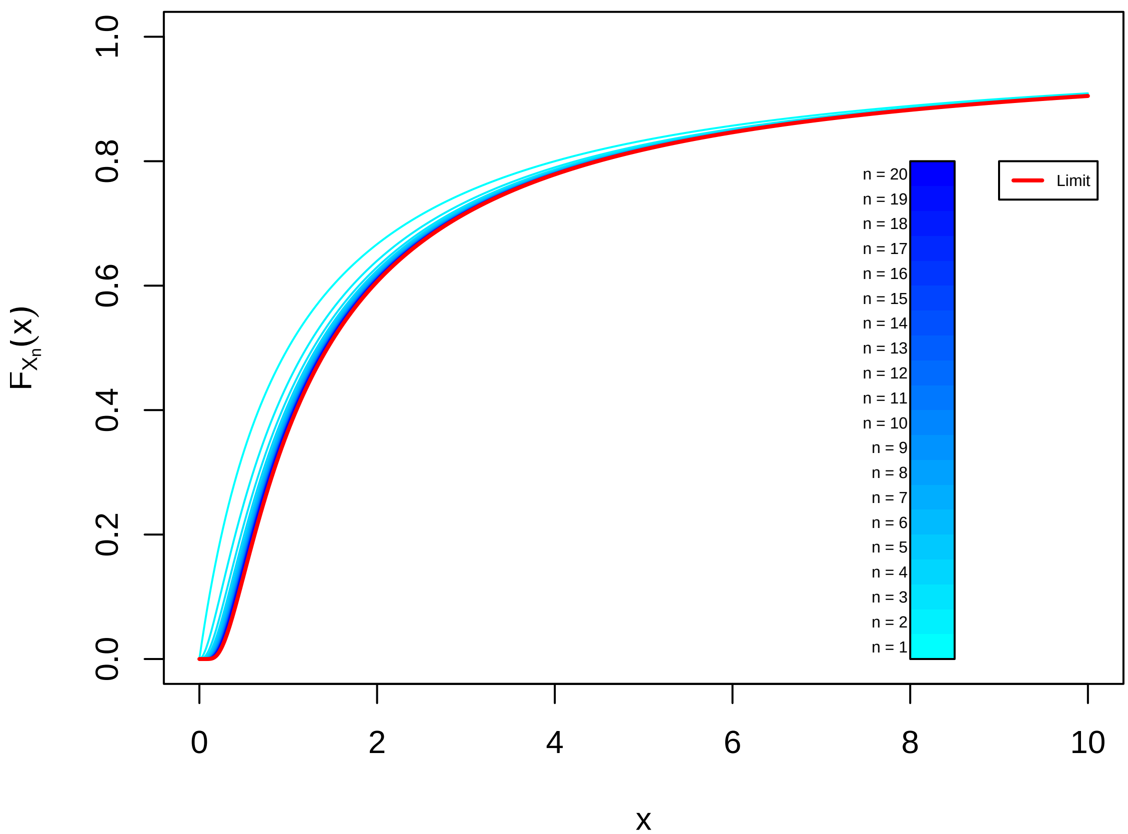
$$F_{V_n}(v) = P_{V_n}[V_n \leq v] = P_{X_n}[X_n/n \leq v] = P_{X_n}[X_n \leq nv] = F_{X_n}(nv) = \left(\frac{nv}{1+nv}\right)^n \qquad v > 0$$

and as $n \longrightarrow \infty$, for all $v > 0$

$$F_{V_n}(v) \longrightarrow e^{-1/v} \qquad \therefore \qquad F_{V_n}(v) \longrightarrow F_V(v) = e^{-1/v}$$

and the limiting distribution of V_n does exist.

```
N<-20
nvec<-c(1:N)
xvec<-seq(0,10,by=0.01)
par(mar=c(4,4,1,1))
plot(xvec,xvec*0,type='n',ylim=range(0,1),xlab='x',ylab=expression(F[X[n]](x)))
for(i in 1:N){
    yvec<-(nvec[i]*xvec/(1+nvec[i]*xvec))^nvec[i]
    lines(xvec,yvec,col=rbc[i])
}
fx<-exp(-1/xvec)
lines(xvec,fx,col='red',lwd=2)
color.legend(8,0.0,8.5,0.8,paste('n =',1:N),rbc,gradient="y",cex=0.5)
legend(9,0.8,c('Limit'),lwd=2,lty=1,col='red',cex=0.5)
```



EXAMPLE 6: Continuous random variable X_n with support $\mathbb{X} \equiv (0, \infty)$ and cdf

$$F_{X_n}(x) = \frac{\exp(nx)}{1+\exp(nx)} \qquad x \in \mathbb{R}$$

and zero otherwise. Then as $n \longrightarrow \infty$

$$F_{X_n}(x) \longrightarrow \begin{cases} 0 & x < 0 \\ \dfrac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \qquad x \in \mathbb{R}$$

This limiting form is **not** a cdf, as it is not right continuous at $x = 0$. However, as $x = 0$ is not a point of continuity, and the ordinary definition of convergence in distribution does not apply. However, it is clear that for $\epsilon > 0$,

$$P_{X_n}[|X_n| < \epsilon] = \frac{\exp(n\epsilon)}{1+\exp(n\epsilon)} - \frac{\exp(-n\epsilon)}{1+\exp(-n\epsilon)} \longrightarrow 1 \text{ as } n \longrightarrow \infty$$

so it is still correct to say $X_n \xrightarrow{d} X$ where $P_X[X = 0] = 1$, and the limiting distribution is **degenerate at** $x = 0$.

```
N<-20;nvec<-c(1:N);xvec<-seq(-10,10,by=0.01)
par(mar=c(4,4,1,1))
plot(xvec,xvec*0,type='n',ylim=range(0,1),xlab='x',ylab=expression(F[X[n]](x)))
for(i in 1:N){
    yvec<-(exp(nvec[i]*xvec)/(1+exp(nvec[i]*xvec)))
    lines(xvec,yvec,col=rbc[i])
}
lines(c(-10,0),c(0,0),col='red',lwd=2)
lines(c(0,10),c(1,1),col='red',lwd=2);points(0,1,pch=19,col='red')
color.legend(7,0.0,8,0.6,paste('n =',c(1:N)),rbc,gradient="y",cex=0.5)
legend(8.5,0.6,c('Limit'),lwd=2,lty=1,col='red',cex=0.5)
```