MATH 556: MATHEMATICAL STATISTICS I

SOME NOTES ON CHARACTERISTIC FUNCTIONS

The characteristic function for a random variable X with pmf/pdf f_X is defined for $t \in \mathbb{R}$ as

$$\begin{aligned}\varphi_X(t) &= \mathbb{E}_X[e^{itX}] = \mathbb{E}_X[\cos(tX) + i\sin(tX)] = \mathbb{E}_X[\cos(tX)] + i\mathbb{E}_X[\sin(tX)] \\ &= \int_{-\infty}^{\infty} e^{itx}\, dF_X(x) = \int_{-\infty}^{\infty} \cos(tx)\, dF_X(x) + i\int_{-\infty}^{\infty} \sin(tx)\, dF_X(x)\end{aligned}$$

using the $dF_X(x)$ notation, where as usual the 'integral' is a sum in the discrete case. As $\cos$ and $\sin$ are bounded functions, the two expectations are finite, so $\varphi_X(t)$ is finite for all t.

Example: Double-Exponential (or Laplace) distribution

$$f_X(x) = \frac{1}{2}e^{-|x|} \qquad x \in \mathbb{R}$$

which is an even function around zero. Then

$$\begin{aligned}\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{2}e^{-|x|}\, dx &= \frac{1}{2}\int_{-\infty}^{0} (\cos(tx) + i\sin(tx))e^{x}\, dx + \frac{1}{2}\int_{0}^{\infty} (\cos(tx) + i\sin(tx))e^{-x}\, dx \\ &\equiv \frac{1}{2}\int_{0}^{\infty} (\cos(-tx) + i\sin(-tx))e^{-x}\, dx + \frac{1}{2}\int_{0}^{\infty} (\cos(tx) + i\sin(tx))e^{-x}\, dx \\ &= \frac{1}{2}\int_{0}^{\infty} (\cos(tx) - i\sin(tx))e^{-x}\, dx + \frac{1}{2}\int_{0}^{\infty} (\cos(tx) + i\sin(tx))e^{-x}\, dx \\ &= \int_{0}^{\infty} \cos(tx)e^{-x}\, dx. \qquad (1)\end{aligned}$$

as $\cos$ is an even function and $\sin$ is an odd function. Integrating (1) by parts we have

$$\begin{aligned}\varphi_X(t) &= \left[-\cos(tx)e^{-x}\right]_0^{\infty} + \int_0^{\infty} t\sin(tx)e^{-x}\, dx \\ &= 1 + \left[-t\sin(tx)e^{-x}\right]_0^{\infty} - \int_0^{\infty} t^2\cos(tx)e^{-x}\, dx = 1 - t^2\varphi_X(t) \qquad \therefore \qquad \varphi_X(t) = \frac{1}{1+t^2}.\end{aligned}$$

Example: Normal distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad x \in \mathbb{R}$$

Then,

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\, dx.$$

Now in the exponent

$$itx - \frac{1}{2}x^2 = -\frac{1}{2}(x-it)^2 + \frac{(it)^2}{2} = -\frac{1}{2}(x-it)^2 - \frac{t^2}{2}$$

so we have

$$\varphi_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-(x-it)^2/2}e^{-t^2/2}\, dx = e^{-t^2/2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-(x-it)^2/2}\, dx = e^{-t^2/2}.$$

as the integral is equal to the standard Normal integral.

General Results: The following results also hold:

- $\varphi_X(t)$ is **continuous** for all t; this follows as $\cos$ and $\sin$ are continuous functions of x, and sums and integrals of continuous functions are also continuous.

In fact, we can prove the stronger result that $\varphi_X(t)$ is **uniformly continuous** on $\mathbb{R}$. Consider, for $h > 0$

$$
\begin{aligned}
|\varphi_X(t+h) - \varphi_X(t)| &\leq \int_{-\infty}^{\infty} |\exp\{i(t+h)x\} - \exp\{itx\}| \, dF_X(x) \\
&= \int_{-\infty}^{\infty} |\exp\{itx\}||\exp\{ihx\} - 1| \, dF_X(x) \\
&\leq \int_{-\infty}^{\infty} |\exp\{ihx\} - 1| \, dF_X(x) && \text{as } |\exp\{itx\}| \leq 1 \\
&\leq 2 && \text{as } |\exp\{ihx\} - 1| \leq 2
\end{aligned}
$$

Further

$$
x > 0 : |\exp\{ihx\} - 1| = \left| \int_0^{hx} e^{iu} \, du \right| \leq \int_0^{hx} \left| e^{iu} \right| \, du = \int_0^{hx} du = hx
$$

$$
x < 0 : |\exp\{ihx\} - 1| = \left| \int_{hx}^{0} e^{iu} \, du \right| \leq \int_{hx}^{0} \left| e^{iu} \right| \, du = \int_{hx}^{0} du = -hx
$$

and hence $|\exp\{ihx\} - 1| \leq |hx|$. Therefore

$$
|\varphi_X(t+h) - \varphi_X(t)| \leq \int_{-\infty}^{\infty} |hx| \, dF_X(x). \tag{2}
$$

Finally, let $\{h_n\}$ be any sequence such that $h_n \longrightarrow 0$ as $n \longrightarrow \infty$. As

$$
\lim_{n \longrightarrow \infty} \int_{-\infty}^{\infty} |\exp\{ih_n x\} - 1| \, dF_X(x) \leq \lim_{n \longrightarrow \infty} \int_{-\infty}^{\infty} |h_n x| \, dF_X(x)
$$

and as $\exp\{ixh\}$ is continuous at zero, we can deduce that

$$
\lim_{h \longrightarrow 0} \int_{-\infty}^{\infty} |\exp\{ihx\} - 1| \, dF_X(x) \leq \int_{-\infty}^{\infty} \lim_{n \longrightarrow \infty} |h_n x| \, dF_X(x) = 0
$$

using the dominated convergence theorem. Therefore $|\varphi_X(t+h) - \varphi_X(t)| \longrightarrow 0$ as $h \longrightarrow 0$, and $\varphi_X(t)$ is uniformly continuous in t as the bound in (2) does not depend on t.

- $\varphi_X(t)$ is **bounded in modulus** by 1: $|\varphi_X(t)| \leq \mathbb{E}_X[|e^{itX}|] = \mathbb{E}_X[1] = 1$.
- If $Y = aX + b$ for real constants a, b, then

$$
\varphi_Y(t) = \mathbb{E}_Y[\exp\{itY\}] = \mathbb{E}_X[\exp\{it(aX+b)\}] = e^{itb}\mathbb{E}_X[\exp\{i(at)X\}] = e^{itb}\varphi_X(at)
$$

- If X is continuous with pdf f_X satisfying $f_X(x) = f_X(-x)$ for all x, then

$$
\varphi_X(t) = \int_{-\infty}^{\infty} (\cos(tx) + i\sin(tx)) f_X(x) \, dx = 2\int_0^{\infty} \cos(tx) f_X(x) \, dx
$$

is entirely real and has no imaginary part.

Inversion Formulae: To compute f_X or F_X from φ_X, we may use an inversion formula. First, recall that for $x_1 \in \mathbb{R}$ we are writing

$$F_X(x_1) = P_X[X \leq x_1] = P_X[X = x_1] + P_X[X < x_1] \equiv \int_{-\infty}^{x_1} dF_X(x)$$

so that the note that the $dF_X(x)$ notation should be interpreted as meaning, for finite x_1

$$\int_{-\infty}^{x_1} dF_X(x) = \begin{cases} F_X(x_1) & F_X(x) \text{ is continuous at } x_1 \\ f_X(x_1) + \lim\limits_{x \longrightarrow x_1^-} F_X(x) & F_X(x) \text{ is not continuous at } x_1 \end{cases}$$

We can state the inversion formula result as follows:

- Let $\overline{F}_X(x)$ be defined by

  $$\overline{F}_X(x) = \frac{1}{2}\left\{F_X(x) + \lim_{y \longrightarrow x^-} F_X(y)\right\}.$$

  Then for $x_0 < x_1$

  $$\overline{F}_X(x_1) - \overline{F}_X(x_0) = \frac{1}{2\pi}\lim_{T\longrightarrow\infty}\int_{-T}^{T}\left(\frac{e^{-ix_0t} - e^{-ix_1t}}{it}\right)\varphi_X(t)\,dt$$

- Alternatively if x_0 and $x_1 = x_0 + h$ for $h > 0$ are continuity points of F_X, then

  $$F_X(x_0 + h) - F_X(x_0) = \frac{1}{2\pi}\lim_{T\longrightarrow\infty}\int_{-T}^{T}\left(\frac{1 - e^{-ith}}{it}\right)e^{-itx_0}\varphi_X(t)\,dt$$

  or equivalently

  $$F_X(x_1) - F_X(x_0) = \frac{1}{2\pi}\lim_{T\longrightarrow\infty}\int_{-T}^{T}\left(\frac{e^{-itx_0} - e^{-itx_1}}{it}\right)\varphi_X(t)\,dt$$

The alternative representation relies on considering continuity points of $F_X(x)$; this is sufficient, as by definition the number of points of discontinuity (that is, where there are masses of probability) must be countable. But even at the discontinuity points, $F_X(x)$ is right-continuous, so we can consider the limit of $F_X(x)$ evaluated at continuity points converging to the discontinuity points from above. Thus the behaviour at the continuity points entirely determines $F_X(x)$.

Note first that by elementary calculus,

$$\frac{e^{-ix_0t} - e^{-ix_1t}}{it} = \int_{x_0}^{x_1} e^{-itu}\,du.$$

Therefore

$$\begin{aligned}\int_{-T}^{T}\left(\frac{e^{-ix_0t} - e^{-ix_1t}}{it}\right)\varphi_X(t)\,dt &= \int_{-T}^{T}\left\{\int_{x_0}^{x_1} e^{-itu}\,du\right\}\left\{\int_{-\infty}^{\infty} e^{itx}dF_X(x)\right\}\,dt \\ &= \int_{-\infty}^{\infty}\int_{-T}^{T}\int_{x_0}^{x_1} e^{-it(u-x)}\,du\,dt\,dF_X(x) \\ &= \int_{-\infty}^{\infty}\int_{-T}^{T}\frac{1}{it}\left(e^{-it(x_0-x)} - e^{-it(x_1-x)}\right)\,dt\,dF_X(x). \qquad (3)\end{aligned}$$

Denote the inner integral by

$$g_1(x_0, x_1, T, x) = \int_{-T}^{T} \frac{1}{it} \left(e^{-it(x_0 - x)} - e^{-it(x_1 - x)} \right) \, dt.$$

Now

$$e^{-it(x_0 - x)} = \cos(t(x_0 - x)) - i \sin(t(x_0 - x)) \qquad e^{-it(x_1 - x)} = \cos(t(x_1 - x)) - i \sin(t(x_1 - x))$$

and splitting the integral range $(-T, 0)$ and $(0, T)$, we have that the integral becomes

$$\int_{-T}^{0} \frac{1}{it} \left(\cos(t(x_0 - x)) - i \sin(t(x_0 - x)) - \cos(t(x_1 - x)) + i \sin(t(x_1 - x)) \right) \, dt$$
$$+ \int_{0}^{T} \frac{1}{it} \left(\cos(t(x_0 - x)) - i \sin(t(x_0 - x)) - \cos(t(x_1 - x)) + i \sin(t(x_1 - x)) \right) \, dt$$
$$= - \int_{0}^{T} \frac{1}{it} \left(\cos(t(x_0 - x)) + i \sin(t(x_0 - x)) - \cos(t(x_1 - x)) - i \sin(t(x_1 - x)) \right) \, dt$$
$$+ \int_{0}^{T} \frac{1}{it} \left(\cos(t(x_0 - x)) - i \sin(t(x_0 - x)) - \cos(t(x_1 - x)) + i \sin(t(x_1 - x)) \right) \, dt$$

$$g_1(x_0, x_1, T, x) = 2 \int_{0}^{T} \left(\frac{\sin((x_1 - x)t)}{t} - \frac{\sin((x_0 - x)t)}{t} \right) = 2g_2(T, x_1 - x) - 2g_2(T, x_0 - x).$$

say, where

$$g_2(T, c) = \int_{0}^{T} \frac{\sin(ct)}{t} \, dt.$$

This is a standard integral: we have (see `https://en.wikipedia.org/wiki/Dirichlet_integral`)

$$\overline{g}_2(c) \equiv \lim_{T \longrightarrow \infty} g_2(T, c) = \int_{0}^{\infty} \frac{\sin(ct)}{t} \, dt = \begin{cases} \pi/2 & c > 0 \\ 0 & c = 0 \\ -\pi/2 & c < 0 \end{cases}$$

and for any fixed x_0, x_1, x, we need to compute when $c = x_0 - x$ and $c = x_1 - x$. Now

(i) If $x < x_0$ or $x > x_1$, then $x_0 - x$ and $x_1 - x$ have the same sign, so

$$2\overline{g}_2(x_1 - x) - 2\overline{g}_2(x_0 - x) = \pm(\pi - \pi) = 0.$$

(ii) If $x_0 = x$, then $x_0 - x = 0$ and $x_1 - x > 0$, so

$$2\overline{g}_2(x_1 - x) - 2\overline{g}_2(x_0 - x) = \pi - 0 = \pi.$$

(iii) If $x_1 = x$, then $x_0 - x < 0$ and $x_1 - x = 0$, so

$$2\overline{g}_2(x_1 - x) - 2\overline{g}_2(x_0 - x) = 0 - (-\pi) = \pi.$$

(iv) If $x_0 < x < x_1$, then $x_0 - x < 0$ and $x_1 - x > 0$, so

$$2\overline{g}_2(x_1 - x) - 2\overline{g}_2(x_0 - x) = \pi - (-\pi) = 2\pi.$$

Therefore

$$\lim_{T\longrightarrow\infty} g_1(x_0,x_1,T,x) = \begin{cases} 0 & x < x_0 \text{ or } x > x_1 \\ \pi & x = x_0 \text{ or } x = x_1 \\ 2\pi & x_0 < x < x_1. \end{cases}$$

Because of this we can deduce that $|g_1(x_0,x_1,T,x)|$ is bounded, and because $g_2(T,c)$ is continuous in T, by the dominated convergence theorem, we can pass the limit under the integral in equation (3), so

$$\begin{aligned}
\lim_{T\longrightarrow\infty}\int_{-T}^{T}\left(\frac{e^{-ix_0t}-e^{-ix_1t}}{it}\right)\varphi_X(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\longrightarrow\infty} g_1(x_0,x_1,T,x)dF_X(x) \\
&= \frac{1}{2\pi}\int_{x_0}^{x_1}\lim_{T\longrightarrow\infty} g_1(x_0,x_1,T,x)dF_X(x) \\
&= \frac{1}{\pi}\int_{x_0}^{x_1}(\overline{g}_2(x_1-x)-\overline{g}_2(x_0-x))dF_X(x) \\
&= \frac{f_X(x_0)}{\pi}(\overline{g}_2(x_1-x_0)-\overline{g}_2(x_0-x_0)) \\
&\qquad + \frac{1}{\pi}\int_{x_0^+}^{x_1^-}(\overline{g}_2(x_1-x)-\overline{g}_2(x_0-x))dF_X(x) \\
&\qquad + \frac{f_X(x_1)}{\pi}(\overline{g}_2(x_1-x_1)-\overline{g}_2(x_0-x_1)) \\
&= \frac{f_X(x_0)}{2} + (F_X(x_1^-)-F_X(x_0^+)) + \frac{f_X(x_1)}{2}
\end{aligned}$$

which follows using the previous results

$$\overline{g}_2(x_1-x_0)-\overline{g}_2(0) = \frac{\pi}{2} = \overline{g}_2(0)-\overline{g}_2(x_0-x_1)$$

$$\overline{g}_2(x_1-x)-\overline{g}_2(x_0-x) = 2\pi \qquad x_0 < x < x_1$$

Now

$$F_X(x_1^-) \equiv P_X[X < x_1] = F_X(x_1) - f_X(x_1)$$

and by right-continuity of $F_X(x)$, $F_X(x_0^+) = F_X(x_0)$, so we can re-write the final expression as

$$\left(F_X(x_1) - \frac{1}{2}f_X(x_1)\right) - \left(F_X(x_0) - \frac{1}{2}f_X(x_0)\right).$$

But note that for arbitrary x

$$F_X(x) - \frac{1}{2}f_X(x) = \frac{1}{2}F_X(x) + \frac{1}{2}\left(F_X(x) - f_X(x)\right) = \frac{1}{2}F_X(x) + \frac{1}{2}\lim_{y\longrightarrow x^-} F_X(y).$$

Thus

$$\lim_{T\longrightarrow\infty}\int_{-T}^{T}\left(\frac{e^{-ix_0t}-e^{-ix_1t}}{it}\right)\varphi_X(t)\,dt = \overline{F}_X(x_1) - \overline{F}_X(x_0)$$

by the definition of $\overline{F}_X(x)$.

In certain circumstances we may compute f_X from φ_X more straightforwardly.

(I) If X is **discrete** taking values on the integers. Then

$$\varphi_X(t) = \sum_{x=-\infty}^{\infty} e^{itx} f_X(x).$$

For integers j and x, note that

$$\int_{-\pi}^{\pi} e^{i(j-x)t}\, dt = \begin{cases} 2\pi & \text{if } x = j \\ 0 & \text{if } x \neq j \end{cases}$$

This follows as if $x \neq j$, setting $k = j - x$, we have that

$$\int_{-\pi}^{\pi} e^{i(j-x)t}\, dt = \int_{-\pi}^{\pi} e^{ikt}\, dt = \int_{-\pi}^{0} e^{ikt}\, dt + \int_{0}^{\pi} e^{ikt}\, dt.$$

Then changing $t \longrightarrow -t$ in the first integral, this equates to

$$\int_0^{\infty} e^{-ikt}\, dt + \int_0^{\pi} e^{ikt}\, dt = 2\int_0^{\pi} \cos(kt)\, dt = 0$$

for $k \neq 0$, by elementary calculus. Thus for any fixed x

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt}\varphi_X(t) dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt}\left\{\sum_{j=-\infty}^{\infty} e^{itj} f_X(j)\right\} dt = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty} f_X(j)\int_{-\pi}^{\pi} e^{i(j-x)t} dt = f_X(x)$$

as the only non-zero term is when $j = x$. Thus for $x \in \mathbb{Z}$

$$f_X(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt}\varphi_X(t)\, dt.$$

(II) If X is **continuous** and $\varphi_X(t)$ is **absolutely integrable**, that is,

$$\int_{-\infty}^{\infty} |\varphi_X(t)|\, dt < \infty$$

then

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_X(t)\, dt$$

Example: Suppose that for $t \in \mathbb{R}$,

$$\varphi_X(t) = e^{-|t|}.$$

Clearly this function is absolutely integrable wrt t, so we have

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} e^{-|t|}\, dt \quad = \quad \frac{1}{\pi}\int_0^{\infty} \cos(tx) e^{-t}\, dt.$$

Now recall that the result in equation (1) states that

$$\int_0^{\infty} \cos(tx) e^{-x}\, dx = \frac{1}{1+t^2}.$$

Therefore we may deduce immediately by exchanging the roles of t and x that

$$\frac{1}{\pi}\int_0^{\infty} \cos(tx) e^{-t}\, dt = \frac{1}{\pi}\frac{1}{1+x^2}$$

Hence we have that $X \sim$ Cauchy.

Diagnosing Discrete or Continuous Distributions

(I) If

$$\limsup_{|t|\longrightarrow\infty} |\varphi_X(t)| = 1$$

then X is often a **discrete** random variable. Technically, X may also have a **singular** distribution – see, for example www.math.mcgill.ca/dstephens/556/Papers/Koopmans.pdf. Such distributions have continuous cdfs which are not absolutely continuous.

(II) If

$$\limsup_{|t|\longrightarrow\infty} |\varphi_X(t)| = 0$$

then X is **continuous**; consequently, if

$$\lim_{|t|\longrightarrow\infty} |\varphi_X(t)| = 0$$

then X is continuous.

Interpreting the characteristic function: To get a further understanding of characteristic function, we consider the inversion formulae. For discrete random variables defined on the integers, we have

$$f_X(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt}\varphi_X(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\cos(xt) - i\sin(xt)]\,\varphi_X(t)\,dt$$

One way to think about this integral is via a discrete approximation; fix

$$t_{j,N} = -\pi + \frac{2\pi j}{N} \qquad j = 0, 1, 2, \ldots, N$$

and write

$$f_X(x) \simeq \frac{1}{2\pi}\left\{\sum_{j=0}^{N}\cos(xt_{j,N})\varphi_X(t_{j,N}) - i\sum_{j=0}^{N}\sin(xt_{j,N})\varphi_X(t_{j,N})\right\}$$

(I) Suppose f_X is **degenerate** at x_0, that is,

$$f_X(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

Then by elementary calculations

$$\varphi_X(t) = \cos(x_0 t) + i\sin(x_0 t)$$

so that

$$\text{Re}(\varphi_X(t)) = \cos(x_0 t) \qquad\qquad \text{Im}(\varphi_X(t)) = \sin(x_0 t)$$

that is, pure sinusoids with period $2\pi/x_0$.

(II) Suppose f_X is **discrete**, then as above

$$\varphi_X(t) = \sum_{j=1}^{\infty}\cos(tx_j)f_X(x_j) + i\sum_{j=1}^{\infty}\sin(tx_j)f_X(x_j)$$

so that

$$\text{Re}(\varphi_X(t)) = \sum_{j=1}^{\infty}\cos(tx_j)f_X(x_j) \qquad\qquad \text{Im}(\varphi_X(t)) = \sum_{j=1}^{\infty}\sin(tx_j)f_X(x_j)$$

that is, a weighted sum of pure sinusoids with period $2\pi/x_1, 2\pi/x_2, \ldots$, with weights determined by f_X

Moments: By a standard series expansion, for $t \in \mathbb{R}$,

$$\exp\{it\} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!}$$

and further, for each $m = 1, 2, \ldots$

$$\exp\{it\} = \sum_{r=0}^{m} \frac{(it)^r}{r!} + R_m(t)$$

where it can be shown that

$$|R_m(t)| \leq \min\left\{\frac{|t|^{m+1}}{(m+1)!}, \frac{2|t|^m}{m!}\right\}.$$

Therefore provided that $\mathbb{E}_X[X^m] < \infty$ is finite (so that $\mathbb{E}_X[X^r] < \infty, r = 1, 2, \ldots, m$) it follows that

$$\varphi_X(t) = \mathbb{E}_X[\exp\{itX\}] = \sum_{r=0}^{m} \frac{(it)^r}{r!}\mathbb{E}_X[X^r] + \mathbb{E}_X[R_m(tX)]$$

$$= 1 + \sum_{r=1}^{m} \frac{(it)^r}{r!}\mathbb{E}_X[X^r] + \mathbb{E}_X[R_m(tX)]$$

It can be shown that as $t \longrightarrow 0$,

$$\frac{\mathbb{E}_X[R_m(tX)]}{|t|^m} \longrightarrow 0$$

and hence

$$\varphi_X(t) = 1 + \sum_{r=1}^{m} \frac{(it)^r}{r!}\mathbb{E}_X[X^r] + \mathrm{o}(t^m)$$

as $t \longrightarrow 0$. This implies that $\varphi_X(t)$ is m times differentiable at $t = 0$, and

$$\varphi_X^{(r)}(0) = \left.\frac{d^r\varphi_X(t)}{dt^r}\right|_{t=0} = i^r\mathbb{E}_X[X^r] \qquad r = 1, 2, \ldots, m.$$

In general, the derivatives of $\varphi_X(t)$ are not guaranteed to be finite; we can consider

$$\varphi_X^{(r)}(t) = \frac{d^r}{dt^r}\{\varphi_X(t)\}$$

but this quantity may not be defined, or finite, at any given t; if $r = 1$

$$\varphi_X^{(1)}(t) = \mathbb{E}_X[-X\sin(tX)] + i\mathbb{E}_X[X\cos(tX)].$$

but there is no guarantee that either expectation is finite. For example, for the Cauchy distribution

$$\varphi_X(t) = e^{-|t|}$$

which has undefined derivative at $t = 0$.