MATH 556 – EXAMPLE MID-TERM EXAMINATION

Marks can be obtained by answering all questions. All questions carry equal marks.

The total mark available is 60, but rescaling of the final mark may occur.

Show your working in all cases. You may quote results from the formula sheet or from lecture notes if you make explicit the result you are quoting.

1. Identify which of the following functions are **not** legitimate specifications for cdfs for a scalar random variable, giving specific reasons as to your conclusions.

   (a)
$$F(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}.$$

3 MARKS

   (b) For some p, $0 < p < 1$
$$F(x) = \begin{cases} 0 & x < 0 \\ p & x = 0 \\ 1 - (1-p)(1 - \exp\{-x^2\}) & x > 0 \end{cases}.$$

3 MARKS

   (c) For $\lambda > 0$
$$F(x) = \frac{\exp\{\lambda(x-2)\}}{1 + \exp\{\lambda(x-2)\}} \qquad x \in \mathbb{R}$$

3 MARKS

   (d)
$$F(x) = e^{-1} \sum_{j=0}^{x} \frac{1}{j!} \qquad x = 0, 1, 2, \ldots$$

   and $F(x) = 0$ otherwise.

3 MARKS

   (e)
$$F(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \le x < 0 \\ x^2/8 + 1/2 & 0 < x \le 2 \\ 1 & x \ge 2 \end{cases}.$$

3 MARKS

2. Suppose that X and Y have a joint probability distribution that is specified using marginal and conditional distributions as follows:

$$X \sim Gamma(\alpha, 1)$$

$$Y|X = x \sim Poisson(x)$$

for some $\alpha > 0$.

(a) Find the marginal pmf for the discrete random variable Y, and find $P_Y[Y = 0]$. 6 MARKS

(b) Find the expectation of Y. 3 MARKS

(c) Suppose that Z is the binary random variable (taking values in the set $\{0, 1\}$ with probability 1), defined by

$$Z = \mathbb{1}_{\{0\}}(Y).$$

where $\mathbb{1}_A(.)$ is the indicator function for set A. Find the expectation of Z.

6 MARKS

3. Suppose that continuous random variable X has a $Uniform(0, a)$ distribution, for some $a > 0$. Let random variable Y be defined by

$$Y = -\log X + \log a$$

where $\log$ denotes natural logarithm.

(a) Find the pdf of Y, and the expectation of Y. 4 MARKS

(b) Find the quantile function for X, and the quantile function for Y. 4 MARKS

(c) Suppose that X_1 and X_2 are independent, and have the same distribution as X. Find the probability

$$P\left[X_1 > X_2\right]$$

4 MARKS

(d) Identify the transformation $g(.)$ such that the transformed variable $Z = g(X)$ has a standard normal distribution (that is, $Z \sim Normal(0, 1)$). 3 MARKS

4. Suppose that Z_1 and Z_2 are independent $Normal(0, 1)$ random variables.

(a) Compute

$$\mathbb{E}_{Z_1,Z_2}[Z_1^6 Z_2^9]$$

5 MARKS

(b) Find the covariance between random variables X_1 and X_2 where

$$X_1 = Z_1 \qquad X_2 = Z_1^2.$$

4 MARKS

(c) Compute the Kullback-Leibler divergence between the pdfs f_0 and f_1, $KL(f_0, f_1)$, if f_0 is the $Normal(\theta_0, 1)$ density, and f_1 is the $Normal(\theta_1, 1)$ density.

6 MARKS