MATH 556 - MID-TERM EXAMINATION 2008

Marks can be obtained by answering all questions. All questions carry equal marks.

The total mark available is 60, but rescaling of the final mark may occur.

Show your working in all cases. You may quote results from the formula sheet or from lecture notes if you make explicit the result you are quoting.

1. Suppose that continuous random variable X has a $Uniform(0,1)$ distribution.

   (a) Find the pdf of Y where
   $$Y = X(1-X)$$
   Find also the expectation of Y.

   6 MARKS

   (b) Suppose that X_1 and X_2 are independent, and have the same distribution as X. Find the probability
   $$\mathrm{P}\left[X_1 X_2 > \frac{1}{2}\right]$$
   and the probability
   $$\mathrm{P}\left[(1-X_1)(1-X_2) > \frac{1}{2}\right]$$

   9 MARKS

2. Suppose that Z_1 and Z_2 are independent $Normal(0,1)$ random variables.

   (a) Find the joint pdf of random variables X_1 and X_2 defined by
   $$X_1 = Z_1 + Z_2 \qquad X_2 = Z_1 - 2Z_2.$$

   6 MARKS

   (b) Find the covariance between random variables Y_1 and Y_2 where
   $$Y_1 = Z_1^2 \qquad Y_2 = Z_1^3$$

   6 MARKS

   (c) Find the moment generating function of
   $$V = \alpha Z_1 + \beta Z_2$$
   for real constants α and β.

   3 MARKS

3. (a) Suppose that X has pdf

$$f_X(x) = \frac{1}{2\sigma}\exp\{-|x/\sigma|\} \qquad -\infty < x < \infty$$

for parameter $\sigma > 0$. Find the characteristic function of X

6 MARKS

(b) Suppose that $X_1, \ldots, X_n$ are independent and identically distributed random variables with characteristic function

$$C_X(t) = \exp\{\mu i t - |2t|^\alpha\} \qquad t \in \mathbb{R}$$

for parameters $0 < \alpha \leq 2$, and $\mu \in \mathbb{R}$.

(i) Are $X_1, \ldots, X_n$ continuous random variables ? Justify your answer. Recall that for complex numbers z_1, z_2

$$|z_1 z_2| = |z_1||z_2|$$

4 MARKS

(ii) Suppose $\mu = 0$. Find real constants a_n and b_n such that T_n defined by

$$T_n = a_n + b_n \sum_{i=1}^{n} X_i$$

has the same distribution as X_1.

5 MARKS

4. (a) Write down the form of an Exponential Family distribution in its *natural* (or *canonical*) parameterization.

4 MARKS

(b) Suppose that X has a one-parameter, natural Exponential Family distribution with natural parameter η, and pmf/pdf $f_X(x|\eta)$. Show that

$$E_{f_X}[X] = \kappa(\eta)$$

for some function κ to be identified.

You may quote without proof properties of the score function.

6 MARKS

(c) Suppose that $X \sim Gamma(\alpha, 1)$. Is the distribution of $Y = 1/X$ an Exponential Family distribution ? If so, find the natural parameter. If not, explain why not.

5 MARKS